Journal of Mathematical Analysis
ISSN: 2217-3412, URL: http://www.ilirias.com
Volume 4 Issue 3(2013), Pages 33-46.

# ON ASYMPTOTICALLY WIJSMAN LACUNARY $\sigma$-STATISTICAL CONVERGENCE OF SET SEQUENCES

BIPAN HAZARIKA, AYHAN ESI, N. L. BRAHA

Abstract. In this paper we presents three definitions which is a natural combination of the definition of asymptotic equivalence, statistical convergence, lacunary statistical convergence, $\sigma$-statistical convergence and Wijsman convergence. In addition, we also present asymptotically equivalent sequences of sets in sense of Wijsman and study some properties of this concept.

## 1. Introduction

Marouf in [20], presented definitions for asymptotically equivalent and asymptotic regular matrices. Pobyvancts in [28], introduced the concepts of asymptotically regular matrices, which preserve the asymptotic equivalence of two nonnegative numbers sequences. Patterson in [26], extend these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. Patterson and Savaş in [27], introduced the concepts of an asymptotically lacunary statistical equivalent sequences of real numbers. The concepts of $\sigma$-asymptotically lacunary statistical equivalent sequences introduced by Patterson and Savas in [32]. Braha, in [7] introduced the concepts of asymptotically $\Delta^m$-lacunary statistical equivalent sequences of real numbers.

The concept of convergence of sequences of points has been extended by several authors to convergence of sequences of sets. The one of these such extensions considered in this paper is the concept of Wijsman convergence. The concept of Wijsman statistical convergence which is implementation of the concept of statistical convergence to sequences of sets presented by Nuray and Rhoades in [25]. Similar to the concept, the concept of Wijsman lacunary statistical convergence presented Ulusu and Nuray in [35]. Hazarika and Esi in [14], introduced the notion asymptotically Wijsman generalized statistical convergence of sequences of sets. For more works on convergence of sequences of sets, we refer to ([1], [2], [3], [4], [5], [36], [37], [38]).

The idea of statistical convergence was formerly given under the name "almost convergence" by Zygmund in the first edition of his celebrated monograph published in Warsaw in 1935 [39]. The concept was formally introduced by Steinhaus [34] and

2010 *Mathematics Subject Classification.* 40A35, 40G15.
*Key words and phrases.* Asymptotic equivalence; statistical convergence; lacunary statistical convergence; strongly $\sigma$-convergence; Wijsman convergence.

Submitted Jun 22, 2013. Published November 8, 2013.

Fast [9] and later was introduced by Schoenberg [33], and also independently by Buck [6]. A lot of developments have been made in this areas after the works of Šalát [29] and Fridy [11]. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Fridy and Orhan [12] introduced the concept of lacunary statistical convergence. Mursaleen and Mohiuddine in [24], introduced the concept of lacunary statistical convergence with respect to the intuitionistic fuzzy normed space. For details related to lacunary statistical convergence, we refer to ([8], [12], [13], [15], [17], [21]).

By a lacunary sequence $\theta = (k_r)$, where $k_0 = 0$ , we shall mean an increasing sequence of non-negative integers with $h_r : k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by $\theta$ will be denoted by $J_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be defined by $q_r$ (see [10]).

In this paper we define asymptotically lacunary $\sigma$-statistical equivalent sequences of sets in sense of Wijsman and establish some basic results regarding the notions asymptotically lacunary $\sigma$-statistical equivalent sequences of sets in sense of Wijsman and asymptotically Wijsman lacunary statistical equivalent sequences of sets.

Now we recall the definitions of statistical convergence, lacunary statistical convergence, $\sigma$-statistical convergence and Wijsman convergence.

**Definition 1.1.** *A real or complex number sequence $x = (x_k)$ is said to be statistically convergent to $L$ if for every $\varepsilon > 0$*

$$\lim_n \frac{1}{n} \left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right| = 0.$$

*In this case, we write $S - \lim x = L$ or $x_k \to L(S)$ and $S$ denotes the set of all statistically convergent sequences.*

**Definition 1.2.** *([12]) A sequence $x = (x_k)$ is said to be lacunary statistically convergent to the number $L$ if for every $\varepsilon > 0$*

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\{k \in J_r : \ |x_k - L| \geq \varepsilon\}\right| = 0.$$

*Let $S_\theta$ denotes the set of all lacunary statistically convergent sequences. If $\theta = (2^r)$, then $S_\theta$ is the same as $S$.*

Let $\sigma$ be a one-to-one mapping from the set of natural numbers into itself. A continuous linear functional $\phi$ on $\ell_\infty$ is said to be an invariant mean or a $\sigma$-mean if and only if

(1) $\phi(x) \geq 0$ when the sequence $x = (x_k)$ is such that $x_k \geq 0$ for all $k$,
(2) $\phi(e) = 1$ where $e = (1, 1, 1, ...)$, and
(3) $\phi(x) = \phi(x_{\sigma(k)})$ for all $x \in \ell_\infty$.

Throughout this paper we shall consider the mapping $\sigma$ has having on finite orbits, that is, $\sigma^m(k) \neq k$ for all nonnegative integers with $m \geq 1$, where $\sigma^m(k)$ is the $m$-th iterate of $\sigma$ at $k$. We denote $V_\sigma$ is the set of bounded sequences all of whose $\sigma$-mean are equal. If $\sigma(k) = k + 1$, then it is the set of almost convergent sequences in [18].

If $x = (x_k)$, the set $Tx = (Tx_k) = (x_{\sigma(k)})$. Savas [30], introduced the following notion.

$$V_\sigma = \left\{x = (x_k) : \lim_n t_{nk}(x) = Le \text{ uniformly in } k, L = \sigma - \lim x\right\},$$

where $t_{nk}(x) = \frac{(x_k + Tx_k + \ldots + T^n x_k)}{n+1}$.

**Definition 1.3.** *([31]) A sequence $x = (x_k)$ is said to be $\sigma$-statistically convergent to the number $L$ if for every $\varepsilon > 0$*

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{k \leq n : \ \left|x_{\sigma^k(m)} - L\right| \geq \varepsilon\right\}\right| = 0, \text{ uniformly on } m.$$

*Let $S_\sigma$ denotes the set of all $\sigma$-statistically convergent sequences.*

**Definition 1.4.** *([30], [32]) A sequence $x = (x_k)$ is said to be strongly lacunary $\sigma$-convergent to the number $L$ if*

$$\mathcal{L}_\theta = \left\{x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in J_r} \left|x_{\sigma^k(m)} - L\right| = 0, \text{ uniformly on } m\right\}.$$

**Definition 1.5.** *([31]) A sequence $x = (x_k)$ is said to be lacunary $\sigma$-statistically convergent to the number $L$ if for every $\varepsilon > 0$*

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in J_r : \ \left|x_{\sigma^k(m)} - L\right| \geq \varepsilon\right\}\right| = 0, \text{ uniformly on } m.$$

*Let $S_{\theta,\sigma}$ denotes the set of all lacunary $\sigma$-statistically convergent sequences. If $\theta = (2^r)$, then $S_{\theta,\sigma}$ is the same as $S_\sigma$.*

Let $(X, \rho)$ be a metric space. For any point $x \in X$ and any non-empty subset $A \subset X$, the distance from x to A is defined by

$$d(x, A) = \inf_{y\in A} \rho(x, y).$$

**Definition 1.6.** *([2]) Let (X,$\rho$) be a metric space. For any non-empty closed subsets $A, A_k \subset X$ $(k \in \mathbb{N})$, we say that the sequence $(A_k)$ is Wijsman convergent to $A$ if $\lim_k d(x, A_k) = d(x, A)$ for each $x \in X$. In this case we write $W - \lim A_k = A$.*

## 2. Definitions and Notations

**Definition 2.1.** *([20]) Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if*

$$\lim_k \frac{x_k}{y_k} = 1,$$

*denoted by $x \sim y$.*

**Definition 2.2.** *([26]) Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent of multiple $L$ provided that for every $\varepsilon > 0$*

$$\lim_n \frac{1}{n} \left|\left\{k \leq n : \left|\frac{x_k}{y_k} - L\right| \geq \varepsilon\right\}\right| = 0,$$

*denoted by $x \sim^{S^L} y$ and simply asymptotically statistical equivalent if $L = 1$.*

Savas and Nuray in [31], defined the asymptotically $\sigma$-statistical equivalent and strongly asymptotically $\sigma$-statistical equivalent sequences as follows:

**Definition 2.3.** *Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically $\sigma$-statistical equivalent of multiple $L$ provided that for every $\varepsilon > 0$*

$$\lim_n \frac{1}{n} \left|\left\{k \leq n : \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \geq \varepsilon\right\}\right| = 0, \text{ uniformly on } m,$$

*denoted by $x \sim^{S^L_\sigma} y$ and simply asymptotically $\sigma$-statistical equivalent if $L = 1$.*

**Definition 2.4.** *Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically $\sigma$-statistical equivalent of multiple $L$ provided that*

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| = 0, \text{ uniformly on } m,$$

*denoted by $x \sim^{[V_\sigma]^L} y$ and simply strongly asymptotically $\sigma$-statistical equivalent if $L = 1$.*

The concepts of Wijsman statistical convergence and boundedless for the sequence $(A_k)$ were given by Nuray and Rhoades [25] as follows:

**Definition 2.5.** *([25]) Let $(X, \rho)$ be a metric space. For any non-empty closed subsets $A, A_k \subset X$ $(k \in \mathbb{N})$, we say that the sequence $(A_k)$ is Wijsman statistical convergent to $A$ if the sequence $(d(x, A_k))$ is statistically convergent to $d(x, A)$, i.e., for $\varepsilon > 0$ and for each $x \in X$*

$$\lim_n \frac{1}{n} \left| \{k \leq n : \ |d(x, A_k) - d(x, A)| \geq \varepsilon\} \right| = 0.$$

*In this case, we write $st - \lim_k A_k = A$ or $A_k \to A\,(WS)$. The sequence $(A_k)$ is bounded if $\sup_k d(x, A_k) < \infty$ for each $x \in X$. The set of all bounded sequences of sets denoted by $L_\infty$.*

Ulusu and Nuray in [36], defined asymptotically equivalent and asymptotically statistical equivalent sequences of sets as follows:

**Definition 2.6.** *Let $(X, \rho)$ be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences $(A_k)$ and $(B_k)$ are asymptotically equivalent (Wijsman sense) if for each $x \in X$,*

$$\lim_k \frac{d(x, A_k)}{d(x, B_k)} = 1,$$

*denoted by $(A_k) \sim (B_k)$.*

**Definition 2.7.** *Let $(X, \rho)$ be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences $(A_k)$ and $(B_k)$ are asymptotically statistical equivalent (Wijsman sense) if for every $\varepsilon > 0$ and for each $x \in X$,*

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right| = 0$$

*denoted by $(A_k) \sim^{WS^L} (B_k)$ and simply asymptotically statistical equivalent (Wijsman sense) if $L = 1$.*

## 3. Asymptotically Wijsman lacunary $\sigma$-statistical equivalent sequences

In this section, we define asymptotically Wijsman $\sigma$-statistical and asymptotically Wijsman lacunary $\sigma$-statistical equivalent sequences of sets and proved some interesting results.

**Definition 3.1.** *Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x,A_k)>0$ and $d(x,B_k)>0$ for each $x\in X$. Two sequences $(A_k)$ and $(B_k)$ are said to be asymptotically Wijsman $\sigma$-statistical equivalent of multiple $L$ provided that for every $\varepsilon>0$*

$$\lim_n \frac{1}{n}\left|\left\{k\le n : \left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})}-L\right|\ge\varepsilon\right\}\right| = 0, \text{ uniformly on } m,$$

*denoted by $A_k \sim^{WS^L_\sigma} B_k$ and simply asymptotically Wijsman $\sigma$-statistical equivalent if $L=1$.*

**Definition 3.2.** *Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x,A_k)>0$ and $d(x,B_k)>0$ for each $x\in X$. Two sequences $(A_k)$ and $(B_k)$ are said to be strongly asymptotically Wijsman $\sigma$-statistical equivalent of multiple $L$ provided that*

$$\lim_n \frac{1}{n}\sum_{k=1}^{n}\left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})}-L\right| = 0, \text{ uniformly on } m,$$

*denoted by $A_k \sim^{W[V_\sigma]^L} B_k$ and simply strongly asymptotically Wijsman $\sigma$-statistical equivalent if $L=1$.*

**Definition 3.3.** *Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x,A_k)>0$ and $d(x,B_k)>0$ for each $x\in X$. We say that the sequences $(A_k)$ and $(B_k)$ are asymptotically Wijsman lacunary $\sigma$-equivalent of multiple $L$ if for each $x\in X$,*

$$\lim_r \frac{1}{h_r}\sum_{k\in J_r}\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})} = L, \text{ uniformly in } m$$

*denoted by $(A_k) \sim^{W(\mathcal{L}_\theta)^L} (B_k)$ and simply asymptotically Wijsman lacunary $\sigma$-equivalent if $L=1$.*

**Definition 3.4.** *Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x,A_k)>0$ and $d(x,B_k)>0$ for each $x\in X$. We say that the sequences $(A_k)$ and $(B_k)$ are strongly asymptotically Wijsman lacunary $\sigma$-statistical equivalent if for every $\varepsilon>0$ and for each $x\in X$,*

$$\lim_r \frac{1}{h_r}\sum_{k\in J_r}\left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})}-L\right| = 0 \text{ uniformly in } m$$

*denoted by $(A_k) \sim^{W[\mathcal{L}_\theta]^L} (B_k)$ and simply strongly asymptotically Wijsman lacunary $\sigma$-equivalent if $L=1$.*

**Definition 3.5.** *Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x,A_k)>0$ and $d(x,B_k)>0$ for each $x\in X$. We say that the sequences $(A_k)$ and $(B_k)$ are asymptotically Wijsman lacunary $\sigma$-statistical equivalent if for every $\varepsilon>0$ and for each $x\in X$,*

$$\lim_r \frac{1}{h_r}\left|\left\{k\in J_r : \left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})}-L\right|\ge\varepsilon\right\}\right| = 0 \text{ uniformly in } m$$

*denoted by $(A_k) \sim^{WS^L_{\theta,\sigma}} (B_k)$ and simply asymptotically Wijsman lacunary $\sigma$-statistical equivalent if $L=1$.*

**Theorem 3.6.** *Let $(X,\rho)$ be a metric space and $A_k, B_k$ be non-empty closed subsets $X$ $(k\in\mathbb{N})$. Then*

(a) $(A_k)\sim^{W[\mathcal{L}_\theta]^L}(B_k)\Rightarrow(A_k)\sim^{WS^L_{\theta,\sigma}}(B_k)$.
(b) $W[\mathcal{L}_\theta]^L$ *is a proper subset of* $WS^L_{\theta,\sigma}$.
(c) *Let* $(A_k)\in L_\infty$ *and* $(A_k)\sim^{WS^L_{\theta,\sigma}}(B_k)$, *then* $(A_k)\sim^{W[\mathcal{L}_\theta]^L}(B_k)$.
(d) $WS^L_{\theta,\sigma}\cap L_\infty = W[\mathcal{L}_\theta]^L\cap L_\infty$.

*Proof.* (a) Let $\varepsilon>0$ and $(A_k)\sim^{W[\mathcal{L}_\theta]^L}(B_k)$. Then we can write

$$\sum_{k\in J_r}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\sum_{\substack{k\in J_r\\ \left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})}-L\right|\geq\varepsilon}}\left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})}-L\right|$$

$$\geq\varepsilon\left|\left\{k\in J_r:\ \left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})}-L\right|\geq\varepsilon\right\}\right|$$

which gives the result.

(b) Suppose that $W[\mathcal{L}_\theta]^L\subset WS^L_{\theta,\sigma}$. Let $(A_k)$ and $(B_k)$ be two sequences defined as follows:

$$A_k=\begin{cases}\{k\}, & \text{if } k_{r-1}<k\leq k_{r-1}+[\sqrt{h_r}], r=1,2,3,...;\\ \{0\}, & \text{otherwise}\end{cases}$$

and

$$B_k=\{0\}\text{ for all }k\in\mathbb{N}.$$

It is clear that $(A_k)\notin L_\infty$ and for $\varepsilon>0$ and for each $x\in X$,

$$\lim_r\frac{1}{h_r}\left|\left\{k\in J_r:\ \left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})}-1\right|\geq\varepsilon\right\}\right|=\lim_r\frac{[\sqrt{h_r}]}{h_r}=0.$$

So $(A_k)\sim^{WS^1_{\theta,\sigma}}(B_k)$, but

$$\lim_r\frac{1}{h_r}\sum_{k\in J_r}\left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})}-L\right|\neq 0.$$

Therefore $(A_k)\not\sim^{W[\mathcal{L}_\theta]^L}(B_k)$.

(c) Suppose that $(A_k)\sim^{WS^L_{\lambda,\sigma}}(B_k)$ and $(A_k)\in L_\infty$. We assume that $\left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})}-L\right|\leq M$ for each $x\in X$ and for all $k\in\mathbb{N}$. Given $\varepsilon>0$, we get

$$\frac{1}{h_r}\sum_{k\in J_r}\left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})}-L\right|$$

$$=\frac{1}{h_r}\sum_{\substack{k\in J_r\\ \left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})}-L\right|\geq\varepsilon}}\left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})}-L\right|+\frac{1}{h_r}\sum_{\substack{k\in J_r\\ \left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})}-L\right|<\varepsilon}}\left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})}-L\right|$$

$$\leq\frac{M}{h_r}\left|\left\{k\in J_r:\ \left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})}-L\right|\geq\varepsilon\right\}\right|+\varepsilon,$$

from which the result follows.

(d) It follows from (a), (b) and (c). □

**Theorem 3.7.** *Suppose for given $\eta_1 > 0$ and every $\varepsilon > 0$ there exist $n_0$ and $m_0$ such that*

$$\frac{1}{n}\left|\left\{0 \le k \le n-1 : \left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right| \ge \varepsilon\right\}\right| < \eta_1$$

*for all $n \ge n_0$ and $m \ge m_0$, then $(A_k) \sim^{WS_\sigma^L} (B_k)$.*

*Proof.* Let $\eta_1$ be given. For every $\varepsilon > 0$, choose $n_1$ and $m_0$ such that

$$\frac{1}{n}\left|\left\{0 \le k \le n-1 : \left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right| \ge \varepsilon\right\}\right| < \frac{\eta_1}{2}, \text{ for all } n \ge n_1; m \ge m_0. \tag{3.1}$$

It is sufficient to show that there exists $n_2$ such that for $n \ge n_1$ and $0 \le m \le m_0$

$$\frac{1}{n}\left|\left\{0 \le k \le n-1 : \left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right| \ge \varepsilon\right\}\right| < \eta_1. \tag{3.2}$$

Let $n_0 = \max\{n_1, n_2\}$. The the relation (3.2) will be true for $n > n_0$ and for all $m$. If $m_0$ chosen fixed, then we get

$$\left|\left\{0 \le k \le m_0-1 : \left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right| \ge \varepsilon\right\}\right| = M.$$

Now for $0 \le m \le m_0$ and $n > m_0$ we have

$$\begin{aligned}
&\frac{1}{n}\left|\left\{0 \le k \le n-1 : \left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right| \ge \varepsilon\right\}\right| \\
&\le \frac{1}{n}\left|\left\{0 \le k \le m_0-1 : \left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right| \ge \varepsilon\right\}\right| \\
&+ \frac{1}{n}\left|\left\{m_0 \le k \le n-1 : \left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right| \ge \varepsilon\right\}\right| \\
&\le \frac{M}{n} + \frac{1}{n}\left|\left\{m_0 \le k \le n-1 : \left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right| \ge \varepsilon\right\}\right| \le \frac{M}{n} + \frac{\eta_1}{2}.
\end{aligned}$$

Thus for sufficiently large $n$

$$\frac{1}{n}\left|\left\{m_0 \le k \le n-1 : \left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right| \ge \varepsilon\right\}\right| \le \frac{M}{n} + \frac{\eta_1}{2} < \eta_1.$$

This established the result. □

**Theorem 3.8.** *For every lacunary sequence $\theta$, $WS_{\theta,\sigma} = WS_\sigma$.*

*Proof.* Let $(A_k) \in WS_{\theta,\sigma}$, then for given $\eta_1 > 0$ there exist $\varepsilon > 0$ and $L$ such that

$$\frac{1}{h_r}\left\{k \in J_r : \left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right| \ge \varepsilon\right\} < \eta_1,$$

for $r \ge r_0$ and $m = k_{r-1} + 1 + v$ where $v \ge 0$.

Let $n \ge h_r$ and write $n = sh_r + u$ with $0 \le u \le h_r$ and $s$ an integer. Since $n \ge h_r$ and $s \ge 0$. We have

$$\begin{aligned}
&\frac{1}{n}\left|\left\{0 \le k \le n-1 : \left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right| \ge \varepsilon\right\}\right| \\
&\le \frac{1}{n}\left|\left\{0 \le k \le (s+1)h_r - 1 : \left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right| \ge \varepsilon\right\}\right|
\end{aligned}$$

$$= \frac{1}{n}\sum_{i=1}^{s}\left|\left\{ih_r \leq k \leq (i+1)h_r - 1 : \left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right| \geq \varepsilon\right\}\right|$$

$$\leq \frac{(s+1)h_r}{n}\eta_1 \leq \frac{2sh_r\eta_1}{n} \text{ for } s \geq 1.$$

For $\frac{h_r}{n} \leq 1$, since $\frac{sh_r}{n} \leq 1$. Therefore

$$\frac{1}{n}\left|\left\{0 \leq k \leq n-1 : \left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right| \geq \varepsilon\right\}\right| \leq 2\eta_1.$$

Hence by Theorem 3.7, we have $WS_{\theta,\sigma} \subseteq WS_\sigma$. Also $WS_\sigma \subseteq WS_{\theta,\sigma}$ for every lacunary sequence $\theta$. This completes the proof of the theorem. □

**Theorem 3.9.** *Let $(X, \rho)$ be a metric space and $A_k, B_k$ be non-empty closed subsets $X$ $(k \in \mathbb{N})$. Let $\theta = (k_r)$ be a lacunary sequence with $\liminf_r q_r > 1$. Then $(A_k) \sim^{WS^L_\sigma} (B_k) \Rightarrow (A_k) \sim^{WS^L_{\theta,\sigma}} (B_k)$.*

*Proof.* Suppose that $\liminf_r q_r > 1$, then there exists a $\alpha > 0$ such that $q_r \geq 1 + \alpha$ for sufficiently large $r$. Then we have

$$\frac{h_r}{k_r} \geq \frac{\alpha}{1+\alpha}.$$

If $(A_k) \sim^{WS^L_\sigma} (B_k)$, then for every $\varepsilon > 0$ and for sufficiently large $r$ we have

$$\frac{1}{h_r}\left|\left\{k \leq k_r : \left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right| \geq \varepsilon\right\}\right|$$

$$\geq \frac{1}{k_r}\left|\left\{k \in J_r : \left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right| \geq \varepsilon\right\}\right|$$

$$\geq \frac{\alpha}{1+\alpha}.\frac{1}{h_r}\left|\left\{k \in J_r : \left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right| \geq \varepsilon\right\}\right|.$$

This completes the proof. □

**Theorem 3.10.** *Let $(X, \rho)$ be a metric space and $A_k, B_k$ be non-empty closed subsets $X$ $(k \in \mathbb{N})$. Let $\theta = (k_r)$ be a lacunary sequence with $\limsup_r q_r < \infty$. Then $(A_k) \sim^{WS^L_{\theta,\sigma}} (B_k) \Rightarrow (A_k) \sim^{WS^L_\sigma} (B_k)$.*

*Proof.* If $\limsup_r q_r < \infty$. Then there exists an $K > 0$ such that $q_r < K$ for all $r \geq 1$. Let $(A_k) \sim^{WS^L_{\theta,\sigma}} (B_k)$ and $\eta_1 > 0$. Then there exists $B > 0$ and $\varepsilon > 0$ such that for every $j \geq B$

$$M_j = \frac{1}{h_j}\left|\left\{k \in J_j : \left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right| \geq \varepsilon\right\}\right| < \eta_1 \text{ for all } m.$$

Also we can find $A > 0$ such that $M_j < A$ for all $j = 1, 2, 3, ...$ Now let $i$ be an integer with satisfying $k_{r-1} < i \leq k_r$, where $r > B$. Then we can write

$$\frac{1}{n}\left|\left\{k \leq i : \left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right| \geq \varepsilon\right\}\right|$$

$$\leq \frac{1}{k_{r-1}}\left|\left\{k \leq k_r : \left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right| \geq \varepsilon\right\}\right|$$

$$= \frac{1}{k_{r-1}}\left|\left\{k \in J_1 : \left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right| \geq \varepsilon\right\}\right|$$

$$+\frac{1}{k_{r-1}}\left|\left\{k\in J_2:\left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})}-L\right|\geq\varepsilon\right\}\right|$$
$$+...+\frac{1}{k_{r-1}}\left|\left\{k\in J_r:\left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})}-L\right|\geq\varepsilon\right\}\right|$$
$$=\frac{k_1}{k_{r-1}k_1}\left|\left\{k\in J_1:\left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})}-L\right|\geq\varepsilon\right\}\right|$$
$$+\frac{k_2-k_1}{k_{r-1}(k_2-k_1)}\left|\left\{k\in J_2:\left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})}-L\right|\geq\varepsilon\right\}\right|$$
$$+...+\frac{k_B-k_{B-1}}{k_{r-1}(k_B-k_{B-1})}\left|\left\{k\in J_B:\left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})}-L\right|\geq\varepsilon\right\}\right|$$
$$+...+\frac{k_r-k_{r-1}}{k_{r-1}(k_r-k_{r-1})}\left|\left\{k\in J_r:\left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})}-L\right|\geq\varepsilon\right\}\right|$$
$$=\frac{k_1}{k_{r-1}M_1}+\frac{k_2-k_1}{k_{r-1}}M_2+...+\frac{k_B-k_{B-1}}{k_{r-1}}M_B$$
$$+\frac{k_{B+1}-k_B}{k_{r-1}}M_{B+1}+...+\frac{k_r-k_{r-1}}{k_{r-1}}M_r$$
$$\leq\left\{\sup_{i\geq1}M_i\right\}\frac{k_B}{k_{r-1}}+\left\{\sup_{i\geq B}M_i\right\}\frac{k_r-k_B}{k_{r-1}}\leq A\frac{k_B}{k_{r-1}}+\eta_1K.$$

This completes the proof of the theorem. □

**Theorem 3.11.** *Let $(X,\rho)$ be a metric space and $A_k, B_k$ be non-empty closed subsets $X$ $(k\in\mathbb{N})$. Let $\theta=(k_r)$ be a lacunary sequence with $1<\liminf_r q_r\leq\limsup_r q_r<\infty$. Then $(A_k)\sim^{WS^L_{\theta,\sigma}}(B_k)\Leftrightarrow(A_k)\sim^{WS^L_{\sigma}}(B_k)$.*

*Proof.* The result is an immediate consequence of Theorem 3.9 and Theorem 3.10. □

**Definition 3.12.** *Let $p\in(0,\infty)$. Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A_k,B_k\subseteq X$ such that $d(x,A_k)>0$ and $d(x,B_k)>0$ for each $x\in X$. We say that the sequences $(A_k)$ and $(B_k)$ are strongly asymptotically Wijsman lacunary $\sigma_p$-statistical equivalent if for every $\varepsilon>0$ and for each $x\in X$,*

$$\lim_r\frac{1}{h_r}\sum_{k\in J_r}\left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})}-L\right|^p=0 \text{ uniformly in } m$$

*denoted by $(A_k)\sim^{W[\mathcal{L}_\theta]^L_p}(B_k)$ and simply strongly asymptotically Wijsman lacunary $\sigma$-equivalent if $L=1$.*

**Definition 3.13.** *Let $p\in(0,\infty)$. Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A_k,B_k\subseteq X$ such that $d(x,A_k)>0$ and $d(x,B_k)>0$ for each $x\in X$. We say that the sequences $(A_k)$ and $(B_k)$ are asymptotically Wijsman lacunary $\sigma_p$-statistical equivalent if for every $\varepsilon>0$ and for each $x\in X$,*

$$\lim_r\frac{1}{h_r}\left|\left\{k\in J_r:\left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})}-L\right|^p\geq\varepsilon\right\}\right|=0 \text{ uniformly in } m$$

*denoted by $(A_k)\sim^{WS^L_{\theta,\sigma_p}}(B_k)$ and simply asymptotically Wijsman lacunary $\sigma$-statistical equivalent if $L=1$.*

The proof of the following theorem follows from Theorem 3.6.

**Theorem 3.14.** *Let $(X,\rho)$ be a metric space and $A_k, B_k$ be non-empty closed subsets $X$ $(k \in \mathbb{N})$. Then*

(a) $(A_k) \sim^{W[\mathcal{L}_\theta]_p^L} (B_k) \Rightarrow (A_k) \sim^{WS_{\theta,\sigma_p}^L} (B_k)$.
(b) $W[L_\theta]_p^L$ *is a proper subset of* $WS_{\theta,\sigma_p}^L$.
(c) *Let* $(A_k) \in L_\infty$ *and* $(A_k) \sim^{WS_{\theta,\sigma_p}^L} (B_k)$, *then* $(A_k) \sim^{W[\mathcal{L}_\theta]_p^L} (B_k)$.
(d) $WS_{\theta,\sigma_p}^L \cap L_\infty = W[L_\theta]_p^L \cap L_\infty$.

## 4. Orlicz asymptotically Wijsman lacunary $\sigma$-statistical equivalent sequences

In this section we define the notion of Orlicz asymptotically Wijsman lacunary $\sigma$-statistical equivalence sequences of sets. An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. An Orlicz function $M$ is said to satisfy the $\Delta_2 -$ *condition* for all values of $u$, if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)$, $u \geq 0$. Note that, if $0 < \lambda < 1$,then $M(\lambda x) \leq \lambda M(x)$, for all $x \geq 0$ (see [16]).

Also we will introduce Cesaro Orlicz asymptotically Wijsman $\sigma$-statistical equivalence between two sequences.

**Definition 4.1.** *Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$. Two sequences $(A_k)$ and $(B_k)$ are said to be $M$-strongly Cesaro Orlicz asymptotically Wijsman $\sigma$-statistical equivalent of multiple $L$ provided that for every*

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} M\left(\left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})} - L\right|\right) = 0,$$

*(denoted by $A_k \overset{|\sigma_1|WM}{\sim} B_k$), and simply $WM$-strongly Cesaro Orlicz asymptotically Wijsman $\sigma$- equivalent if $L = 1$.*

**Definition 4.2.** *Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$. Two sequences $(A_k)$ and $(B_k)$ are said to be $M$-lacunary strongly Cesaro Orlicz asymptotically Wijsman $\sigma$-statistical equivalent of multiple $L$ provided that for every*

$$\lim_r \frac{1}{h_r} \sum_{k \in J_r} M\left(\left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})} - L\right|\right) = 0,$$

*(denoted by $A_k \overset{|\sigma_1|LWM}{\sim} B_k$), and simply $LWM$-strongly Cesaro Orlicz asymptotically Wijsman $\sigma$- equivalent if $L = 1$.*

**Theorem 4.3.** *Let $(X,\rho)$ be a metric space and $A_k, B_k$ be non-empty closed subsets of $X$, $(k \in \mathbb{N})$. $M$ be an Orlicz function which satisfies the $\Delta_2$ conditions and $M(1) = 1$. Then for two sequences $(A_n)$ and $(B_n)$ are said to be $A_k \overset{WSM_{\theta,\sigma}^L}{\sim} B_k$ of multiple $L$ provided that for every $\epsilon > 0$,*

$$\lim_r \frac{1}{h_r} \left\{ \textit{the number of} \quad k \in J_r, : M\left(\left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})} - L\right|\right) \geq \epsilon \right\} = 0.$$

*If* $\lim_r \inf q_r > 1$, *then* $A_k \overset{WS^L_{\theta,\sigma}}{\sim} B_k$ *implies* $x \overset{WSM^L_{\theta,\sigma}}{\sim} B_k$.

*Proof.* Suppose that $\lim_r \inf q_r > 1$, then there exists a $\delta > 0$, such that $q_r \geq 1+\alpha$ for sufficiently large. Then we have

$$\frac{h_r}{k_r} \geq \frac{\alpha}{1+\alpha}.$$

If $A_k \overset{WS^L_{\theta,\sigma}}{\sim} B_k$, then for every $\epsilon > 0$ and for sufficiently large $r$, we have

$$\frac{1}{h_r}\left\{\text{the number of} \quad k \in J_r : M\left(\left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right|\right) \geq \epsilon\right\} =$$

$$\frac{k_r}{h_r} \cdot \frac{1}{k_r}\left\{\text{the number of} \quad k \in J_r : M\left(\left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right|\right) \geq \epsilon\right\} \leq$$

$$\frac{1+\alpha}{\alpha} \cdot \frac{1}{k_r}\left\{\text{the number of} \quad k \in J_r : M\left(\left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right|\right) \geq \epsilon\right\}. \tag{4.1}$$

In other side from fact that $M$ satisfies the $\Delta_2$ conditions it follows that

$$M\left(\left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right|\right) \leq K \cdot \left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right|,$$

for some constant $K > 0$, in both cases where $\left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})} - L\right| \leq 1$ and $\left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})} - L\right| \geq 1$. Really in first case it follows directly from definition of the Orlicz function. In second case we have

$$\left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right| = 2 \cdot L^{(1)} = 2^2 \cdot L^{(2)} = \cdots = 2^s \cdot L^{(s)},$$

such that $L^{(s)} \leq 1$. Now taking into consideration $\Delta_2$ conditions of Orlicz functions, we get the following estimation:

$$M\left(\left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right|\right) \leq T \cdot L^{(s)} \cdot M(1) = K \cdot \left|\frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L\right|, \tag{4.2}$$

where $T$ and $K$ are constants. Now proof of Theorem follows from relations (4.1) and (4.2). □

**Theorem 4.4.** *Let* $(X, \rho)$ *be a metric space and* $A_k, B_k$ *be non-empty closed subsets of* $X$, *for any* $k \in \mathbb{N}$ *and* $M$ *be an Orlicz function. If* $\limsup_r q_r < \infty$, *then from* $A_k \overset{WSM^L_{\theta,\sigma}}{\sim} B_k$ *implies* $x \overset{WS^L_{\theta,\sigma}}{\sim} B_k$.

*Proof.* Proof of the theorem is similar to Theorem 3.10, so we omit them. □

The proof of the following results we omit, because they are similar to the fact which we have prove above.

**Theorem 4.5.** *Let* $(X, \rho)$ *be a metric space and* $A_k, B_k$ *be non-empty closed subsets of* $X$, *for any* $k \in \mathbb{N}$ *and* $M$ *be an Orlicz function. The following relations are valid:*

**(i):** *If* $\lim\inf_r q_r > 1$, *then* $A_k \overset{|\sigma_1|WM}{\sim} B_k \Rightarrow A_k \overset{|\sigma_1|LWM}{\sim} B_k$.

**(ii):** *If* $\limsup_r q_r < \infty$, *then* $A_k \overset{|\sigma_1|LWM}{\sim} B_k \Rightarrow A_k \overset{|\sigma_1|WM}{\sim} B_k$.

**(iii):** *If* $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$, *then* $A_k \overset{|\sigma_1|LWM}{\sim} B_k \Longleftrightarrow A_k \overset{|\sigma_1|WM}{\sim} B_k$.

**Theorem 4.6.** *Let* $(X, \rho)$ *be a metric space and* $A_k, B_k$ *be non-empty closed subsets of* $X$, *for any* $k \in \mathbb{N}$ *and* $M$ *be an Orlicz function. Then*

**1:** *(a)If* $A_k \overset{|\sigma_1|WM}{\sim} B_k$ *then* $A_k \overset{WS_\sigma^L}{\sim} B_k$.
**:** *(b)* $WS_\sigma^L$ *is a proper subset of* $|\sigma_1|WM$.

**2:** *If* $M$ *satisfies the* $\Delta_2$ *condition and* $A_k \in l_\infty(|\sigma_1|WM)$ *such that* $A_k \overset{WS_\sigma^L}{\sim} B_k$, *then* $A_k \overset{|\sigma_1|WM}{\sim} B_k$.

**3:** *If* $M$ *satisfies the* $\Delta_2$ *condition, then* $SW_\sigma^L \cap l_\infty(|\sigma_1|WM) = (|\sigma_1|WM) \cap l_\infty(|\sigma_1|WM)$, *where* $l_\infty(|\sigma_1|WM) = \{(A_k) : M(d(x, A_{\sigma^k(m)})) \in l_\infty, x \in X\}$

*Proof.* 1: (a) If $\epsilon > 0$ and $A_k \overset{|\sigma_1|WM}{\sim} B_k$, then

$$\epsilon \frac{1}{n} \left\{ \text{the number of} \quad k \leq n : \left| \frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L \right| \geq \epsilon \right\} \leq$$

$$\epsilon \frac{1}{n} \left\{ \text{the number of} \quad k \leq n : M\left( \left| \frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L \right| \right) \geq M(\epsilon) \right\}$$

$$\leq \frac{1}{n} \sum_{\substack{k=1 \\ M\left(\left|\frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})} - L\right|\right) \geq M(\epsilon)}}^{n} M\left( \left| \frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L \right| \right) \leq \frac{1}{n} \sum_{k=1}^{n} M\left( \left| \frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L \right| \right)$$

(b) To prove this fact it is enough to find a sequence of sets $(A_k)$ such that $A_k \in WS_\theta^L$ and $A_k \notin |\sigma_1|WM$. Let $L = 1$ and $M = x$. We define $d(x, A_{\sigma^k(m)})$ to be

$$1, 2, \cdots, [\sqrt{n}]$$

for $k = i$, for $i \in \{1, 2, \cdots, [\sqrt{n}]\}$, and $d(x, A_{\sigma^k(m)}) = 1$ otherwise. $d(x, B_{\sigma^k(m)}) = 1$ for all $k$. It seems that $(A_k)$ is not $d-$metric bounded. In what follows we will prove that $A_k \in WS_\theta^L$ and $A_k \notin |\sigma_1|WM$.

Let $\epsilon > 0$ be given, then we have:

$$\frac{1}{n} \left\{ \text{the number of} \quad k \leq n : \left| \frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L \right| \geq \epsilon \right\} = \frac{[\sqrt{n}]}{n} \to 0,$$

when $n \to \infty$, hence $A_k \in WS_\theta^L$. On the other hand

$$\frac{1}{n} \sum_{k=1}^{n} M\left( \left| \frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L \right| \right) = 0 + 1 + 2 + \cdots + \left([\sqrt{n}] - 1\right) = \frac{[\sqrt{n}]\left([\sqrt{n}] - 1\right)}{2n},$$

from which follows that $A_k \notin |\sigma_1|WN$.

2: Proof of this part is similar to proof of the above Theorems.

3: Proof follows from part 1 and 2. □

**Theorem 4.7.** *Let* $M$ *be an Orlicz function and* $\sup q_r < \infty$. *Then*

**1:** *(a)If* $A_k \overset{|\sigma_1|LWM}{\sim} B_k$ *then* $A_k \overset{WS_{\theta,\sigma}^L}{\sim} B_k$.
**:** *(b)* $S_{\theta,\sigma}^L$ *is a proper subset of* $|\sigma_1|LWM$.

**2:** *If* $M$ *satisfies the* $\Delta_2$ *condition and* $A_k \in l_\infty(|\sigma_1|LWM)$, *such that* $A_k \overset{S_{\theta,\sigma}^L}{\sim} B_k$, *then* $A_k \overset{|\sigma_1|LWM}{\sim} B_k$.

**3:** *If $M$ satisfies the $\Delta_2$ condition, then $S^L_{\theta,\sigma} \cap l_\infty(|\sigma_1|LWM) = |\sigma_1|LWM \cap l_\infty(|\sigma_1|LWM)$.*

## References


[1] J. P. Aubin, H. Frankowska, Set-valued analysis, Boston: Birkhauser (1990).

[2] M. Baronti, P. Papini, Convergence of sequences of sets, Methods of functional analysis in approximation theory (pp. 133-155). Basel: Birkhauser (1986).

[3] G. Beer, Convergence of continuous linear functionals and their level sets, Archiv der Mathematik, 52(1989), 482-491.

[4] G. Beer, On convergence of closed sets in a metric space and distance functions, The Bulletin of the Australian Mathematical Society, 31(1985), 421-432.

[5] J. M. Borwein, J. D. Vanderwerff, Dual Kadec-Klee norms and the relationship between Wijsman, slice and Mosco convergence, Michigan Math. Journal, 41(1994), 371-387.

[6] R. C. Buck, Generalized asymptotic density, Amer. J. Math. 75 (1953) 335-346.

[7] N. L. Braha, On asymptotically $\Delta^m$-lacunary statistical equivalent sequences, Appl. Math. Comput. 219 (2012), no. 1, 280-288.

[8] H. Çakalli, Lacunary statistical convergence in topological groups, Indian J. Pure Appl. Math. 26 (2)(1995), 113-119, MR 95m:40016.

[9] H. Fast, Sur la convergence statistique, Colloq. Math. 2(1951) 241-244.

[10] A. R. Freedman, J. J. Sember, M. Raphael, Some Cesàro-type summability spaces, Proc. London Math. Soc., 37(3) (1978) 508-520.

[11] J. A. Fridy, On statistical convergence, Analysis, 5(1985) 301-313.

[12] J. A. Fridy, C. Orhan, Lacunary statistical convergence, Pacific J. Math., 160, No. 1, (1993), 43-51, MR 94j:40014.

[13] J. A. Fridy, C. Orhan, Lacunary statistical summability, J. Math. Anal. Appl., 173, (1993), 497-504, MR 95f :40004.

[14] B, Hazarika, A. Esi, On asymptotically Wijsman generalized statistical convergence of sequences of sets (preprint).

[15] B. Hazarika, E. Savaş, Lacunary statistical convergence of double sequences and some inclusion results in $n$-normed spaces, Acta Mathematica Vietnamica (accepted for publication)

[16] M. A. Krasnoselskii, Y. B. Rutitsky, Convex function and Orlicz spaces, Groningen, Netherland, 1961.

[17] J. Li, Lacunary statistical convergence and inclusion properties between lacunary methods, Internat. J. Math. Math. Sci. 23(3) (2000), 175–180, S0161171200001964.

[18] G. G. Lorentz, A contribution to the theory of divergent sequences, Acta Math. 80(1948), 167-190.

[19] I. J. Maddox, A new type of convergence, Math. Proc. Cambridge Phil. Soc., 83(1978), 61-64.

[20] M. S. Marouf, Asymptotic equivalence and summability, Internat. J. Math. Math. Sci., 16(4) (1993) 755-762.

[21] S. A. Mohiuddine, E. Savas, Lacunary statistical convergent double sequences in probabilistic normed spaces, Ann. Univ. Ferrara, doi 10.1007/s11565-012-0157-5.

[22] S. A. Mohiuddine, M. A. Alghamdi, Statistical summability through lacunary sequence in locally solid Riesz spaces, Jour. Inequa. Appl.,Vol 2012, 2012:225 doi:10.1186/1029-242X-2012-225.

[23] M. Mursaleen, Matrix transformations between some new sequence spaces, Houton J. Math., 9(1983), 505-509.

[24] M. Mursaleen, S. A. Mohiuddine, On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space, Jour. Comput. Appl. Math., 233(2)(2009) 142-149.

[25] F. Nuray and B. E. Rhoades, Statistical convergence of sequences of sets, Fasciculi Mathematici, 49(2012), 1-9.

[26] R. F. Patterson, On asymptotically statistically equivalent sequences, Demonstratio Math., 36(1) (2003) 149-153.

[27] R. F. Patterson, E. Savaş, On asymptotically lacunary statistically equivalent sequences. Thai Journal of Mathematics, 4(2006), 267-272.

[28] I. P. Pobyvancts, Asymptotic equivalence of some linear transformation defined by a nonnegative matrix and reduced to generalized equivalence in the sense of Cesaro and Abel, Mat. Fiz. 28(1980) 83-87.

[29] T. Šalát, On statistical convergence of real numbers, Math. Slovaca, 30(1980), 139-150.
[30] E. Savas, On lacunary strong $\sigma$-convergence, Indian J. Pure Appl Math., 21(1990), 359-365.
[31] E. Savaş, F. Nuray, On $\sigma$-statistically convergence and lacunary $\sigma$-statistically convergence, Math. Slovaca, 43(3)(1993), 309-15.
[32] E. Savas, R. F. Patterson, $\sigma$-asymptotically lacunary statistical equivalent sequences, Central European Jour. Math., 4(4)(2006), 648-655.
[33] I. J. Schoenberg, The integrability of certain functions, Central European Journal of Mathematics 4(4)(2006), 648-655 and related summability methods, Amer. Math. Monthly, 66(1959) 361-375.
[34] H. Steinhaus, Sr la convergence ordinate et la convergence asymptotique, Colloq. Math., 2 (1951) 73-84.
[35] U. Ulusu, F. Nuray, Lacunary statistical convergence of sequence of sets. Progress in Applied Mathematics, 4(2)(2012), 99-109.
[36] U. Ulusu, F. Nuray, On asymptotically lacunary statistical equivalent set sequences, Journal of Mathematics
[37] R. A. Wijsman, Convergence of sequences of convex sets, cones and functions, Bull. Amer. Math. Soc.,70(1964), 186-188.
[38] R. A. Wijsman, Convergence of sequences of convex sets, cones and functions II, Trans. Amer. Math. Soc.,123-1(1966), 32-45.
[39] A. Zygmund, Trigonometric Series, Cambridge Univ. Press, Cambridge, UK, (1979).

Bipan Hazarika
Department of Mathematics, Rajiv Gandhi University, Rono Hills, Doimukh 791112, Arunachal Pradesh, India.
*E-mail address*: bh_rgu@yahoo.co.in

Ayhan Esi
Adiyaman University, Science and Art Faculty, Department of Mathematics, 02040, Adiyaman, Turkey.
*E-mail address*: aesi23@hotmail.com

N. L. Braha
Department of Mathematics and Computer Sciences, Avenue "Mother Theresa", 5, Prishtinë, 10000, Kosova.
*E-mail address*: nbraha@yahoo.com