Journal of Mathematical Analysis
ISSN: 2217-3412, URL: http://www.ilirias.com
Volume 5 Issue 1(2014), Pages 20-35.

GEOMETRY OF AN $\mathbb{A}$-BILINEAR FORM, DARBOUX THEOREM: A LAGRANGIAN PERSPECTIVE

A. MALLIOS*, A. CONTE-THRASYVOULIDOU, Z. DAOULTZI-MALAMOU

Abstract. The present account constitutes a further scrutiny and still amelioration and extension of the "*geometric*" perspective, selon Lagrange, of the classical "*Geometric Algebra*" (E. Artin), started already by the first papers of the senior author of this study with P. P. Ntumba [11, 12], within a *sheaf-theoretic* context; the latter was motivated by a similar treatise in [9], followed by potential physical applications in theoretical physics: e.g. *gauge theories* and *quantum gravity* [10]. In all this the classical aspect of a background, so-called "*space-time*" manifold has been *replaced by* a *sheaf-theoretic context.* An echo of the above with extensions/generalizations of fundamental aspects, pertaining to what we may call "*geometry of a bilinear form*", as rooted already in Lagrange work, is presented herewith. The values of the "*forms*" employed are taken, in view still of potential physical applications, in suitable real/complex (commutative unital associative) algebras.

0. Introduction

As already hinted at in the *Abstract*, the subsequent discussion aims at getting in the more general context possible, basic notions and fundamental aspects (as e.g. Darboux Theorem) of nowadays *Symplectic Geometry*, by looking at the very roots/ideas of the theory that go back, of course, already to J.-L. Lagrange (1808). The same actually belong, in recent terms, to "*Geometric Algebra*", as presented for instance in the classic of E. Artin [2]. Indeed, one is concerned here in fact with what one may consider, in principle, as the "*geometry of a bilinear form*"; a notion very akin still to the spirit of the initiator of the theory, as above. Whence, the employed phraseology, as "*Lagrangian $\mathbb{A}$-planes*" (see Definition 1.1 below), a concept of fundamental rôle for the sequel, along with the synonym, in practice, of the particular case of a "*hyperbolic*" one (cf. (1.11), as follows). In this context, the celebrated "*Darboux decomposition*" is accomplished herewith, in terms of "*hyperbolic Lagrangian $\mathbb{A}$-planes* (see Theorem 3.1 in the sequel). Thus, to be in accord with our intention of generality (cf. *Abstract*), suggested now even by

*The corresponding author

Mathematics Subject Classification (2000) 15A63, 15A33,
Keywords: $\mathbb{A}$-module, (pre)symplectic $\mathbb{A}$-module, symmetric/skew symmetric $\mathbb{A}$-bilinear form, isotropic element, hyperbolic elements/space, (hyperbolic) Lagrangian $\mathbb{A}$-plane, *geometry of an* ($\mathbb{A}$)-*bilinear form*, orthogonal geometry, (generalized) Darboux theorem, Darboux decomposition, Darboux matrix, PID-algebras.

Submitted January 11, 2014. Published March 23, 2014.

concrete *physical applications*, "*wave*" functions e.g. in Quantum Theory (see [10], yet Note 3.1 below), one is led to consider as "*domain of coefficients*" of our linear maps no more scalars (the standard fields $\mathbb{R}$, $\mathbb{C}$), but a suitable more general (real/complex) *algebra* $\mathbb{A}$; in other words, to deal with $\mathbb{A}$-*modules*: this implies confrontation with delicate subtleties just forfeit to consider "*vectors*" (from $\mathbb{A}$), in place of usual scalars; therefore, the appearance in crucial cases of particular, suitable indeed thus far, *algebras*, as for instance, PID ("*principal ideal domain*") ones (see e.g. (3.16) below). In this context, cf. also e.g. [11]. We take thus throughout a particular care to point out the real essence of each problem at issue, so that to assume the minimal possible hypothesis so far to overcome it, and to recast the classical case. See, for instance, Section 5, NB in the sequel. This helps, of course, to have further a "*perspicuous view of the foundations of a typical building*", as Geometric Algebra/Symplectic Geometry, following thus also, in that context, a classical demand (L. Wittgenstein).

1. Preliminaries

In the sequel $\mathbb{A}$ denotes a *unital commutative* (linear associative) *algebra* over a *field* $\mathbb{K}$. (We tacitly assume that the *characteristic* of $\mathbb{K}$ is *different from* 2). Yet, by assumption, we also consider throughout, *left* $\mathbb{A}$-*modules*. So given an $\mathbb{A}$-*module* E, suppose that

$$\phi : E \times E \longrightarrow \mathbb{A} \tag{1.1}$$

is an $\mathbb{A}$-*bilinear map* ("*form*") on E, which we further assume to be *skew symmetric*; thus, by definition, for any pair (x, y) in E, one has the relation $\phi(x, y) = -\phi(y, x)$. Hence, due to our hypothesis for $\mathbb{K}$, ϕ *is* identically *zero* on the *diagonal* $\Delta \subseteq E \times E$. Now, the following fact is *crucial throughout the ensuing discussion*: Namely, we have that;

$$\begin{aligned} &\text{any two } \textit{elements} \;\; x, y \;\; \textit{in} \;\; E, \textit{for which} \;\; \phi(x, y) \in \mathbb{A} \;\; (\text{cf. (1.1)}), \\ &\text{is } \textit{not a zero divisor}, \text{ are } \;\; \mathbb{A} - \textit{linearly independent.} \end{aligned} \tag{1.2}$$

The proof is routine, based strictly on our hypothesis for the element $\phi(x, y) \in \mathbb{A}$ (viz. *not zero divisor*); see also e.g. N. Bourbaki [4: Chap. II, p. 25, comments following Definition 10]. The previous result enables now one to consider the next *basic notion* for all that follows. That is, we set.

Definition 1.1. Consider a pair (E, ϕ) consisting of an $\mathbb{A}$-*module* E, and a *skew symmetric* $\mathbb{A}$-*bilinear form* ϕ on it. Then, one speaks of a *Lagrangian* $\mathbb{A}$-*plane* of E (being, as we shall see a *sub*-$\mathbb{A}$-*module* of E), any time there exists a *pair* (x, y) *of* (elements of) E, *as in (1.2)*. We denote it by,

$$L \equiv \mathbb{A}[x, y] (\cong \mathbb{A}^2), \tag{1.3}$$

the isomorphism, as before, being an $\mathbb{A}$-*isomorphism.* Yet, for convenience, we shall speak of (1.3), occasionally, just, of a *Lagrangian plane* of E, as well.

Thus, a *Lagrangian* $\mathbb{A}$-*plane*, whenever it exists is, by definition, that *sub*-$\mathbb{A}$-*module* of E, generated by the pair $\{x, y\}$ of elements of E, as in (1.2); we also call x, y, as before, the "*axes*" of L, as in (1.3). Now, one speaks of a

$$\begin{aligned} &\textit{pre-symplectic} \;\; \mathbb{A} - \text{module, for a pair } \;\; (E, \phi), \;\; \text{as in Definition 1.1,} \\ &\text{which (at least) contains a } \textit{Lagrangian} \;\; \mathbb{A} - \text{plane, as in (1.3).} \end{aligned} \tag{1.4}$$

Thus, *by* the very *definition*,

every pre-symplectic $\mathbb{A}$-module has, at least, "dimension" 2 (cf. (1.3)). Yet, we can say that, (1.5)

pre-symplectic $\mathbb{A}$*-modules are, in fact, characterized by* the condition *(1.2).* (1.5.1)

As a result, one thus concludes that;

there are no actually "*1-dimensional pre-symplectic* $\mathbb{A}$*-modules",* when $\mathbb{A}$-*appropriate*: take e.g. $\mathbb{A}$, an "*integral domain*" (cf. thus (1.2)). (1.6)

Thus, one further infers that, *only symmetric* $\mathbb{A}$*-bilinear forms* are allowed *on such* $\mathbb{A}$*-modules*, with $\mathbb{A}$ for instance *as above*. Yet, we can say that,

on "*1-dimensional*" (pre-symmplectic), $\mathbb{A}$*-modules*, having for instance $\mathbb{A}$, as before, *only "symmetry" is allowed*! (1.7)

In this context, we still note that, *by assumption,* $\mathbb{A}$*-bilinear forms* on a given $\mathbb{A}$-*module* E, that define our "*geometry*" on E (cf. for instance, E. Artin [2: p. 105], are *non-trivial* (: non identically zero). So, based on the preceding, we conclude that:

pairs (E, ϕ), *such that* $E \cong \mathbb{A}$, within an $\mathbb{A}$-*isomorphism*, having the $\mathbb{A}$-*bilinear form* ϕ, *skew symmetric, do not* actually *exist*(!); of course (see e.g. (1.2)), under *suitable restrictions on* (the algebra) $\mathbb{A}$, our "*domain of coefficients/scalars*" of the $\mathbb{A}$-modules considered. (1.8)

As we shall see, *the above* is *still valid* in general *for* all *odd dimensional* (free) "*symplectic*" $\mathbb{A}$*-modules*; thus, under further appropriate *conditions for* ϕ, apart from (1.2), concerning $\mathbb{A}$ as well. See the next Definition 1.2.

Now, according to the definition of a *skew symmetric* $\mathbb{A}$*-bilinear form* ϕ on a given $\mathbb{A}$*-module* E, every element $x \in E$ is "*isotropic*", in the sense that one has, $\phi(x, x) = 0$; yet, alias, *nilpotent*, by setting:

$$\phi(x, x) \equiv x \cdot x = x^2 = 0, \tag{1.9}$$

extending thus in our case standard terminology: see, for instance E. Artin, as above, p. 118. Thus, extending further the *classical theory*, viz. "*Geometric Algebra*", still cf. E. Artin [2], for $\mathbb{A} = \mathbb{K}$, see (1.1), one can consider "*hyperbolic pairs*", (x, y) of elements of E, as before, for which one has, by definition, that

$$\phi(x, y) = 1. \tag{1.10}$$

See also ibid., p. 119, Definition 3.8. Thus, one meets here an *important*, as we shall see, *particular case* of our previous *condition (1.2)*. So, in other words,

every hyperbolic pair of elements of E, say (x, y), *provides* (cf. Definition 1.1), a *Lagrangian* ($\mathbb{A}-$)plane of E (see (1.3)); we still name it, "*hyperbolic*". (1.11)

Such planes of E are going to play an important rôle in the sequel (cf., for instance, Theorems 2.1 and 3.1 below).

Note 1.1. — We can also follow herewith the terminology of E. Artin, ibid., and look at $\phi(x, y) \in \mathbb{A}$, for a given pair (x, y) in E, as the ("*vectorial*") "*angle*" of the elements ("*vectors*") x, y in E, denoted by,

$$\sphericalangle(x, y) := \phi(x, y) \in \mathbb{A}. \tag{1.12}$$

See also ibid., p. 105. Therefore, we can say that:

$$\begin{aligned}&\text{given a } \textit{hyperbolic Lagrangian } \mathbb{A}\textit{-plane} \text{ (see(1.11))}, \textit{its}\\&\text{“}\textit{axes are orthogonal}\text{”; that is, one has, by the very}\\&\text{definitions, cf. (1.11)/(1.12), and (1.10):}\end{aligned} \tag{1.13}$$

$$\sin(\sphericalangle(x, y) \equiv \phi(x, y)) = 1. \quad (1.13.1)$$

Now, we come to the main objective of this article; namely, the study of (skew symmetric) *non-degenerate* $\mathbb{A}$-*bilinear forms*. Thus, following *Artin's classic* again, as above, to consider "*symplectic goemetry*", proper; precisely speaking, a "*generalized*" one, due to the presence of the *algebra* $\mathbb{A}$, as a domain of scalars, not necessarily a field. So we further set the next.

Definition 1.2. Let (E, ϕ) be a pair consisting of an $\mathbb{A}$-*module* E and a (*skew symmetric*) $\mathbb{A}$-*bilinear form* ϕ on it (cf. (1.1)). Now, we say that the given map ϕ is *non-degenerate*, whenever the following condition holds true:

$$\begin{aligned}&\textit{for any } x \in E, \ \textit{for which} \text{ one has,}\\&\phi(x, y) = 0, \ \textit{for any } y \in E, \qquad (1.14.1)\\&\textit{then}, \ x = 0, \ \text{as well.}\end{aligned} \tag{1.14}$$

Equivalently, one obtains:

$$\begin{aligned}&\textit{for any } x \neq 0 \text{ in } E, \ \textit{there exists } y \in E\\&\text{(necessarily, } \textit{non-zero}\text{)}, \ \textit{such that}, \ \phi(x, y) \neq 0.\end{aligned} \tag{1.15}$$

Now, by an obvious *abuse of language*, and just for convenience of the terminology, we are going to apply further on, *a pair* (E, ϕ), as in the previous Definition 1.2, viz. *with* ϕ still *non-degenerate*, will be also, occasionally, called, just, *non-degenerate* $\mathbb{A}$-*module*. In particular, we are now ready to come to the following *basic notion* for all that follows. That is, we set.

Definition 1.3. We call *symplectic* $\mathbb{A}$-*module*, a pair (E, ϕ), for which E is an $\mathbb{A}$-*module* and ϕ a *non-degenerate skew symmetric* $\mathbb{A}$-*bilinear form*, which still *satisfies cond. (1.2)*; that is, *we assume* herewith, *by definition*/(cf. also (1.4)), that:

$$E \text{ is a } \textit{non-degenerate pre-symplectic } \mathbb{A}\textit{-module}. \tag{1.16}$$

On the other hand, suppose more generally that we have,

$$L \equiv \mathbb{A}[x, y] \cong \mathbb{A}^2, \tag{1.17}$$

within an $\mathbb{A}$-*isomorphism*; thus, by assumption, L is a *free* $\mathbb{A}$ - *module*, of *rank/ dimension* 2, *sub*-$\mathbb{A}$-*module* of a given $\mathbb{A}$-*module* $E \equiv (E, \phi)$, see e.g. Definition 1.2, such that the elements x, y *in* E are $\mathbb{A}$-*linearly independent* (:"*axes*" of L). We don't call L a "*Lagrangian* $\mathbb{A}$-*plane*" (cf. Definition 1.1), since *we didn't assume cond. (1.2)*; namely, that $\phi(x, y) \in \mathbb{A}$ is *not a zero divisor* (cf. also e.g. (1.5.1)). However, we do have certain useful informations, concerning (1.17). That is, suppose that,

$$\phi|_L \ \text{ is } \textit{non-degenerate}. \tag{1.18}$$

Then, *equivalently*, one gets at the following condition;

$$\begin{aligned}&\textit{for any } a \in L, \text{ as in (1.17), } \textit{with}\\ &\phi(a,x)=\phi(a,y)=0, \qquad\qquad (1.19.1)\\ &\textit{one concludes that } a=0.\end{aligned} \tag{1.19}$$

Indeed, the assertion about the *equivalence of (1.18)/(1.19)* is *immediate*, by the very definitions (see also Definition 1.2). So as a consequence, the aforementioned equivalence yields for instance that: given a pair (E,ϕ) as above, and an $\mathbb{A}$-*plane* (in fact, *sub*-$\mathbb{A}$-*module*) of E, *as in (1.17)*, that is actually a pair $(L,\phi|_L)$, one concludes that;

> *the condition* $\phi(x,y)=0$ *excludes* the *non-degeneracy of* ϕ.
> That is, in other words, if (the *skew symmetric* $\mathbb{A}$-*bilinear form*) $\phi : E\times E \longrightarrow \mathbb{A}$ (cf. (1.1)) is still *non-degenerate*, then *one* gets at the following relation; (1.20)
>
> $$\phi(x,y)\neq 0, \qquad (1.20.1)$$
>
> *for the "axes" of any* $\mathbb{A}$ *-plane* L *of* E, *as in (1.17).*

The assertion follows, straightforwardly, in view of (1.19)⇔(1.18), and (1.15)⇔(1.14). On the other hand,

> the condition of being the "*axes*" $\{x,y\}$ of a given $\mathbb{A}$-*plane* $L\equiv\mathbb{A}[x,y]$ *of* E (cf. (1.17)), "*hyperbolic*", that is, *satisfy the relation* (1.10), see also (1.11) for the terminology applied, can be reduced via an appropriate change of the basis $\{x,y\}$ *of* L, to the condition, (1.21)
>
> $$\phi(x,y)\in\mathbb{A}^{\bullet}; \qquad (1.21.1)$$
>
> that is, $\phi(x,y)$ is an *invertible* element, alias a "*unit*", of $\mathbb{A}$.

Of course, now L is still a *Lagrangian* $\mathbb{A}$-*plane* of E (cf. Definition 1.1). Yet, in view of (1.10), we also occasionally call the given $\mathbb{A}$-bilinear form ϕ on E, "*normalized*", with respect to the *basis* (: *axes*) of L, as above. So we also speak, herewith of a *hyperbolic Lagrangian* $\mathbb{A}$-*plane* of E, still a *fundamental notion* for the sequel. Thus, first, based on the preceding (see e.g. (1.18)⇔(1.19), and Definition 1.3), *one* actually *gets at the following.*

> every *hyperbolic Lagrangian* $\mathbb{A}$-*plane* L of E (cf. (1.17)) *is*, in fact, "*symplectic*": that is, the pair,
>
> $$(L,\phi|_L), \qquad (1.22.1) \qquad\qquad (1.22)$$
>
> yields a *symplectic* (sub-) $\mathbb{A}$-*module* (of E, cf. Definition 1.3); yet, otherwise, $\phi|_L$ *is non-degenerate*(!), as well.

2. "Orthogonal" complements through $\mathbb{A}$-bilinear forms

We briefly fix first the terminology, we are going to apply in the sequel, according to the title of this Section.

Thus, let (E,ϕ) be a given pair, consisting of an $\mathbb{A}$-*module* E and an $\mathbb{A}$-*bilinear form* ϕ on E (cf. (1.1)). So along with E, one can consider the *dual* $\mathbb{A}$-*module* of E, denoted by E^*, and given as follows;

$$E^* := Hom_{\mathbb{A}}(E,\mathbb{A}), \tag{2.1}$$

that is the $\mathbb{A}$-*module* of all $\mathbb{A}$-*linear forms* on E. Yet, given ϕ, as before one still defines the map,

$$\begin{gathered}\hat{\phi} : E \longrightarrow E^*, \ \textit{with}\ \hat{\phi}(x) \equiv \phi_x \in E^*, \ x \in E, \ \textit{such that} \\ \phi_x(y) \equiv (\hat{\phi}(x))(y) := \phi(x, y), \quad y \in E.\end{gathered} \tag{2.2}$$

As a result, for any *subset* $M \subseteq E$, one can define its "*orthogonal*" (sub-)$\mathbb{A}$-*module* of E, with respect to ϕ, denoted by $M^{\perp_\phi} \equiv M^\perp$, and defined by the relation,

$$\begin{aligned} M^\perp &:= \{x \in E : \phi(x, y) \equiv \phi_x(y) = 0, \quad \text{for any} \quad y \in M\} \\ &\equiv \{x \in E : x \perp M\} = \{x \in E : \ M \subseteq \ker \phi_x\};\end{aligned} \tag{2.3}$$

that is, *those* $x \in E$, that are "*orthogonal*" *to all of* M, with respect to ϕ.

Now, based on the previous jargon, we get at the following useful/convenient *alternative* of the notion *of non-degeneracy of* ϕ, as above. That is, we have the next.

Lemma 2.1. Let (E, ϕ) be a pair, consisting of an $\mathbb{A}$-*module* E and an $\mathbb{A}$-*bilinear form* ϕ on it. Then, ϕ is *non-degenerate* (viz. just, *(1.14) is valid), if, and only if,* $\hat{\phi}$ (cf. (2.2)) *is one-to-one.*

Proof. Clear, according to the very definitions, as above: see (1.14) and (2.2). □

In this context, and *in conjunction with (2.2)*, we also remark that:

$$\begin{gathered}\text{if the given } \mathbb{A}\text{-}\textit{bilinear form}\ \phi \text{ is (either} \textit{symmetric} \text{ or)} \textit{skew symmetric},\\ \text{and } \textit{there exists}\ x\ (\neq 0) \text{ in } E,\ \textit{such that} \text{ the corresponding (partial)}\\ \text{map } \phi_x \equiv \hat{\phi}(x) \text{ (cf. (2.2))} \textit{is one-to-one}, \text{ then, } \hat{\phi}\ \textit{is}\ 1-1,\ \textit{as well}; \text{ viz.}\\ \text{(Lemma 2.1), } \phi\ \textit{is non-degenerate.}\end{gathered} \tag{2.4}$$

On the other hand, based on (2.3), and taking, therein, $M = E$, one obtains a useful relation for the sequel, that is, we get;

$$\ker \hat{\phi} = E^\perp \subseteq \text{ (by definition, cf. (2.3)) } E, \tag{2.5}$$

such that *one* obviously (ibid.) *obtains* $(\ker \hat{\phi})^\perp \equiv E^{\perp\perp} = E$; the last equation under suitable conditions for $\mathbb{A}$: e.g. there are *no* "*isotropic*" elements in E (cf. [2: p. 118]) ; thus, take, for instance, $\mathbb{A}$, an *integral domain.*

We come now to a *basic conclusion*, pertaining to *hyperbolic Lagrangian* ($\mathbb{A}$-) *planes*; viz. consequences of ϕ being "*normalized*": cf. (1.10), however, see also (1.21.1). So one gets now at the following.

Theorem 2.1. Let (E, ϕ) be a pair of an $\mathbb{A}$-*module* E, and ϕ a *skew symmetric* $\mathbb{A}$-*bilinear form* on E (cf. (1.1)). Moreover, suppose that,

$$L \equiv \mathbb{A}[x, y]\, (\cong \mathbb{A}^2) \tag{2.6}$$

is a *hyperbolic Lagrangian* $\mathbb{A}$-*plane* of E (cf. (1.11)). Then, *one gets the relation*;

$$E = L \oplus L^\perp. \tag{2.7}$$

That is, in other words, one concludes that;

$$\begin{gathered}\textit{every hyperbolic Lagrangian } \mathbb{A}\textit{-plane of } E \textit{ is "complemented" in } E,\\ \text{with respect to } \phi : \ \text{viz. } \textit{(2.7) holds true.}\end{gathered} \tag{2.8}$$

Furthermore, *the restriction of* ϕ *to each one of the two members of (2.7)* yields *symplectic* (sub-)$\mathbb{A}$-*modules* of E.

Proof. Given that $L + L^{\perp} \subseteq E$, as $\mathbb{A}$-*modules, we* first *prove that*;

$$L \cap L^{\perp} = (0). \tag{2.9}$$

Indeed, let $z \in L \cap L^{\perp}$, so that since $\{x, y\}$ is, by hypothesis, a *basis* of $L \equiv \mathbb{A}[x, y]$, one has, $z = \lambda x + \mu y$, with λ, μ in $\mathbb{A}$. Therefore, we obtain,

$$\phi(z, x) = \phi(\lambda x + \mu y, x) = \lambda\phi(x, x) + \mu\phi(y, x) = -\mu = 0;$$

see also (1.10) (hypothesis for (E, ϕ)), and the fact that, still $z \in L^{\perp}$ along with (2.3). Similarly, since $\phi(z, y) = 0$, one still gets at $\lambda = 0$, so that $z = \lambda x + \mu y = 0$, which *proves (2.9)*. So, we actually have $L + L^{\perp} = L \oplus L^{\perp} \subseteq E$. *We* now *prove* the reverse relation,

$$E \subseteq L \oplus L^{\perp}. \tag{2.10}$$

Taking $z \in E$, we first remark that,

$$z = z + \phi(z, x)y - \phi(z, y)x - \phi(z, x)y + \phi(z, y)x,$$

such that $\phi(z, y)x - \phi(z, x)y \in L$, while *we* further *prove* that

$$v \equiv z + \phi(z, x)y - \phi(z, y)x \in L^{\perp}. \tag{2.11}$$

Indeed, *we prove* that $\phi(v, \beta) = 0$, *for any* $\beta = \lambda x + \mu y$, with λ, μ in $\mathbb{A}$, an arbitrary, in fact, *element of* L. The assertion is immediate, due to $\mathbb{A}$-*bilinearity of* ϕ*, and the* assumption, by *hypothesis, that* $\phi(x, y) = 1$*, which* thus *yields* now *(2.7)*. So, *this completes the first part of the theorem.* Now our resertion, concerning the pair,

$$(L, \phi|_L), \tag{2.12}$$

is already a *consequence of* our *hypothesis for L, and* relevant remarks about *(1.22.1)* (cf. also Definition 1.3). On the other hand *supposing that* $L^{\perp} \neq (0)$ (otherwise, *(2.8) is* already *reduced to (2.7)*), *let* $z \neq 0$ *in* $L^{\perp}$. Thus, based further on the *hypothesis that* ϕ *is non-degenerate*, we can systematically *employ (1.19)*: So, since $z \neq 0$, *there exists* v $(\neq 0)$ *in E, with* $\phi(z, v) \neq 0$; yet, in view of (2.7), one gets $v = \alpha + \beta$, such that $\alpha \in L$ and $\beta \in L^{\perp}$. Hence, one obtains;

$$0 \neq \phi(z, v) = \phi(z, \alpha + \beta) = \phi(z, \alpha) + \phi(z, \beta) = \phi(z, \beta), \tag{2.13}$$

while we also remark that *one has* $\beta \in L^{\perp}$, *with* $\beta \neq 0$. The last relations yield now exactly the *non-degeneracy of* ϕ *on* $L^{\perp}$, so that the pair

$$(L^{\perp},\ \phi|_{L^{\perp}}) \tag{2.14}$$

is still a "*symplectic*" (sub-)$\mathbb{A}$-*module of* E, in what we may also call, in the "*general sense*"; viz. *(2.14) is* a pair, just, *as the* initially *given one* (E, ϕ), and we are done. □

By repeating, actually extending, the argument in the last part of the previous theorem, one gets, in fact, at the following more general conclusion. That is, we have.

Corollary 2.1. Let (E, ϕ) be a *symplectic* $\mathbb{A}$-*module* (in the "*general sense*", viz. like in Theorem 2.1), and $M \subseteq E$ a *sub*-$\mathbb{A}$-*module* of E, *complemented in* E, *through* $M^{\perp}$ *(: "orthogonal" complement of* M in E); *that is*, we *suppose that*,

$$E = M \oplus M^{\perp}. \tag{2.15}$$

Then, *the restriction of* ϕ *to* $M^{\perp}$ *is* also *non-degenerate*; alias, *the pair*

$$(M^{\perp}, \phi|_{M^{\perp}}) \tag{2.16}$$

is still a *symplectic* $\mathbb{A}$*-module* (in the "*general sense*").

Note that in the previous result, one does not need ϕ to be "*normalized*"; thus we have *not* actually *employed (1.10)*. Instead, we *assumed (2.15)*.

3. Darboux Theorem

We are ready now to embark on the proof of the classical result in the title, *within our framework*. What we are actually going to prove, roughly speaking, is that;

$$\begin{array}{c}\textit{every symplectic (free) A-module (of finite rank), acquires} \\ \textit{a "Darboux decomposition", being also of even dimension.}\end{array} \tag{3.1}$$

That is, formally, one gets at the following *fundamental result*.

Theorem 3.1. Let E be a *free* $\mathbb{A}$*-module* of *finite rank*, say $n \in \mathbb{N}$ (viz. we assume that $E \cong \mathbb{A}^n$, within an $\mathbb{A}$-*isomorphism*), and $\phi : E \times E \longrightarrow \mathbb{A}$ a *skew symmetric, non-degenerate* $\mathbb{A}$*-bilinear form* of E, which we further assume to be "*hyperbolic*" (cf. (1.10)/(1.21)), relative to "*Lagrangian* $\mathbb{A}$*-planes*" of E (see Definition 1.1). We express the above, succinctly, by just saying that:

$$(E, \phi) \text{ is a "}\textit{symplectic}\text{" } \mathbb{A}\textit{-module} \text{ with "}\textit{hyperbolic Lagrangian } \mathbb{A}\textit{-planes}\text{".} \tag{3.2}$$

Then, the "*Darboux Theorem*" says that (E, ϕ), or simply E, *is a* "*hyperbolic space*". That is, E *is* (modulo an $\mathbb{A}$-*isomorphism*) *the* ($\phi-$) *orthogonal* (see (2.3)/(2.7)) *direct sum of its* (hyperbolic) *Lagrangian* $\mathbb{A}$*-planes*; namely, one obtains the following ("*Darboux*") *decomposition*:

$$E = L_1 \oplus L_2 \oplus \cdots \oplus L_s, \tag{3.3}$$

within an $\mathbb{A}$-*isomorphism*, where L_i, $1 \leq i \leq s \in \mathbb{N}$, are *pairwise* ($\phi$)-*orthogonal* $\mathbb{A}$*-planes* of E, as above. Moreover, one concludes that,

$$n = 2s, \tag{3.4}$$

that is, *the* (finite) *rank* n *of* E *is* (always) *even*.

Note 3.1. (Terminological). — The assumption of "*normalization*" of ϕ (cf. (1.10)), *relative to Lagrangian* $\mathbb{A}$*-planes*, as in the statement of the previous theorem, is certainly *crucial for the proof* (see below). Now, in the *classical case* this is redundant, due to the fact that the "*domain of coefficients*" of the ($\mathbb{A}-$)modules involved in effect *vector spaces*, is, in general, just a *field*! However, by taking into account important applications of *symplectic geometry*, even in physics (e.g. *quantum theory*, see, for instance, A. Mallios [10: p. 148, (2.1)]), this is *no more in force* (ibid.). Hence, we are thus compelled to consider appropriate *more general* domains of *coefficients*, than "*scalars*". In this context, we also notice that a *similar situation* appears in A. Mallios - P. P. Ntumba [12: pp. 179, 183], where actually a suitable form/"*echo*" of the analogous condition with (1.10)/(1.21.1), as above, still appears therein! So the present account constitutes also, in fact (see e.g. (1.11) and (1.22) a *further scrutinized "distillation"* of the assumptions, we usually employ in *more general aspects* of the above classical result, at issue.

Proof of Theorem 3.1. Let L_1 be a (*hyperbolic*) "*Lagrangian* $\mathbb{A}$-*plane*" of E (cf. (1.11)). Therefore (Theorem 2.1; (2.7)/(2.8)), one obtains;

$$E = L_1 \oplus L_1^{\perp}, \tag{3.5}$$

while we also conclude (ibid., (2.14)), that the pair,

$$(L_1^{\perp},\ \phi|_{L_1^{\perp}}) \tag{3.6}$$

is also *symplectic*. Hence one obtains that (still (1.11)),

$$L_1^{\perp} = L_2 \oplus L_2^{\perp}, \tag{3.7}$$

where $L_2 \equiv \mathbb{A}[x_1, y_1]$, a *Lagrangian* $\mathbb{A}$-*plane of* $L_1^{\perp}$ (same Theorem 2.1, as before). Thus, in view of (3.5)/(3.7), one gets at the relation;

$$E = L_1 \oplus L_2 \oplus L_2^{\perp}, \tag{3.8}$$

within an $\mathbb{A}$-*isomorphism*, of the $\mathbb{A}$-*modules* involved. Hence, proceeding now *by induction*, one arrives at the following decomposition of E; that is,

$$E = L_1 \oplus \cdots \oplus L_{s-1} \oplus L_{s-1}^{\perp}. \tag{3.9}$$

Now, by virtue of Theorem 2.1 (see (2.14), along the comments after it) *one gets* that,

$$L_{s-1}^{\perp} \equiv L_s \cong \mathbb{A}^2, \tag{3.10}$$

within an $\mathbb{A}$-*isomorphism*; this actually *proves* already *(3.3)*, while we still *obtain*, thereby, $n = 2s$. Yet, the ($\phi-$)*orthogonality*", pair-wise, *of the* $\mathbb{A}$-*planes* involved in (3.3), has been also secured during the same proof, as above (see e.g. (3.5)/(2.7)), and *we are done*. □

3.1. **Darboux "generalized" (: Darboux decomposition with a "residue").** We consider in the sequel the more general case that a given pair (E, ϕ), as in the preceding, has the $\mathbb{A}$-*bilinear form* ϕ, still *skew symmetric*, however, now *not* in general *non-degenerate*. Thus, we examine the "*Darboux decomposition*" of E, as in (3.3), in that *more general case*: That is, *equivalently*, by virtue of *Lemma 2.1*, we consider the case that *the* corresponding *map* $\hat{\phi}$ (cf. (2.2)) is *not*, necessarily, $1-1$. So, by setting,

$$N \equiv \ker \hat{\phi} = E^{\perp}, \tag{3.11}$$

(see (2.5)), we further consider the map,

$$\tilde{\phi} : E/N \times E/N \longrightarrow \mathbb{A}, \tag{3.12}$$

in such a manner that *one defines*,

$$\tilde{\phi}(\dot{x}, \dot{y}) := \phi(x, y), \tag{3.13}$$

for any $(x, y) \in E \times E$; of course *we set* e.g. here, $\dot{x} := [x] \in E/N$, *with* $x \in E$. It is routine to check that $\tilde{\phi}$ as above is *well-defined*, and, in fact a *skew symmetric* $\mathbb{A}$-*bilinear form* on the (*quotiend*) $\mathbb{A}$-*module* E/N, as before; see also e.g. N. Bourbaki [4: Chap. II, p. 7], or T. S. Blyth [3: p. 32ff]. So, according to the very definitions, the *pair*,

$$(E/\ker \hat{\phi}, \tilde{\phi}), \tag{3.14}$$

is a *symplectic* $\mathbb{A}$-*module*; that is, $\tilde{\phi}$ is a *skew symmetric non-degenerate* $\mathbb{A}$-*bilinear form*. See thus, (2.2)/(2.3), (2.5) and (3.13), along with (1.15).

Now, concerning (3.14), and in view of eventual "*peculiarities*" of $\mathbb{A}$, we should remark that,

$$\text{\textit{even if} } E \text{ \textit{is a free} } \mathbb{A}\text{\textit{-module, this is not} always \textit{the case for the quotient} } \mathbb{A}\text{\textit{-module} } E/N; \text{ any way, the latter will be, at least, \textit{finitely generated}, if } E \text{ is \textit{free of finite rank}.} \tag{3.15}$$

See, for instance, N. Bourbaki [4: Chap. II, p. 27]. On the other hand, *the quotient* $\mathbb{A}$*-module*,

$$E/\ker\hat{\phi} \equiv E/N \text{ is still \textit{free of finite rank} (at most, that one of } E\text{), in the case, for instance, that } \mathbb{A} \text{ is a } PID\text{\textit{-algebra} (:``\textit{principal ideal domain}''), and of course with } E \text{ \textit{free of finite rank}.} \tag{3.16}$$

See e.g. T. S. Blyth [3: p. 265, Theorem 18.3]. It is actually the situation, as in *(3.16)*, that we usually adopt! However, cf. Theorem 3.2 in the sequel. Namely, we have that;

$$\text{by \textit{assuming (3.16)}, the } \mathbb{A}\text{-module \textit{(3.14) is free of finite rank}, say } k\ (\leq n \equiv \dim E), \text{ being also \textit{symplectic}.} \tag{3.17}$$

On the other hand, we *further assume* that,

the $\mathbb{A}$-*bilinear map* $\tilde{\phi}$ (cf. (3.12)) is "*normalized*", relative to *Lagrangian* $\mathbb{A}$*-planes of* E/N; viz. *one has*,

$$\tilde{\phi}(\dot{x},\dot{y}) = 1, \tag{3.18.1}$$

for any (Lagrangian) $\mathbb{A}$-*plane* $\mathbb{A}[\dot{x},\dot{y}]$ *of* E/N (see (3.14), and (1.11)). Here we also remark that, in view of (3.13), one concludes that;

$$\tilde{\phi} \text{ is \textit{normalized, if, and only if,} } \phi \text{ \textit{is}.} \tag{3.18.2}$$

(3.18)

In this context, we still notice for use right away, that;

$$\text{the elements } \dot{x},\dot{y} \text{ \textit{in} } E/N \text{ are } \mathbb{A}\text{\textit{-linearly independent, if,} and only if, } x, y \text{ \textit{in} } E \text{ \textit{are}.} \tag{3.19}$$

The assertion is clear, by the very definitions; of course, we set above, $\dot{x} \equiv [x] \in E/N$, *with* $x \in E$. Yet, here E is just an $\mathbb{A}$-*module*; see Section 2, as above. Therefore, by considering a *Lagrangian* $\mathbb{A}$-*plane of* E (take e.g. E now, *pre-symplectic*, cf. (1.11), one gets at the relation;

$$\dot{L} := \mathbb{A}[\dot{x},\dot{y}] \cong \mathbb{A}^2 \cong \mathbb{A}[x,y] \equiv L \subseteq E, \tag{3.20}$$

within $\mathbb{A}$-*isomorphisms* where, of course, L stands for the corresponding, in view of (3.19), *Lagrangian* $\mathbb{A}$-*plane* of E/N. It is still clear, see (3.13), that

$$E/N \text{ is \textit{pre-symplectic, if, and only if,} } E \text{ \textit{is}.} \tag{3.21}$$

Thus, we are now ready to embark on "*Darboux generalized*". That is, one gets at the following *basic result*.

Theorem 3.2. Let (E,ϕ) be a *pre-symplectic A-module* (cf. Definition 1.1), with "*hyperbolic*" *Lagrangian* $\mathbb{A}$*-planes* (viz., *(1.10) is valid* for Lagrangian $\mathbb{A}$-planes (see also (1.5)). Moreover, *assume that*:

$$E/N \text{ (see (3.11)/(3.14)) is a \textit{free A- module} of \textit{finite rank}, say, } k \in \mathbb{N}. \tag{3.22}$$

Then, one obtains the following ("*generalized Darboux*") decomposition of E.

$$E = (L_1 \oplus L_2 \oplus \cdots \oplus L_s) \oplus E^{\perp}, \tag{3.23}$$

within an $\mathbb{A}$-*isomorphism*, such that $k = 2s$; or yet, one has,

$$\dim(E/N) = k = 2s. \tag{3.24}$$

Proof. Based on (3.14) and our hypothesis for E/N, we first remark that, according to Theorem 3.1, specialized now to E/N, one gets at the following "*Darboux decomposition*";

$$E/\ker\hat{\phi} = \dot{L}_1 \oplus \cdots \oplus \dot{L}_s, \quad \textit{with} \;\; 2s = k, \tag{3.25}$$

modulo an $\mathbb{A}$-*isomorphism*. Therefore (*1st Isomorphism Theorem*), one obtains (cf. also (3.20)/(2.5)),

$$E = (\dot{L}_1 \oplus \cdots \oplus \dot{L}_s) \oplus \ker\hat{\phi} = (L_1 \oplus \cdots \oplus L_s) \oplus E^{\perp}, \tag{3.26}$$

within $\mathbb{A}$-*isomorphisms*, and the *proof is finished*. □

4. Matrix representation

By analogy with the classical case of *vector spaces*, viz. when, in particular, one takes $\mathbb{A} = \mathbb{K}$, we can consider, what we may call, the "*matrix analogue*" of the preceding; that is, the form of the above, in terms of *matrices over* $\mathbb{A}$ (see Section 1).

Thus, by looking at a pair,

$$(E, \phi), \tag{4.1}$$

as in Definition 1.3, we can further take a *Lagrangian* $\mathbb{A}$-*plane of* E, say (cf. (1.3)),

$$L \equiv \mathbb{A}[x, y] \cong \mathbb{A}^2. \tag{4.2}$$

Hence, by assumption, if $\phi(x, y) \equiv t \in \mathbb{A}$ (a *non-zero divisor*, by hypothesis, cf. (1.2)), the *matrix of* L is given by,

$$\begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} = t \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.3}$$

Thus, *by taking, in particular* $t \in \mathbb{A}^{\bullet}$ (see (1.21.1)), we can consider the *pair* $(x, t^{-1}y)$, as a *basis of* L, in place of (x, y), as in (4.2). So one arrives at the *classical form*, of the so-called "*Darboux matrix*" of L, viz. one gets (4.3) in the *equivalent form*,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.4}$$

That is, in other words, *under the hypothesis* $t \in \mathbb{A}^{\bullet}$; with $t \equiv \phi(x, y)$, as in (4.2), one concludes that;

> *every Lagrangian* $\mathbb{A}$-*plane of* E "*is*" (viz. can be reduced to) a *hyperbolic* one. (4.5)

See also (1.10)/(1.11) in the preceding. So, in view of (4.5), and Theorem 2.1, concerning the *basic property (2.7)*, for *hyperbolic* $\mathbb{A}$-*planes* (viz. their "*orthogonal complementation*", via ϕ), one comes to the conclusion that;

> every pair (E, ϕ), as above, is the "$(\phi-)$*orthogonal direct sum* of its *hyperbolic* (Lagrangian) $\mathbb{A}$-*planes*, yet in other words a *hyperbolic space*. (4.6)

See Theorem 3.1; (3.3). For the classical case, see e.g. E. Artin [2: p. 119, Definition 3.9, along with Theorem 3.7].

Now, based further on the previous "*matrix analogue*" *of (4.5)*, as in (4.4) (see also (1.21)), one gets at the following *matrix representation of (3.3)*; that is, we obtain.

$$\begin{pmatrix} 0 & 1 & & & & & & & \\ -1 & 0 & & & & & \mathrm{O} & & \\ & & 0 & 1 & & & & & \\ & & -1 & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & & \ddots & & & \\ & \mathrm{O} & & & & & \ddots & & \\ & & & & & & & 0 & 1 \\ & & & & & & & -1 & 0 \end{pmatrix} \tag{4.7}$$

as the "*matrix form*", in terms of $\mathbb{A}$, of the *hyperbolic/(Darboux) decomposition of* (E, ϕ), as in Theorem 3.1.

On the other hand, by looking at the "*generalized Darboux decomposition*", as in (3.26), it is clear that, similarly to (4.7), one defines the corresponding *supplemented matrix to (4.7)*, by just adding to the latter the *matrix* 0, *associated with* $\ker \hat{\phi} = E^{\perp} \subseteq E$. So one gets at the following "*matrix form*" *of (3.26)* (:"*generalized" Darboux*).

$$\begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & \mathrm{O} & & \\ & & \ddots & & & & \\ & & & \ddots & & & \\ & & & & 0 & 1 & \\ & \mathrm{O} & & & -1 & 0 & \\ & & & & & & \mathrm{O} \end{pmatrix} \tag{4.8}$$

5. Orthogonal (generalized) geometry and matrix representation

We consider in the sequel, still an $\mathbb{A}$-*bilinear form*,

$$\phi : E \times E \longrightarrow \mathbb{A}, \tag{5.1}$$

on E, a *free* $\mathbb{A}$-*module* of *finite rank*, say $n \in \mathbb{N}$. That is, *we assume* that,

$$E = \mathbb{A}^n, \tag{5.2}$$

within an $\mathbb{A}$-*isomorphism.* Yet, we suppose this time that ϕ is *symmetric*; that is, we accept that,

$$\phi(x, y) = \phi(y, x), \tag{5.3}$$

for any pair (x, y) of elements *in* E. On the other hand, we say that ϕ, as in (5.1), is *non-degenerate*, whenever *the map* $\hat{\phi} : E \longrightarrow E^*$ (see (2.2)) *is one-to-one*: Thus, we take here Lemma 2.1 in the preceding, as a definition of the "*non-degeneracy*" *of* ϕ; besides, ϕ may be either symmetric or skew symmetric. We further note herewith that by the above definition of the *non-degeneracy of* ϕ, we do generalize against the so-called "*strong non-degeneracy*" of ϕ (: $\hat{\phi}$, as in (2.2), is a *bijection*; see e.g. J. Milnor - D. Husemoller [11: p. 1]). Of course, following here, *classical standards*, extended by considering throughout, an appropriate $\mathbb{A}$: (: *unital commutative algebra* over a *field* $\mathbb{K}$), as in the preceding, for our (generalized) "*domain of coefficients*", we may speak of a pair (E, ϕ), with E an $\mathbb{A}$-*module*, and ϕ a *non-degenerate symmetric* $\mathbb{A}$-*bilinear form* on it (cf. (5.1)), as referring to our (generalized) *orthogonal geometry* on E; see e.g. N. Jacobson [7: p. 361], yet, E. Artin [2: p. 111].

Now, suppose we are given an $\mathbb{A}$-*module* E, and $x \in E$, which is *not* a "*torsion*" *element*: that is, we accept that, the relation,

$$ax = 0, \ \textit{implies}\ a = 0, \tag{5.4}$$

for any element $a \in \mathbb{A}$. *Equivalently*, we thus consider $x \in E$, as before, as an $\mathbb{A}$-*linearly independent* element of E. See also e.g. W. A. Adkins - S. H. Weintraub [1: p. 117]. Hence, one can further look at the *associated with* x, *as in (5.4)*, with that, which one might call an $\mathbb{A}$-*line* of E; that is we set:

$$L \equiv \mathbb{A}[x] \subseteq E. \tag{5.5}$$

Thus, we come first to the following *basic result* for the sequel. That is, one has the next.

Lemma 5.1. Suppose we are given a pair (E, ϕ), with E an $\mathbb{A}$-*module* and ϕ an $\mathbb{A}$-*bilinear form* on E (cf. (5.1)). Moreover, let x be an element of E, *such that*,

$$\phi(x, x) = 1. \tag{5.6}$$

Then, one gets at the following "*orthogonal*" *decomposition* of E (see also (5.5)),

$$E = L \oplus L^{\perp}; \tag{5.7}$$

so, in other words, L is "*complemented*" in E, through $L^{\perp}$, as the latter *sub*-$\mathbb{A}$-*module* of E is given by (2.3).

Proof. First, *we prove that*,

$$L \cap L^{\perp} = (0). \tag{5.8}$$

Indeed, this is an *immediate consequence of (5.5)*, and *the fact that*

$$\left.\begin{array}{l} \textit{for any } z \in E, \text{ one gets;} \\ \phi(z, x) = 0, \ \textit{if, and only if}, \ z \in L^{\perp}. \qquad (5.9.1) \end{array}\right. \tag{5.9}$$

The latter assertion *(5.9) is valid, still by the very definitions* (see also (5.5)). On the other hand, *we also have*;

$$E \subseteq L + L^{\perp}. \tag{5.10}$$

Indeed, if $z \in E$, then it is clear that,

$$z = z + \phi(z, x)x - \phi(z, x)x. \tag{5.11}$$

So, first one has $\phi(z, x)x \in L$ (cf. (5.5)), while *we* still *prove that*, $z - \phi(z, x)x \in L^{\perp}$; namely, in view of (5.9.1), it suffices to see that $\phi(z - \phi(z, x)x, x) = 0$, which is *true*, of course, still *by the very definitions and (5.6)*, and *this finishes the proof of the Lemma.* □

Now, the following lemma is also a *basic result* for the sequel; see thus e.g. Theorem 5.1 below. So one gets at the next.

Lemma 5.2. Let (E, ϕ) be a given pair, consisting of an $\mathbb{A}$-*module* E and an $\mathbb{A}$-*bilinear form* ϕ on it, *such that the following* (orthogonal decomposition of E) *holds true*:

$$E = L \oplus L^{\perp}, \tag{5.12}$$

see (5.5) and (2.3). *Then*,

$$\phi|_L \text{ is } \textit{non-degenerate}. \tag{5.13}$$

See, for instance, *(1.14), valid for any* ϕ, as above.

Proof. Consider an element $y \in L \subseteq E$, such that;

$$\phi(y, z) = 0, \quad \text{for any} \quad z \in L. \tag{5.14}$$

Therefore (cf. (2.3)), $y \in L^{\perp}$, as well, that is, $y \in L \cap L^{\perp} = 0$ (see (5.12)), *which proves* (1.14), hence, *the assertion*. □

As *a spin-off* of the previous lemma, thus in effect of the validity *of (5.12)*, one still gets at the following.

Corollary 5.1. Suppose we are given *a pair* (E, ϕ), *as in Lemma 5.2, so that* (ibid.), *(5.12) holds true. Then*,

$$\phi|_{L^{\perp}} \ \text{ is } \textit{non-degenerate}. \tag{5.15}$$

Proof. According to our hypothesis, the given $\mathbb{A}$-*bilinear form* ϕ on E *is* already *non-degenerate* (Lemma 5.2). Now, to prove (5.15), one may further employ the *criterion/definition* (1.15) (*being* actually *stated, for any* ϕ, as e.g. the given one here). So let $y \in L^{\perp} \subseteq E$, with $y \neq 0$. Hence, by the *non-degeneracy of* ϕ*, there exists* $z \in E = L \oplus L^{\perp}$ (cf. (5.12)), *such that* $z \neq 0$, *and* $\phi(y, z) \neq 0$. Thus, taking e.g. $z = z_1 + z_2$ (cf. (5.12)), one gets,

$$\phi(y, z) = \phi(y, z_1 + z_2) = \phi(y, z_1) + \phi(y, z_2) = \phi(y, z_2) \neq 0, \tag{5.16}$$

which proves (1.15), hence, the assertion as well. □

> **N.B.** Suppose we have a pair (E, ϕ), with E an $\mathbb{A}$-*module* and ϕ an $\mathbb{A}$-*bilinear form* on it. Moreover, let $x \in E$, such that
>
> $$\phi(x, x) \in \mathbb{A} \ \text{ is } \textit{not} \text{ a } \textit{zero divisor}. \tag{5.17}$$
>
> Our hypothesis in *(5.6) meets*, of course, *the previous condition*; yet, the same is certainly valid, automatically by the very definitions, in case $\mathbb{A}$ is e.g. an "*integral domain*" (: no zero divisors at all). In this latter case, (5.6) *still ensures* x, as a *torsion free* element of E; *hence*, as already noted in the preceding (*equivalently*) an $\mathbb{A}$-*linearly independent* one, as well, so that *(5.5) has a meaning*! In this context, we further note that *in an integral domain, zero* is *the only "nilpotent"* element. Yet, we still note, *in conjunction with* (5.4), as well as, *with (5.17)*, that: *over an integral domain, every free module is torsion free.* See, for example, Ref. following (5.4) above, p. 131, Proposition 4.8. Anyway, we still note that;
>
> $$\begin{array}{l}\text{(5.8) } \textit{is valid, by only supposing} \text{ that } \phi(x,x), \\ \textit{with } x \textit{ as in } \text{(5.5)}, \textit{ is not a zero divisor.}\end{array} \tag{5.18}$$

See also (5.9.1), as above. Moreover, a *reinforcement of (5.13)*: That is, one obtains that,

$$\text{(5.13) } \textit{is valid, if and only if, } L \cap L^{\perp} = (0). \tag{5.19}$$

Indeed, just a *consequence of the hypothesis* that,

$$\phi(x, x) \textit{ is not a zero divisor, with } x \textit{ as in (5.5).} \tag{5.20}$$

On the other hand, assuming that (E, ϕ) is a pair, as e.g. in Lemma 5.1, suppose that an *element* $x \in E$, is *such that*:

$$\phi(x, x) \in \mathbb{A}, \text{ is } \textit{not a zero divisor}. \tag{5.21}$$

Then, one still concludes that x *is* $\mathbb{A}$-*linearly independent*; hence, as already explained in the preceding, a *torsion free* element, as well. Therefore,

$$\textit{under} \text{ the hypothesis } \textit{(5.21)}, \text{ the relation } \textit{(5.5) has a meaning.} \tag{5.22}$$

To prove the previous *assertion, concerning the element* $x \in E$, *as in (5.21)*, we remark that; if $ax = 0$, then $\phi(ax, x) = a \cdot \phi(x, x) = 0$, hence, in view of (5.21), one gets $a = 0$.

We are now in the position to state and prove the *main result* of this section. Namely, one gets at the next.

Theorem 5.1. Let (E, ϕ) be a pair with E a *free* $\mathbb{A}$-*module* of *finite rank*, say $n \in \mathbb{N}$, and ϕ an $\mathbb{A}$-*bilinear form* on E. Moreover, *assume that* E has ("*hyperbolic*") $\mathbb{A}$-*lines* of the form (5.5): *we posit* that there exist "*hyperbolic elements*" of E, thus, by definition, *elements* $x \in E$, for which *(5.18) is valid*; hence, in particular, *in effect*, the relation *(5.6)*. Then, E *admits the* following "*orthogonal decomposition*";

$$E = \mathbb{A}[x_1] \oplus \cdots \oplus \mathbb{A}[x_n], \tag{5.23}$$

within an $\mathbb{A}$-*isomorphism, where* $x_i \in E$, *with* $1 \leq i \leq n$, stand for a "(*hyperbolic*) *basis*" of E.

Proof. The proof is made inductively, on the basis of Lemmas 5.1 and 5.2, employing a *similar argument* to the case of ϕ being, skew-symmetric: see Theorem 3.1 in the preceding. So we may omit the details. □

On the other hand, suppose that;

$$\begin{aligned} &\phi \text{ is a } \textit{symmetric } \mathbb{A}\textit{-bilinear form, not} \text{ necessarily } \textit{non-degenerate};\\ &\text{viz. there exists } \ker\hat{\phi} \text{ (cf. (2.2)). That is, one has,}\\ &\qquad\qquad \ker\hat{\phi} \neq \{0\}. \qquad (5.24.1) \end{aligned} \tag{5.24}$$

Then, one gets at the *analogous decomposition of* E, *as in* Theorem 3.2 (see (3.26)).

For the *proof* we can apply a similar argument to that one for Theorem 3.2. Furthermore, one obtains *analogous "matrix representations"*, as in the *skew-symmetric* case; cf. (4.7)/(4.8). Concerning the classical case, one finds e.g. a corresponding account in K. W. Gruenberg and A. J. Weir [6: p. 102, Section 5.4].

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A. Mallios,
Department of Mathematics, University of Athens,, Panepistimioupolis, Athens 15784, Greece
E-mail address: amallios@math.uoa.gr

A. Conte - Thrasyvoulidou,
Department of Mathematics, University of Athens,, Panepistimioupolis, Athens 15784, Greece
E-mail address: aconte@math.uoa.gr

Z. Daoultzi - Malamou,
Department of Mathematics, University of Athens,, Panepistimioupolis, Athens 15784, Greece
E-mail address: zmalamou@math.uoa.gr