Journal of Mathematical Analysis
ISSN: 2217-3412, URL: http://www.ilirias.com
Volume 5 Issue 1(2014), Pages 11-15.

COUNTEREXAMPLES IN ROTUND AND LOCALLY UNIFORMLY ROTUND NORM

F. HEYDARI, D. BEHMARDI [1]

Abstract. In this paper we investigate some properties of Banach spaces with rotund and locally uniformly rotund norm and introduce some space with no rotund norm. In particular we will show that there exists a compact space K such that $K^{(\omega_1+1)} = \emptyset$ but $C(K)$ does not admit an equivalent rotund norm. We introduce a subspace of $\ell^c_\infty(\Gamma)$ which does not contain $\ell^c_\infty(\Gamma')$ for any $\Gamma' \subseteq \Gamma$. Also we investigate some relation between rotundity of X, X^* and X^{**}.

1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space. Recall the set $S_X = \{x \in X : \|x\| = 1\}$ and $B_X = \{x \in X : \|x\| \leq 1\}$ as unit sphere and closed unit ball of X, respectively. Define $\varphi : X \to X^{**}$ by

$$\varphi(x)(f) = f(x) \quad x \in X, f \in X^*.$$

φ is an isometry linear map and $\varphi(B_X)$ is w^*-dense in $B_{X^{**}}$[2]. For $x \in X$ we denote $\varphi(x)$ by $\hat{x}$ and $\varphi(X)$ by $\hat{X}$.

The norm $\|\cdot\|$ on X is rotund (R) if $\|x\| = \|y\| = \|\frac{x+y}{2}\|$ implies $x = y$. The norm on X is locally uniformly rotund (LUR) if for $\{x_n\}_{n\in\mathbb{N}} \subseteq S_X, x \in S_X$, $\|x_n + x\| \to 2$ implies $\|x_n - x\| \to 0$. Equivalently, if $2\|x_n\|^2 + 2\|x\|^2 - \|x_n + x\|^2 \to 0$, then $\|x_n - x\| \to 0$, where $x \in X, \{x_n\}_{n\in\mathbb{N}} \subseteq X$. In definition of LUR norm, we can use net instead of sequence.

We know that separable Banach space admits an equivalent LUR norm but non-separable Banach spaces may not admit an equivalent R norm. Let Γ be an uncountable set. It is shown in [1] that $\ell^c_\infty(\Gamma)$ does not admit an equivalent R norm where $\ell^c_\infty(\Gamma)$ is the space of all bounded real-valued functions with countable support defined on Γ. Also, for each uncountable subset Γ' of Γ, the space $\ell^c_\infty(\Gamma')$ is subspace of $\ell^c_\infty(\Gamma)$ that does not admit an equivalent R norm.

In next section we introduce an uncountable set Γ and a subspace X of $\ell^c_\infty(\Gamma)$ such that X does not admit an equivalent R norm and does not contain $\ell^c_\infty(\Gamma')$ for any uncountable subset Γ' of Γ.

It is obvious, if X and Y are two Banach spaces and $|\cdot|_1$, $|\cdot|_2$ are equivalent R

2010 *Mathematics Subject Classification.* 46B20, 46B45.
Key words and phrases. Locally uniformly rotund norm; rotund norm.
©2014 Ilirias Publications, Prishtinë, Kosovë.
Submitted November 27, 2013. Published February 11, 2014.
[1] Corresponding author.

norm on X and Y respectively, then the norm $\|\cdot\|$ on $X \times Y$ is an equivalent R norm where $\|(x,y)\|^2 = |x|_1^2 + |x|_2^2$.
Let X and Y be two Banach spaces and $T : X \to Y$ be a bounded linear injective map. If $|\cdot|_1$ is a norm on X and $|\cdot|_2$ is a norm on Y, then norm $\|\cdot\|$ on X defined by $\|x\|^2 = |x|_1^2 + |Tx|_2^2$, is an equivalent norm on X. If $|\cdot|_2$ is R, then $\|\cdot\|$ is R. Also if T is an isometry and $|\cdot|_2$ is LUR then $\|\cdot\|$ is LUR.
There are many relations between rotund renorming in X and X^*, X^{**}. It is proved in [7] that if norm on X is LUR, then second dual norm on X^{**} is LUR on members of $\widehat{X}$ in X^{**}, that is, if $x \in S_X$ and $\{x_n^{**}\} \subseteq S_{X^{**}}$, then $\|\widehat{x} + x_n^{**}\|^{**} \to 2$ implies $\|\widehat{x} - x_n^{**}\|^{**} \to 0$, but this is not true for R norm. Let X be a non-reflexive Banach space such that X^* is separable. We will show in next section that X admits an equivalent norm $|\cdot|$ such that the norm $|\cdot|^{**}$ on X^{**} is R but the norm $|\cdot|^{****}$ on X^{****} is not R on $\widehat{Y}$ where $Y = X^{**}$.
Haydon has shown the relation between LUR renorming in X^* and X, in the following Theorem:

Theorem 1.1 ([4]). *Let X be a Banach space and X^* admits an equivalent dual LUR norm, then X admits an equivalent LUR norm.*

We will prove in next section that this Theorem is not true, if we do not have duality condition.
Let X be a topological space and $X^{(0)} = X$. Let $X^{(1)} = X'$ denotes the set of accumulation points of X. For an ordinal α, the α-th derived set $X^{(\alpha)}$ is defined by transfinite induction as $X^{(\alpha+1)} = (X^{\alpha})'$. $X^{(\beta)} = \bigcap\limits_{\alpha<\beta} X^{(\alpha)}$ if β is a limit ordinal.
A topological space (X, τ) is scattered if for every nonempty subset A of X, there is a relatively open subset U of A which contains exactly one point. Clearly X is scattered , if for some ordinal α, $X^{(\alpha)} = \emptyset$.

Theorem 1.2 ([6]). *Let K be a compact space such that $K^{(\omega_1)} = \emptyset$. Then $C(K)$ admits an equivalent LUR norm.*

This Theorem is not true when $K^{(\alpha)} = \emptyset$, for $\alpha > \omega_1$, even if, $\alpha = \omega_1 + 1$.
A tree is a partially ordered set $(T, \leq)$ such that for every $t \in T$, the set $\{s \in T, s \leq t\}$ is well ordered by $\leq$. We consider two elements 0 and ∞, which are not in T, such that $0 < t < \infty$ for every $t \in T$. We also consider intervals. If $s, t \in T$, then for instance $(s,t] = \{u \in T; s < u \leq t\}$ while $(0,t] = \{u \in T; u \leq t\}$. For each $t \in T$, we denote by $r(t)$ the unique ordinal which has the same order type as $(0,t)$. we will always assume that the tree T is Hausdorff, that is, if $(0,t) = (0,t')$ and $r(t) = r(t')$ is a limit ordinal, then $t = t'$. For any $t \in T$, We write t^+ for the set of immediate successors of t, that is, $t^+ = \{u \in T : s < u, \; if \; and \; only \; if \; s \leq t\}$. We equip T with the weakest topology τ for which all intervals $(0,t]$ are open and closed. The tree is locally compact and scattered space. (T, τ) is Hausdorff since T is a Hausdorff tree. Let $C_0(T)$ be the set of all real-valued functions f on T which are continuous for τ and for all $\varepsilon > 0$ the set $\{t \in T : |f(t)| \geq \varepsilon\}$ is compact. The space $C_0(T)$ is the closed linear span in $L_\infty(T)$ of the indicator function $1_{(0,t]}(t \in T)$.

Theorem 1.3 ([1]). *Let T be a tree, $f \in C_0(T)$ and $\delta > 0$. For all but finitely many $t \in T$, there exists $u \in t^+$ such that $|f(t) - f(u)| < \delta$.*

If T is a tree, we denote the one point compactification of T by $\widehat{T}$. In next section we introduce a tree T_0 where $C(\widehat{T}_0)$ does not admit an equivalent R norm while $C(\widehat{T}_0)^*$ admits an equivalent LUR norm and $\widehat{T}_0^{(\omega_1+1)} = \emptyset$.

2. Results

Theorem 2.1. *Let X be a non-reflexive Banach space such that X^* is separable. Let $\|\cdot\|$ be norm on X and $\{f_n\}_{n\in\mathbb{N}}$ be dense subset of S_{X^*}. The norm $|\cdot|$ defined by $|x|^2 = \|x\|^2 + \sum_{i\geq 1} 2^{-i} f_i^2(x)$, for $x \in X$, is an equivalent norm on X such that the norm $|\cdot|^{**}$ on X^{**} is R but the norm $|\cdot|^{****}$ on X^{****} is not R on $\widehat{Y}$ where $Y = X^{**}$.*

Proof. It is obvious that the norm $|\cdot|$ is an equivalent norm on X. Let $x^{**}, y^{**} \in S_{X^{**}}$ such that the norm $|x^{**}|^{**} = |y^{**}|^{**} = |\frac{x^{**}+y^{**}}{2}|^{**} = 1$. Since $\varphi(B_X)$ is w^*-dense in $B_{X^{**}}$ there exists two nets $\{x_\alpha\}_{\alpha\in\Gamma} \subseteq B_X$ and $\{y_\alpha\}_{\alpha\in\Gamma} \subseteq B_X$ such that $x_\alpha \to x^{**}, y_\alpha \to y^{**}$ in w^*-topology, then $\widehat{x}_\alpha + \widehat{y}_\alpha \to x^{**} + y^{**}$ in w^*-topology. The norm $|\cdot|^{**}$ is w^*-lower semicontinuous then for arbitrary $\varepsilon > 0$ there exists $\alpha_0 \in \Gamma$ such that for every $\alpha > \alpha_0$ we have $2 - \varepsilon < |\widehat{x}_\alpha + \widehat{y}_\alpha|^{**} = |x_\alpha + y_\alpha| \leq 2$, therefore $|x_\alpha + y_\alpha| \to 2$ that implies

$$2|x_\alpha|^2 + 2|y_\alpha|^2 - |x_\alpha + y_\alpha|^2 \to 0,$$

or

$$2\|x_\alpha\|^2 + 2\sum_{i\geq 1} 2^{-i} f_i^2(x_\alpha) + 2\|y_\alpha\|^2 + 2\sum_{i\geq 1} 2^{-i} f_i^2(y_\alpha)$$
$$-\|x_\alpha + y_\alpha\|^2 - \sum_{i\geq 1} 2^{-i} f_i^2(x_\alpha + y_\alpha) \to 0.$$

For every $\alpha \in \Gamma$ and every $n \in \mathbb{N}$ we have

$$2\|x_\alpha\|^2 + 2\|y_\alpha\|^2 - \|x_\alpha + y_\alpha\|^2 \geq 0, \tag{2.1}$$

$$2f_n^2(x_\alpha) + 2f_n^2(y_\alpha) - f_n^2(x_\alpha + y_\alpha) = f_n^2(x_\alpha - y_\alpha) \geq 0. \tag{2.2}$$

Therefore $f_n(x_\alpha - y_\alpha) \to 0$ for every $n \in \mathbb{N}$.
Since $\{f_n\}_{n\in\mathbb{N}}$ is dense in S_{X^*} we have $f(x_\alpha - y_\alpha) \to 0$ for every $f \in X^*$ and consequently $x^{**} = y^{**}$. Therefore $|\cdot|^{**}$ is R norm in X^{**}. It is proved in [3] that the norm $|\cdot|^{****}$ is not R on $\widehat{Y}$ in X^{****} where $Y = X^{**}$. □

Let $T_0 = \bigcup_{\alpha<\omega_1} \{0,1\}^\alpha$. Define an order on T_0 by $t \leq s$ if $dom(t) \leq dom(s)$ and $s\,|_{dom(t)} = t$. We can regard T_0 as a tree by this order .

Theorem 2.2. *The space $C(\widehat{T}_0)^*$ admits an equivalent LUR norm but $C(\widehat{T}_0)$ does not admit an equivalent R norm.*

Proof. Since T_0 is scattered, $\widehat{T}_0$ is also scattered. Therefore $C(\widehat{T}_0)^*$ is isometrically isomorphic to $\ell_1(\Gamma)$ for some Γ [2] and consequently admits an equivalent LUR norm [1]. $C_0(T_0)$ is closed subspace of $C(\widehat{T}_0)$. It is proved in [1] that $C_0(T_0)$ does not admit an equivalent R norm. Hence $C(\widehat{T}_0)$ does not admit an equivalent R norm.

□

Theorem 2.3. *There exists a compact space K such that $K^{(\omega_1+1)} = \emptyset$ but $C(K)$ does not admit an equivalent R norm.*

Proof. Consider $K = \widehat{T}_0$. $C(K)$ does not admit an equivalent R norm. To show that $K^{(\omega_1+1)} = \emptyset$, it is enough to show that $T_0^{(\omega_1)} = \emptyset$. We proceed by transfinite induction on α to show that if $t \in T_0$ and $dom(t) = \alpha < \omega_1$, then $t \notin T_0^{(\alpha+1)}$. We note that if $\alpha = 1$, then $(0, t] = \{t\}$, so $t \notin T_0^{(\alpha+1)}$. Next we assume that $\alpha > 1$ and $t \notin T_0^{(dom(t)+1)}$ for every $t \in T$ such that $dom(t) < \alpha$. Now, if the result fail for α, that is if there exists t such that $dom(t) = \alpha$ and $t \in T_0^{(\alpha+1)}$ then there is $s \in (0, t] \cap T_0^{(\alpha)}$ such that $s \neq t$ and thus $s < t$. Therefore $\beta = dom(s) < dom(t) = \alpha$ and by the induction hypothesis $s \notin T_0^{(\beta+1)}$. Since $\beta+1 \leq \alpha$, we have $T_0^{(\alpha)} \subseteq T_0^{(\beta+1)}$ that implies $s \notin T_0^{(\alpha)}$, which is a contradiction. Consequently, $T_0^{(\omega_1)} = \emptyset$. $\square$

Let Λ be the set of all injective maps t of ordinals $\alpha = dom(t)$(necessarily $\alpha < \omega_1$) into ω_0 , such that $\omega_0 \setminus Im(t)$ is infinite. The ordering on Λ is given by $t' \leq t$ if $dom(t') \leq dom(t)$ and $t |_{dom(t)} = t'$. Clearly Λ is a tree and for every $t \in \Lambda$ the set t^+ is countably infinite. We can define another tree Υ containing Λ as a subtree as follows.
For any $t \in \Lambda$ we partition $t^+ = t_1^+ \cup t_2^+$ into two infinite subset, and we define $\Upsilon = \Lambda \cup (\Lambda \times \{1, 2\})$ such that $t < (t, i) < u$ whenever $u \in t_i^+, i = 1, 2$. Λ is closed subset of Υ, hence $\Gamma = \Upsilon \setminus \Lambda$ is open and discrete. $(0, t]$ contains at most countable element for every $t \in \Upsilon$. In Υ we have $t^+ = \{(t, 1), (t, 2)\}, (t, i)^+ = t_i^+, i = 1, 2$ for every $t \in \Lambda$. Let $e_t = (0, t]$ for $t \in \Upsilon$. For $f \in C_0(\Upsilon)$, there exist $\{\alpha_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ and $\{t_n\}_{n\in\mathbb{N}} \subseteq \Upsilon$, such that $\alpha_n \to 0$ and $f = \sum_{n\in\mathbb{N}} \alpha_n e_{t_n}$, hence $f |_\Gamma \in \ell_\infty^c(\Gamma)$. Λ is closed in Υ and therefore $f |_\Lambda \in C_0(\Lambda)$. Define $\varphi : C_0(\Upsilon) \to C_0(\Lambda) \times \ell_\infty^c(\Gamma)$ by $\varphi(f) = (f |_\Lambda, f |_\Gamma)$. Then φ is an injective bounded linear map.

Theorem 2.4. *Let $X = \{f |_\Gamma : f \in C_0(\Upsilon)\}$. X is subspace of $\ell_\infty^c(\Gamma)$ which does not admit an equivalent R norm and does not contain $\ell_\infty^c(\Gamma')$ for any uncountable subset Γ' of Γ.*

Proof. It is shown in [5] that $C_0(\Upsilon)$ does not admit an equivalent R norm. Since $\varphi(C_0(\Upsilon)) \subseteq C_0(\Lambda) \times X$ and φ is injective, $C_0(\Lambda) \times X$ does not admit an equivalent R norm. But $C_0(\Lambda)$ admits an equivalent R norm [5]. Therefore, X does not admit an equivalent R norm.
For $(t, i) \in \Gamma$ define $d((t, i)) = dom(t)$. Let Γ' be an arbitrary uncountable subset of Γ. There exists $A = \{t_n\}_{n\in\mathbb{N}} \subseteq \Gamma'$ such that $t_n \neq t_m$ if $m \neq n$ and either there exists $\alpha < \omega_1$ such that $d(t_n) = \alpha$ for every $n \in \mathbb{N}$ or $d(t_n) + 1 < d(t_{n+1})$ for every $n \in \mathbb{N}$. Let $x \in \ell_\infty^c(\Gamma)$ where $x(s) = 1$ for $s \in A$ and $x(s) = 0$ for $s \notin A$, hence $x \in \ell_\infty^c(\Gamma')$. Suppose that $x \in X$. Then there exists $f \in C_0(\Upsilon)$ such that $f |_\Gamma = x$. Let $K = \{s \in \Upsilon : f(s) \geq \frac{1}{2}\}$. K is compact subspace of Υ and $A \subseteq K$. If $f = \sum_{n\in\mathbb{N}} \alpha_n 1_{(0,s_n]}$, then $K \subseteq \bigcup_{n\in\mathbb{N}} (0, s_n]$. Therefore, there exist $n_1, n_2, \cdots n_k \in \mathbb{N}$ such that $K \subseteq (0, s_{n_1}] \cup (0, s_{n_2}] \cup \ldots \cup (0, s_{n_k}]$. If $d(t_n) = \alpha$ for some $\alpha < \omega_1$ and for every $n \in \mathbb{N}$, then $(0, s_{n_1}] \cup (0, s_{n_2}] \cup \ldots \cup (0, s_{n_k}]$ does not cover A, a contradiction. Let $d(t_n) + 1 < d(t_{n+1})$ for every $n \in \mathbb{N}$. There exists infinite subset $A' = \{t_{n_i}\}_{i\in\mathbb{N}}$ of A and $1 \leq i \leq k$ such that $A' \subseteq (0, s_{n_i}]$. By Theorem 1.3 there exist $t_{n_i} \in A'$ and $i \in \{1, 2\}$ such that $\frac{1}{2} < f((s, i))$ for some $s \in t_{n_i}^+$. $(s, i) \in \Gamma$ but $(s, i) \notin A$

since $d(t_{n_i}) + 1 = d((s,i))$, then $x((s,i)) = 0$, a contradiction. In each case $x \notin X$. Therefore X does not contain $\ell_\infty^c(\Gamma')$.

$\square$

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

References

[1] R. Deville, G. Godefroy, V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monogr. Surveys Pure Appl. Math. 64, Longman Scientific and Technical, (1993).
[2] M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucía, J. Pelant, V. Zizler, *Functional analysis and infinite-dimensional geometry*, CMS Books in Mathematics, Springer-Verlag, New York, (2001).
[3] J. R. Giles, *A non-reflexive Banach space has non-smooth third conjugate space*, Canad. Math. Bull **17** **1** (1974) 117–119.
[4] R. Haydon, *Locally uniformly convex norms in Banach spaces and their duals*, Journal of Functional Analysis **254** (2008) 2023–2039.
[5] R. Haydon, *Trees in renorming theory*, Proceedings of the London Mathematical Society **78** (1999) 541–584.
[6] R. Haydon, C. A. Rogers, *A locally uniformly convex renorming for certain $C(K)$*, Mathematica **37** (1990) 1–8.
[7] A. C. Yorke, *Differentiability and local rotundity*, Journal of the Australian Mathematical Society (Series A) **28** **02** (1979) 205–213.

F. Heydari
Mathematics Dept., Alzahra Univ., Vanak, Tehran, Iran
E-mail address: `fatemeh_heydari@alzahra.ac.ir`

D. Behmardi
Mathematics Dept., Alzahra Univ., Vanak, Tehran, Iran
E-mail address: `behmardi@alzahra.ac.ir`