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A UNIQUENESS THEOREM OF L-FUNCTIONS WITH RATIONAL MOVING TARGETS

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Abstract. We prove a uniqueness theorem with rational moving targets for a class of $L-$functions, which generalizes a Steuding's theorem in [3]. The class contains the Selberg class, as well as the Riemann-zeta function.

We define the class $\mathbb{M}$ to be the collection of functions

$$L(s) = \sum_{n=1}^{\infty} a(n)/n^s \qquad \text{with} \quad a(1) = 1$$

satisfying Ramanujan hypothesis, analytic continuation, functional equation and having multiplicative coefficients. We also denote the degree of a function $L \in \mathbb{M}$ by d_L which is a non-negative real number. We refer the reader to Chapter six of [3] for a complete definition. Obviously, the class $\mathbb{M}$ contains the Selberg class. Also every function in the class $\mathbb{M}$ is an L function and the Riemann-zeta function is in the class. Steuding ([4]) proved that if $L_1, L_2 \in \mathbb{M}$, c is a constant, and if the roots of the equation $L_1(s)-c=0$ are the same roots of the equation $L_2(s)-c=0$, counting multiplicities, then $L_1 \equiv L_2$. In this short note, as a continuous work of [1], we prove the following theorem.

Theorem. Let $L_1, L_2 \in \mathbb{M}$ and R is a rational function with $\lim_{s\to\infty} R(s) \neq 1$. If the roots of the equation $L_1(s) - R(s) = 0$ are the same roots of the equation $L_2(s) - R(s) = 0$ with counting multiplicities, then $L_1 \equiv L_2$.

Proof. Let T be a positive real number and $N_L(T)$ the number of zeros of $L(s) - R(s) = 0$ in the region $\{s = \sigma + it : \ |t| < T\}$, counting multiplicities. Since L satisfies the functional equation, i.e.,

$$\Phi(s) = \overline{\Phi(1-\bar{s})},$$

with $\Phi(s) = \epsilon Q^s \prod_{j=1}^{\nu} \Gamma(\lambda_j s+\mu_j)L(s)$, where $|\epsilon| = 1$, $Q > 0$, $\lambda_j > 0$, and $\mathrm{Re}\mu_j > 0$; we obtain

$$N_L(T) = 2\sum_{n=1}^{\infty} N_L^*(2^{-n}T),$$

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where $N_L^*(2^{-n}T)$ counts the number of zeros of $L-R$ in the region $\{s=\sigma+it : 2^{-n}T \le t < 2^{-(n-1)}T\}$, counting multiplicities. A theorem in ([3], Page 145) gives

$$N_L^*(2^{-n}T) = \frac{d_L}{2\pi}(2^{-n}T)\log\left(\frac{4\cdot 2^{-n}T}{e}\right) + O(T),$$

so

$$2N_L^*(2^{-n}T) = \left(\frac{d_L}{\pi}T\log\frac{T}{e}\right)\frac{1}{2^n} + \frac{d_L}{\pi}T\left(\frac{\log 4 - n\log 2}{2^n}\right) + O(T).$$

Summing over n, we see

$$N_L(T) \quad = \quad \left(\frac{d_l}{\pi}T\log\frac{T}{e}\right)\sum_{n=1}^{\infty}\frac{1}{2^n} + \frac{d_L}{\pi}T\sum_{n=1}^{\infty}\frac{\log 4 - n\log 2}{2^n} + O(T).$$

By a little computing, the first infinite sum is 1 and the second infinite sum is 0. Consequently,

$$N_L(T) = \frac{d_L}{2\pi}T\log\frac{T}{e} + O(T) \quad \text{as} \quad T\to\infty. \tag{1}$$

Applying (1), which is called the Riemann-von Mangoldt formula for rational moving targets, to L_1 and L_2, we have

$$N_{L_1}(T) = \frac{d_{L_1}}{2\pi}T\log\frac{T}{e} + O(T) \quad \text{and} \quad N_{L_2}(T) = \frac{d_{L_2}}{2\pi}T\log\frac{T}{e} + O(T).$$

Since $N_{L_1}(T) = N_{L_2}(T)$, then, $d_L := d_{L_1} = d_{L_2}$. We first assume that L_1, L_2 are entire, and define the auxiliary function ℓ by

$$\ell(s) = \frac{L_1(s)-R(s)}{L_2(s)-R(s)}.$$

Since the equations $L_1(s) = R(s)$ and $L_2(s) = R(s)$ have the same solutions, counting multiplicities, $\ell(s)$ is non-zero and entire. Let $T(L,r)$ is the Nevanlinna height or Nevanlinna characteristic function of L. Then, Theorem 7.9 in [3] states that

$$T(L_j,r) = \frac{d_L}{\pi}r\log r + O(r), \qquad j=1,2. \tag{2}$$

Furthermore, by Nevanlinna's First Main Theorem (e.g. see pg 148 in [3]), and (2), we obtain

$$\begin{aligned} T(\frac{1}{L_j-R},r) \quad &= \quad T(L_j-R,r)+O(1) \\ &= \quad T(L_j,r)+T(R,r)+O(1) \\ &= \quad \frac{d_L}{\pi}r\log r + O(r). \end{aligned} \tag{3}$$

The second to last equality is due to the computation $T(R,r) = d\log r + O(1)$ where

$$R(s) = \frac{a_p s^p + \cdots + a_0}{b_q s^q + \cdots b_0},$$

and $d = \max\{p,q\}$ (e.g. see 1.3 of [2]). We now compute the growth order $\sigma(L_1 - R)$ for the function $L_1 - R$ in light of (3). Nevanlinna's First Main Theorem implies that

$$\sigma(L_1-R,r) = \quad \limsup_{r\to\infty}\frac{\log T(L_1-R,r)}{\log r} = 1.$$

Now for f,g meromorphic, the inequality

$$\log T(fg,r) \le \log\left(T(f,r)+T(g,r)\right) \le \log\left(\max\{T(f,r),T(g,r)\}\right) + \log 2$$

implies that $\sigma(fg) \le \max\{\sigma(f), \sigma(g)\}$. Thus, $\sigma(\ell) \le 1$. Since ℓ is also non-vanishing and entire, we conclude by the Hadamard Factorization Theorem (e.g. see [5], for example) that $\ell(s) = \exp(P(s))$ where P is a polynomial of degree at most 1.

We now derive upper and lower asymptotic bounds for ℓ as $s \to \infty$ along the positive real axis. Since L_j $(j = 1, 2)$ satisfies Ramanujan hypothesis, i.e., $|a(n)| < n^\delta$ for any $\delta > 0$ and $a(1) = 1$, there exist $\sigma_0 > 0, C_0$ such that

$$|L_j(s) - 1| < C_0 2^{-\mathrm{Re}s}$$

for $\mathrm{Re}s > \sigma_0$. Now for $j = 1, 2$, we have for $\mathrm{Re}s > \sigma_0$ and $|t| \le T_0$ for some $T_0 > 0$ that

$$\begin{aligned} |L_j(s) - R(s)| &\le |1 - L_j(s)| + |R(s) - 1| \\ &\le C_0 2^{-\mathrm{Re}s} + |R(s) - 1| \end{aligned}$$

as well as

$$\begin{aligned} |L_j(s) - R(s)| &\ge |R(s) - 1| - |L(s) - 1| \\ &\ge |R(s) - 1| - C_0 2^{-\mathrm{Re}s}. \end{aligned}$$

Combining these estimates for $j = 1, 2$ as appropriate, we see the following inequality holds uniformly in region $\mathrm{Re}s > \sigma_0$, $|t| \le T_0$:

$$\frac{|R(s) - 1| - C_0 2^{-\mathrm{Re}s}}{|R(s) - 1| + C_0 2^{-\mathrm{Re}s}} \le \ell(s) \le \frac{|R(s) - 1| + C_0 2^{-\mathrm{Re}s}}{|R(s) - 1| - C_0 2^{-\mathrm{Re}s}}.$$

Letting $\mathrm{Re}s \to \infty$ through this region shows that $\ell(s) \to 1$ as $\mathrm{Re}s \to \infty$. Consequently, the polynomial P vanishes identically, showing that $\ell \equiv 1$. Therefore, $L_1 \equiv L_2$. The theorem is therefore proved in this case.

On the other hand, assume that L_1, L_2 have poles at $s = 1$ of orders m_1, m_2, respectively. Then, we may replace L_j in the proof above with the product $(s - 1)^m L_j(s)$, where $m = m_1 + m_2$. Our estimate of $\sigma(\ell)$ does not change in this case, and our asymptotic estimates can be likewise completed by allowing the rational function R to absorb the additional $(s - 1)^m$ factor as in

$$\ell(s) = \frac{L_1(s) - R(s)(s - 1)^{-m}}{L_2(s) - R(s)(s - 1)^{-m}}.$$

Since $R(s)(s - 1)^{-m}$ is again a rational function, the estimates in the first case hold identically, and we have proved the theorem completely.

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