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EXPONENTIAL STABILITY OF SOME WAVE COUPLED SYSTEMS

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Abstract. In this work, we consider coupled systems of wave equations for internal (bounded feedback) and boundary (unbounded feedback) dissipation in a one-dimensional domain. We will establish the exponential decay rate of energy of these systems.

1. Introduction and main results

In this paper, we study the stability of coupled wave equations systems with some dissipation types.
First we consider the following initial and boundary value problem with bounded feedback which is given by an internal dissipation

$$\begin{cases} u_{tt}(x,t) - \partial_x^2 u(x,t) + a(x)u_t(x,t) + \gamma\partial_{xt}^2 v(x,t) = 0, \ (0,1)\times\mathbb{R}_+, \\ v_{tt}(x,t) - \partial_x^2 v(x,t) + \gamma\partial_{xt}^2 u(x,t) = 0, \ (0,1)\times\mathbb{R}_+, \\ \partial_x u(0,t) = \partial_x u(1,t) = v(0,t) = v(1,t) = 0, \ t > 0, \\ u(x,0) = u^0(x), \ u_t(x,0) = u^1(x), \ x \in (0,1), \\ v(x,0) = v^0(x), \ v_t(x,0) = v^1(x), \ x \in (0,1), \end{cases} \tag{1.1}$$

where $a \in L^\infty([0,1],\mathbb{R}_+)$ satisfying the following condition: there exists a nonzero Lebesgue measure interval $\omega \subset (0,1)$ and $a_0 > 0$ such that

$$a(x) \geq a_0, \qquad \forall\, x \in \omega. \tag{1.2}$$

$\gamma \neq 0$ is small enough.
The damping $a(x)u_t$ is applied in the first equation and the second is indirectly damped through the coupling between the two equations.

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We also consider the same initial and boundary value problem but with boundary dissipation that is an unbounded feedback

$$\begin{cases} u_{tt}(x,t) - \partial_x^2 u(x,t) + \gamma \partial_{xt}^2 v(x,t) = 0, \ \ (0,1) \times \mathbb{R}_+, \\ v_{tt}(x,t) - \partial_x^2 v(x,t) + \gamma \partial_{xt}^2 u(x,t) = 0, \ \ (0,1) \times \mathbb{R}_+, \\ a\partial_x u(0,t) - \partial_t u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0, \\ v(0,t) = \partial_x v(1,t) = 0, \quad t > 0, \\ u(x,0) = u^0(x), \ u_t(x,0) = u^1(x), \ x \in (0,1), \\ v(x,0) = v^0(x), \ v_t(x,0) = v^1(x), \ x \in (0,1). \end{cases} \tag{1.3}$$

Where γ and a are two real numbers strictly positive and γ is small enough.

The stabilization of coupled systems was already studied in the literature see [4, 5, 7, 20, 9, 15, 19]. A method for that is to use a frequency domain approach based on the growth of the resolvent on the imaginary axis, see [15]. This method does not seem adapable to our systems.
In order to obtain a characterization of the decay of our damped coupled systems of wave equations, we will use the method given in [3]. This method uses a time domain approach reducing the exponential stability to an observability inequality for the associated conservative problem. This approach, given for a coupled system, is based on the result of K. Ammari and M. Tucsnak who gave in [9] a sufficient and necessary condition for the exponential stability for a class of second order evolution equations with unbounded feedbacks. This result has common points with that of Haraux in [11] for feedbacks which are bounded in the energy space, where the assumption of boundedness of the feedbacks will be replaced by an assumption on the transfer function.
This paper is organized as follows. In this first section we give precise statements of the main results. Section 2 is devoted to the proof of wellposedness. The last section contains the proof of the main results of stability.

The solutions of our systems above are of the form (u,v).
We define the energy of the solutions of (1.1) and (1.3) at time t by the same identity as

$$E_1(t) = E_2(t) =$$

$$\frac{1}{2}\left(\|\partial_x u(.,t)\|^2_{L^2(0,1)} + \|\partial_t u(.,t)\|^2_{L^2(0,1)} + \|\partial_x v(.,t)\|^2_{L^2(0,1)} + \|\partial_t v(.,t)\|^2_{L^2(0,1)}\right). \tag{1.4}$$

By the integration by part's formula and using the boundary condition, the solutions of (1.1) and (1.3) check the energy identity

$$\frac{dE_1(t)}{dt} = -\int_0^1 a(x)|\partial_t u(x,t)|^2 dx, \qquad \frac{dE_2(t)}{dt} = -\frac{1}{a}|\partial_t u(0,t)|^2, \qquad \forall t > 0. \tag{1.5}$$

Therefore, these energies are nonincreasing functions of the variable time t and the systems (1.1) and (1.3) are dissipative.

By denoting

$$L_0^2(0,1)=\left\{u\in L^2(0,1);\quad \int_0^1 u(x)dx=0\right\},\quad V=H^1(0,1)\cap L_0^2(0,1),$$

$$V_1=\left\{u\in H^1(0,1):u(1)=0\right\},\quad V_2=\left\{v\in H^1(0,1):v(0)=0\right\},$$

we define the energy spaces as follows

$$\mathcal{H}_1=V\times L_0^2(0,1)\times H_0^1(0,1)\times L^2(0,1), \tag{1.6}$$

$$\mathcal{H}_2=V_1\times L^2(0,1)\times V_2\times L^2(0,1), \tag{1.7}$$

which are equiped with the same usual inner product given by

$$(U_1,U_2)_1=(U_1,U_2)_2=\int_0^1(\partial_x u_1\partial_x\overline{u}_2+y_1\overline{y}_2+\partial_x v_1\partial_x\overline{v}_2+z_1\overline{z}_2)dx,$$

for all $U_1=(u_1,y_1,v_1,z_1)\in\mathcal{H}_i$ and $U_2=(u_2,y_2,v_2,z_2)\in\mathcal{H}_i$ $i=1,2$. The norm associated to this inner product is defined by

$$\|U_1\|^2_{\mathcal{H}_1}=\|U_1\|^2_{\mathcal{H}_2}=\|\partial_x u_1\|^2+\|y_1\|^2+\|\partial_x v_1\|^2+\|z_1\|^2,$$

where $\|.\|=\|.\|_{L^2(0,1)}$.
We define the unbounded operators $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively on $\mathcal{H}_1$ and $\mathcal{H}_2$ by

$$\mathcal{A}_1U=(y,\partial_x^2u-a(x)y-\gamma\partial_x z,z,\partial_x^2v-\gamma\partial_x y)^T \tag{1.8}$$

with

$$D(\mathcal{A}_1)=\left\{\begin{array}{c} U=(u,y,v,z)\in\left[H^2(0,1)\cap V\right]\times V\times\left[H^2(0,1)\cap H_0^1(0,1)\right]\times H_0^1(0,1); \\ \partial_x u(0)=\partial_x u(1)=0,\qquad \int_0^1 a(x)ydx=0\end{array}\right\}$$

and

$$\mathcal{A}_2U=(y,\partial_x^2u-\gamma\partial_x z,z,\partial_x^2v-\gamma\partial_x y)^T \tag{1.9}$$

with

$$D(\mathcal{A}_2)=\left\{\begin{array}{c} U=(u,y,v,z)\in(H^2(0,1)\cap V_1)\times V_1\times(H^2(0,1)\cap V_2)\times V_2; \\ a\partial_x u(0)-y(0)=0,\ \partial_x v(1)=0\end{array}\right\}.$$

Then, setting $U=(u,u_t,v,v_t)^T$,
we rewrite the initial boundary value problems (1.1) and (1.3) into an abstract evolution problem

$$\left\{\begin{array}{l}\frac{dU}{dt}=\mathcal{A}_iU,\\ \\ U(0)=U_0,\end{array}\right.$$

with $i\in\{1,2\}$ and $U_0=(u^0,u^1,v^0,v^1)^T$.
The existence and uniqueness of the solutions of (1.1) and (1.3) can be obtained by standard semigroup method. More precisely we shall prove the following proposition

Proposition 1.1. (1) *Let $(u^0,u^1,v^0,v^1)\in D(\mathcal{A}_1)$. The system* (1.1) *admits a unique solution in $\mathcal{H}_1$ with the regularity*

$$(u,u_t,v,v_t)\in C\left([0,+\infty[,D(\mathcal{A}_1)\right)\times C^1\left([0,+\infty[,\mathcal{H}_1\right). \tag{1.10}$$

Moreover, (u,u_t,v,v_t) satisfies the first energy estimate in (1.5).
For $(u^0,u^1,v^0,v^1)\in\mathcal{H}_1$, our problem admits a unique solution with

$$(u,u_t,v,v_t)\in C\left([0,+\infty[,V\times L_0^2(0,1)\times H_0^1(0,1)\times L^2(0,1)\right). \tag{1.11}$$

(2) *The system* (1.3) *is well-posed. More precisely, for every* $(u^0, u^1, v^0, v^1) \in \mathcal{H}_2$*, there exists a unique mild solution of our system* (1.3)

$$(u, u_t, v, v_t) \in C([0, +\infty[, V_1 \times L^2(0,1) \times V_2 \times L^2(0,1)). \tag{1.12}$$

For $(u^0, u^1, v^0, v^1) \in D(\mathcal{A}_2)$*, the problem* (1.3) *admits a unique classical solution having the regularity*

$$(u, u_t, v, v_t) \in C([0, +\infty[, D(\mathcal{A}_2)) \cap C^1([0, +\infty[, \mathcal{H}_2). \tag{1.13}$$

The main results of this paper are given by the following theorem

Theorem 1.2. (1) *The system described by* (1.1) *is exponentially stable in the energy space* $\mathcal{H}_1$*, i.e* $\exists c_1,\ c_2 > 0$*, such that*

$$E_1(t) \leq c_1 e^{-c_2 t} \left\|(u^0, u^1, v^0, v^1)\right\|^2_{\mathcal{H}_1}, \quad \forall t > 0, \forall (u^0, u^1, v^0, v^1) \in \mathcal{H}_1. \tag{1.14}$$

(2) *The energy of the solution* (u, v) *of* (1.3) *is exponentially stable in the space* $\mathcal{H}_2$ *i.e. there exist two psitive constants* C *and* α *such as the solution* (u, v) *of* (1.3) *satisfies*

$$E_2(t) \leq C e^{-\alpha t} \left\|(u^0, u^1, v^0, v^1)\right\|_{\mathcal{H}_2}, \quad \forall t > 0,\ \forall (u^0, u^1, v^0, v^1) \in \mathcal{H}_2. \tag{1.15}$$

2. Well-posedness results

For the well-posedness of the system (1.1), we show that the operator $\mathcal{A}_1$ defined by (1.8) generates a C_0-semigroup of contractions on the Hilbert space $\mathcal{H}_1$. We have the following fundamental result.

Theorem 2.1. *The operator* $\mathcal{A}_1$ *is a maximal dissipative operator on the energy space* $\mathcal{A}_1$*. Moreover, it generates a* C_0-*semigroup* $e^{t\mathcal{A}_1}$ *of contractions on* $\mathcal{H}_1$.

Proof. Let $U = (u, y, v, z) \in D(\mathcal{A}_1)$. By an integration by parts and using the boundary conditions, we have

$$(\mathcal{A}_1 U, U) = \int_0^1 (\partial_x y \partial_x \overline{u} - \partial_x u \partial_x \overline{y} - a(x) y \overline{y} + \partial_x z \partial_x \overline{v} - \partial_x v \partial_x \overline{z} + \gamma \partial_x \overline{y} z - \gamma \partial_x y \overline{z}) dx.$$

Thus, we get

$$Re(\mathcal{A}_1 U, U) = -\int_0^1 a(x) |y|^2 dx \leq 0. \tag{2.1}$$

Then, $\mathcal{A}_1$ is a dissipative operator on $\mathcal{H}_1$.
Now let $F = (f_1, f_2, f_3, f_4) \in \mathcal{H}_1$. We look for an element $U = (u, y, v, z) \in D(\mathcal{A}_1)$ such that

$$(I - \mathcal{A}_1) U = F. \tag{2.2}$$

Equivalently, we consider the following system

$$y = u - f_1, \quad z = v - f_3, \tag{2.3}$$

$$(1 + a(x)) u - \partial_x^2 u + \gamma \partial_x v = (1 + a(x)) f_1 + f_2 + \gamma \partial_x f_3, \tag{2.4}$$

$$v - \partial_x^2 v + \gamma \partial_x u = \gamma \partial_x f_1 + f_3 + f_4, \tag{2.5}$$

$$\partial_x u(0) = \partial_x u(1) = v(0) = v(1) = 0. \tag{2.6}$$

Take $\phi \in V$ and $\psi \in H_0^1(0,1)$, multiplying (2.4) by $\overline{\phi}$ and (2.5) by $\overline{\psi}$, we get the following variational problem

$$\int_0^1 ((1 + a(x)) u \overline{\phi} + \partial_x u \partial_x \overline{\phi} + \gamma \partial_x v \overline{\phi} + v \overline{\psi} + \partial_x v \partial_x \overline{\psi} + \gamma \partial_x u \overline{\psi}) dx$$

$$= \int_0^1 ((1+a(x))f_1\overline{\phi} + \gamma\partial_x f_1\overline{\psi} + f_2\overline{\phi} + f_3\overline{\psi} + \gamma\partial_x f_3\overline{\phi} + f_4\overline{\psi})dx. \tag{2.7}$$

Let us denote by

$$S((u,v),(\phi,\psi)) = \int_0^1 ((1+a(x))u\overline{\phi} + \partial_x u\partial_x\overline{\phi} + \gamma\partial_x v\overline{\phi} + v\overline{\psi} + \partial_x v\partial_x\overline{\psi} + \gamma\partial_x u\overline{\psi})dx$$

and

$$L(\phi,\psi) = \int_0^1 ((1+a(x))f_1\overline{\phi} + \gamma\partial_x f_1\overline{\psi} + f_2\overline{\phi} + f_3\overline{\psi} + \gamma\partial_x f_3\overline{\phi} + f_4\overline{\psi})dx.$$

It is clear that S is a sesquilinear map in $(V \times H_0^1(0,1))^2$.
By a simple calculation, we obtain

$$|S((u,v),(\phi,\psi))| \leq C\left(\|\partial_x u\|^2 + \|\partial_x v\|^2\right)^{\frac{1}{2}}\left(\|\partial_x\phi\|^2 + \|\partial_x\psi\|^2\right)^{\frac{1}{2}}.$$

Then S is a continuous map in $(V \times H_0^1(0,1))^2$. Moreover,

$$S((u,v),(u,v)) = \int_0^1 \left[(1+a(x))|u|^2 + |\partial_x u|^2| + |v|^2 + |\partial_x v|^2 + \gamma(\partial_x v\overline{u} + \partial_x u\overline{v})\right]dx,$$

however,

$$\left|\int_0^1 \gamma(\partial_x v\overline{u} + \partial_x u\overline{v})dx\right| \leq \frac{\gamma}{2}\left(\|\partial_x v\|^2 + \|v\|^2 + \|\partial_x u\|^2 + \|u\|^2\right).$$

Then,

$$\begin{aligned}
|S((u,v),(u,v))| &\geq \left|\int_0^1 (1+a(x))|u|^2 + |\partial_x u|^2| + |v|^2 + |\partial_x v|^2 dx\right| \\
&- \gamma\left|\int_0^1 (\partial_x v\overline{u} + \partial_x u\overline{v})dx\right| \\
&\geq (1-\tfrac{\gamma}{2})\left(\|\partial_x v\|^2 + \|v\|^2 + \|\partial_x u\|^2 + \|u\|^2\right) \\
&\geq (1-\tfrac{\gamma}{2})\|(u,v)\|^2_{V\times H_0^1(0,1)}.
\end{aligned}$$

Then, since γ is small, S is a coercive form in $(V \times H_0^1(0,1)) \times (V \times H_0^1(0,1))$.
It is easy to check that L is a linear and continuous form on $V \times H_0^1(0,1)$.
Thus, thanks to Lax-Milgram lemma [16, Theorem 2.9.1], the variational problem (2) admits a unique solution $(u,v) \in V \times H_0^1(0,1)$. Using some integrations by parts, we easily check that (u,v) satisfies

$$(1+a(x))u - \partial_x^2 u = (1+a(x))f_1 + f_2 + \gamma\partial_x f_3 - \gamma\partial_x v \in L^2(0,1), \tag{2.8}$$

$$v - \partial_x^2 v = \gamma\partial_x f_1 + f_3 + f_4 - \gamma\partial_x u \in L^2(0,1), \tag{2.9}$$

Then the weak solution (u,v) of (2.8)-(2.9) associated with boundary conditions belongs to the space $H^2(0,1) \times H^2(0,1)$. Therefore, $(u,y,v,z) \in D(\mathcal{A}_1)$ once we have set $y = u - f_1$, $z = v - f_3$. Finally, thanks to Lumer-Phillips Theorem [17, Theorem 1.4.3], we conclude that $\mathcal{A}_1$ generates a C_0-semigroup of contractions on $\mathcal{H}_1$. The proof is thus completed. □

In order to show the second part of the Proposition 1.3, we give the following theorem that is proved in the same way as Theorem 2.1 .

Theorem 2.2. *The operator $\mathcal{A}_2$ generates a C_0-semigroup of contractions on $\mathcal{H}_2$.*

3. Proof of the stability results

Before showing the results of the exponential stability, we start by a study of the strong stability of our systems. For that, we have the following theorem

Theorem 3.1. *The energies associated to the system* (1.1) *and* (1.3) *defined in* (1.4) *satisfy*

$$E_i(t) \to 0 \quad as \quad t \to +\infty \tag{3.1}$$

for all initial data $U_0 \in \mathcal{H}_i$, *i=1,2.*

Proof: The operator $\mathcal{A}_1$ generates a C_0-semigroup of contractions in $\mathcal{H}_1$, then to prove that $\lim\limits_{t\to+\infty} E_1(t) = 0$ is sufficient (see [12],[13]) to check that the imaginary axis is included in the resolvent set; i.e.,

$$\{i\lambda, \quad \lambda \in \mathbb{R}\} \subset \rho(\mathcal{A}_1). \tag{3.2}$$

To check (3.2) we use the contradiction argument. Suppose that (3.2) is not true, then there is a $\lambda \in \mathbb{R}$ with $\lambda \neq 0$, such that $i\lambda$ is in the spectrum of $\mathcal{A}_1$. Since $\mathcal{A}_1$ has a compact resolvent, $i\lambda$ must be an eigenvalue of $\mathcal{A}_1$. It turns out there is a vector function

$$U = (u, y, v, z) \in D(\mathcal{A}_1), \qquad \|U\| = 1, \tag{3.3}$$

such that

$$i\lambda U - \mathcal{A}_1 U = 0. \tag{3.4}$$

In addition, we have

$$(i\lambda U - \mathcal{A}_1 U, U)_1 = i\lambda \left\|\partial_x u\right\|^2 - (\partial_x y, \partial_x \overline{u}) + i\lambda \left\|y\right\|^2 - (\partial_x^2 u, \overline{y}) + \int_0^1 a(x) \left|y(x)\right|^2 dx + \gamma(\partial_x z, \overline{y}) + i\lambda \left\|\partial_x v\right\|^2 - (\partial_x z, \partial_x \overline{v}) + i\lambda \left\|z\right\|^2 - (\partial_x^2 v, \overline{z}) + \gamma(\partial_x y, \overline{z}).$$

Taking the real part of the above expression and using (3.4) yields

$$\int_0^1 a(x) \left|y(x)\right|^2 dx = 0, \tag{3.5}$$

which gives

$$ay = 0 \ \text{ in } \ L^2(0,1). \tag{3.6}$$

Thus, one can write (3.4) as follows

$$i\lambda u - y = 0, \tag{3.7}$$

$$i\lambda y - \partial_x^2 u + \gamma \partial_x z = 0, \tag{3.8}$$

$$i\lambda v - z = 0, \tag{3.9}$$

$$i\lambda z - \partial_x^2 v + \gamma \partial_x y = 0. \tag{3.10}$$

Substituting (3.7), (3.9) in (3.8) and (3.7), (3.9) in (3.10), we obtain

$$-\lambda^2 u - \partial_x^2 u + i\lambda\gamma \partial_x v = 0, \tag{3.11}$$

$$-\lambda^2 v - \partial_x^2 v + i\lambda\gamma \partial_x u = 0. \tag{3.12}$$

By making the sum and subtraction of (3.11) and (3.12), we obtain the two equations below

$$\partial_x^2 w_1 - i\lambda\gamma \partial_x w_1 + \lambda^2 w_1 = 0, \tag{3.13}$$

$$\partial_x^2 w_2 + i\lambda\gamma \partial_x w_2 + \lambda^2 w_2 = 0, \tag{3.14}$$

with

$$w_1 = u + v, \qquad w_2 = u - v \tag{3.15}$$

$$\partial_x(w_1+w_2)(0) = \partial_x(w_1+w_2)(1) = 0 \text{ and } (w_1-w_2)(0) = (w_1-w_2)(1) = 0. \quad (3.16)$$

The solutions of the equation $r^2 - i\lambda\gamma r + \lambda^2 = 0$ are $r_\pm = \frac{i\lambda\gamma \pm i\lambda\sqrt{\gamma^2+4}}{2}$.
Hence, the solutions of (3.13) and (3.14) are in the form

$$w_1 = A_1 e^{r_+ x} + B_1 e^{r_- x}, \quad w_2 = A_2 e^{-r_+ x} + B_2 e^{-r_- x}.$$

The second and the third boundary condition in (3.16) give

$$A_1 - A_2 + B_1 - B_2 = 0, \quad (3.17)$$

$$(A_1 - A_2)r_+ + (B_1 - B_2)r_- = 0. \quad (3.18)$$

The determinant of these equations is equal to $r_+ - r_- = i\lambda\sqrt{\gamma^2+4} \neq 0$, then

$$\begin{cases} A_1 = A_2 = A, \\ B_1 = B_2 = B. \end{cases}$$

Since (1.2) and (3.6) , there are two real numbers $x_0,\ y_0$, where $0 < x_0 < y_0 < 1$, such that

$$y = 0 \text{ on }]x_0, y_0[.$$

Therefore, since $u,\ v \in H^2(0,1),\ y \in V,\ z \in H^1(0,1)$ and using the equations (3.7)-(3.10), we can show that

$$u = \partial_x u = v = \partial_x v = 0 \text{ on } [x_0, y_0].$$

Then, we obtain the following equations

$$A\cosh(r_+x_0) + B\cosh(r_-x_0) = 0, \qquad Ar_+\cosh(r_+x_0) + Br_-\cosh(r_-x_0) = 0,$$

$$A\sinh(r_+x_0) + B\sinh(r_-x_0) = 0, \qquad Ar_+\sinh(r_+x_0) + Br_-\sinh(r_-x_0) = 0,$$

who are give $A = B = 0$.

Thus, we obtain $w_1 = 0$ and $w_2 = 0$.
Consequently, we have

$$u = 0 \text{ and } v = 0$$

i.e., $U = 0$. A contradiction to (3.3).
Finally, we conclude that $\{i\lambda, \quad \lambda \in \mathbb{R}\} \subset \rho(\mathcal{A}_1)$.
For the same reason, we can prove the result for $E_2(t)$. Which completes this proof. □

Proof of Theorem 1.2

1. For the proof of our first result concerning the exponential stability of the system (1.1), we use the result given in [10, Theorem 1]. For that, we rewrite the system (1.1) in the following form

$$\begin{cases} \frac{d}{dt}U(.,t) = AU(.,t) + BU(.,t), \\ \\ U(.,0) = U_0, \end{cases} \quad (3.19)$$

in the Hilbert space $\mathcal{H}_1$, where

$$A = \begin{pmatrix} 0 & I & 0 & 0 \\ \partial_x^2 & 0 & 0 & -\gamma\partial_x \\ 0 & 0 & 0 & I \\ 0 & -\gamma\partial_x & \partial_x^2 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -a(x) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.20)$$

The operator A is a skew-adjoint operator in $\mathcal{H}_1$ with compact resolvent. Then, an immediate consequence is that the operator A has a complet orthonormal set of eigenvectors

$$\begin{cases} \{U_n;\ \ n \in \mathbb{Z}^*\} \\ \|U_n\|_{\mathcal{H}_1} = 1, \quad (U_n, U_m) = \delta_{mn}, \end{cases}$$

where $(,)$ is the inner product in $\mathcal{H}_1$ and δ_{mn} is the Kronecker delta symbol, such that

$$\begin{cases} AU_n = i\lambda_n U_n, \quad n \in \mathbb{Z}^* \\ \lambda_n \in \mathbb{R}, |\lambda_n| \to +\infty, \\ 0 < |\lambda_1| \le |\lambda_2| \le ... \le |\lambda_n| < ..., \lambda_j \ne \lambda_k \ \text{ if } \ j \ne k. \end{cases}$$

Then, according to the result [10, Theorem 1], to establish the exponential stability for the system (1.1) it is enough to verify the following conditions:
The spectrum $\{\lambda_i;\ i \in \mathbb{Z}^*\}$ of A satisfies the following gap property

$$inf\,\{|\lambda_j - \lambda_k|\,; j,k \in \mathbb{Z}^*, j \ne k\} \equiv \alpha > 0, \tag{3.21}$$

B is dissipative i.e.,

$$Re(BU,U) \le 0 \ \text{ for all } \ U \in \mathcal{H}_1, \tag{3.22}$$

if any sequence $\{U_n \in \mathcal{H}_1; n = 1,2,...\}$ satisfies

$$\lim_{n\to+\infty}(BU_n, U_n) = 0 \ \text{ then } \ \lim_{n\to+\infty} BU_n = 0, \tag{3.23}$$

and there exists $\delta > 0$ such that

$$\|BV\| \ge \delta \ \text{ for any unit eigenvector } \ V \ \text{ of } \ A. \tag{3.24}$$

Thus, we calculate the spectrum of A and its eigenvectors.
Let the following eigenvectors and eigenvalues problem

$$AU_n = i\lambda_n U_n, \tag{3.25}$$

where $U_n = (u_n, y_n, v_n, z_n) \in D(A)$ and $(\lambda_n) \subset \mathbb{R}$. This equation is equivalent to

$$\begin{aligned} &y_n = i\lambda_n u_n, \\ &\partial_x^2 u_n - \gamma\partial_x z_n = i\lambda_n y_n, \\ &z_n = i\lambda_n v_n, \\ &\partial_x^2 v_n - \gamma\partial_x y_n = i\lambda_n z_n. \end{aligned}$$

Then,

$$\begin{aligned} &\lambda_n^2 u_n + \partial_x^2 u_n - i\lambda_n\gamma\partial_x v_n = 0, \\ &\lambda_n^2 v_n + \partial_x^2 v_n - i\lambda_n\gamma\partial_x u_n = 0, \\ &\partial_x u_n(0) = \partial_x u_n(1) = v_n(0) = u_n(1) = 0. \end{aligned}$$

Let us put $w_{n1} = u_n + v_n$ and $w_{n2} = u_n - v_n$. These functions satisfy the following equations

$$\lambda_n^2 w_{n1} + \partial_x^2 w_{n1} - i\lambda_n\gamma\partial_x w_{n1} = 0, \tag{3.26}$$

$$\lambda_n^2 w_{n2} + \partial_x^2 w_{n2} + i\lambda_n\gamma\partial_x w_{n2} = 0, \tag{3.27}$$

$$\partial_x(w_{n1} + w_{n2})(0) = \partial_x(w_{n1} + w_{n2})(1) = 0,\ (w_{n1} - w_{n2})(0) = (w_{n1} - w_{n2})(1) = 0. \tag{3.28}$$

Let $r_+,\ r_-$ be the roots of the equation $r^2 - i\gamma\lambda_n r + \lambda_n^2 = 0$, which are $i\frac{\gamma\lambda_n \pm \lambda_n\sqrt{\gamma^2+4}}{2}$.
Then the solution of the equations (3.26) and (3.27) is given by

$$w_{n1}(x) = A_1 e^{r_+ x} + B_1 e^{r_- x}, \quad w_{n2}(x) = A_2 e^{-r_+ x} + B_2 e^{-r_- x} \qquad x \in (0,1). \tag{3.29}$$

As the proof of Theorem 3.1, we have $A_1 = A_2 = A$ and $B_1 = B_2 = B$.
Therefore, the second and the fourth boundary condition give

$$A(e^{r_+} - e^{-r_+}) + B(e^{r_-} - e^{r_-}) = 0,$$
$$Ar_+(e^{r_+} - e^{-r_+}) + Br_-(e^{r_-} - e^{r_-}) = 0.$$

Since the conditions $w_{n1} \neq 0, \;\; w_{n2} \neq 0$, the determinant of these equations

$$(r_+ - r_-)(e^{r_+} - e^{-r_+})(e^{r_-} - e^{-r_-}) = 0.$$

Then, $e^{r_+} - e^{-r_+} = 0$ or $e^{r_-} - e^{-r_-} = 0$ who give that λ_n is in the form λ_n^+ or λ_n^- i.e,

$$\lambda_n = \lambda_n^{\pm} = \frac{2n\pi}{\gamma \pm \sqrt{\gamma^2+4}}, \qquad n \in \mathbb{Z}^*, \tag{3.30}$$

and the solutions are

$$w_{n1}^{\pm} = Ce^{in\pi x}, \quad w_{n2}^{\pm} = Ce^{-in\pi x}.$$

Taking the following conditions

$$\|(w_{n1}, i\lambda_n w_{n1})\| = 1 \;\; \text{and} \;\; \|(w_{n2}, i\lambda_n w_{n2})\| = 1$$

then we have the condition

$$\|U_n\|_{\mathcal{H}_1} = 1.$$

Then

$$C = \frac{1}{\sqrt{n^2\pi^2 + \lambda_n^2}} = \alpha_n.$$

Consequently

$$u_n = \alpha_n \cos(n\pi x), \qquad y_n = i\lambda_n \alpha_n \cos(n\pi x),$$

$$v_n = \alpha_n \sin(n\pi x), \qquad z_n = i\lambda_n \alpha_n \sin(n\pi x).$$

Then, it is clear that (3.21) is obtained from (3.30).
Now let the damping perturbation, B, be a bounded operator on $\mathcal{H}_1$. And we verify the conditions which are related to it.
Firstly, we have

$$(BU, U) = -\int_0^1 a(x) \left|y(x)\right|^2 dx \leq 0, \tag{3.31}$$

then (3.22) is checked.
Secondly, let $(U_n) \subset \mathcal{H}_1$ satisfying $\lim\limits_{n\to+\infty} (BU_n, U_n) = 0$ i.e., $\int_0^1 a(x) \left|y_n(x)\right|^2 dx \longrightarrow 0$.
Therefore

$$\|BU_n\|_{\mathcal{H}_1}^2 = \int_0^1 (a(x))^2 \left|y_n(x)\right|^2 dx \leq \left(\sup_{x\in[0,1]} a\right) \int_0^1 a(x) \left|y_n(x)\right|^2 dx \longrightarrow 0.$$

Finally, to check the last condition, we note that

$$\begin{aligned} \|BU_n\|_{\mathcal{H}_1}^2 &= \int_0^1 \left|a(x)\lambda_n\alpha_n \cos(n\pi x)\right|^2 dx \\ &\geq C_{a,\gamma} \int_s^{s'} |\cos(n\pi x)|^2 dx \\ &\geq \delta > 0, \end{aligned}$$

where $s,\ s' \in]0,1[$ such that $s < s'$ and $[s,s'] \subset Supp\, a$. This ends the proof. $\square$

Secondly, we will study the stability of the system (1.3) where the dissipation is located on a part of the boundary. For that, we prove the second result of the Theorem 1.2 using the approach given in [3] for the unbounded feedbacks stabilization.

2. In order to show the result (1.15), one can raise the damping, applied on the solution u, at the level of the first equation (see [20] for more details). Thus, the system (1.3) will be equivalent to

$$\begin{cases} u_{tt}(x,t) - \partial_x^2 u(x,t) + \frac{1}{a} u_t(0,t)\delta_0 + \gamma \partial_{xt}^2 v(x,t) = 0, \ \ (0,1)\times\mathbb{R}_+, \\ v_{tt}(x,t) - \partial_x^2 v(x,t) + \gamma \partial_{xt}^2 u(x,t) = 0, \ \ (0,1)\times\mathbb{R}_+, \\ \partial_x u(0,t) = 0, \quad u(1,t) = 0, \ t>0, \\ v(0,t) = 0, \quad \partial_x v(1,t) = 0, \ t > 0, \\ u(x,0) = u^0(x), \ \ u_t(x,0) = u^1(x), \ \ v(x,0) = v^0(x), \ \ v_t(x,0) = v^1(x). \end{cases} \tag{3.32}$$

We consider the spaces $H_1 = H_2 = L^2(0,1)$ and the operators $A_1 = A_2 = -\partial_x^2$, with domains

$$D(A_1) = \left\{ u \in H^2(0,1) \cap V_1; \ \ \frac{du}{dx}(0) = 0 \right\}, \quad D(A_1^{\frac{1}{2}}) = V_1,$$

$$D(A_2) = \left\{ v \in H^2(0,1) \cap V_2; \ \ \frac{dv}{dx}(1) = 0 \right\}, \quad D(A_2^{\frac{1}{2}}) = V_2.$$

Consider the operator B and its adjoint given by

$$Bk = \frac{1}{\sqrt{a}} k\delta_0, \quad \forall k \in U = \mathbb{R} \ \text{ and } \ B^*\varphi = \frac{1}{\sqrt{a}}\varphi(0), \quad \forall \varphi \in V_1$$

and finally, let

$$C = \gamma \frac{d}{dx} \ \text{ with } \ \ D(C) = V_2 \ \text{ and } \ \ C^* = -\gamma\frac{d}{dx} \ \text{ with } \ \ D(C^*) = V_1.$$

So, the system (3.32) can be written in the abstract form

$$\ddot{w}_1(t) + (A_1)_{-1} w_1(t) + BB^*\dot{w}_1(t) + C\dot{w}_2(t) = 0, \quad t \geq 0, \tag{3.33}$$

$$\ddot{w}_2(t) + A_2 w_2(t) - C^*\dot{w}_1(t) = 0, \quad t \geq 0, \tag{3.34}$$

$$w_i(0) = w_i^0, \ \dot{w}_i(0) = w_i^1, \ i = 1,2 \tag{3.35}$$

where $(A_1)_{-1}$ is the extension of A_1 on the space $H_1 = L^2(0,1)$.
According to [3, Theorem 3.10], under the following condition of boudedness of the transfer function: for a fixed $\gamma > 0$

$$\sup_{Re\,\lambda=\gamma} \left\| \lambda B^* [\lambda^2 I + A_1 + \lambda^2 C(\lambda^2 + A_2)^{-1} C^*]^{-1} B \right\|_{\mathcal{L}(U)} < +\infty, \tag{3.36}$$

the system (3.33)-(3.35) is exponentially stable in $\mathcal{H}_1$ if and only if there exists a constant $c > 0$ such that

$$\int_0^1 \left\| B^* \dot{\phi}(t) \right\|_U^2 dt \geq c \left\| (\phi^0, \phi^1, \psi^0, \psi^1) \right\|_{\mathcal{H}_1}^2 \tag{3.37}$$

where (ϕ, ψ) is a solution of the following conservative adjoint system

$$\ddot{\phi} + A_1\phi + C\dot{\psi} = 0,$$
$$\ddot{\psi} + A_2\psi - C^*\dot{\phi} = 0,$$
$$\phi(0) = \phi^0,\ \dot{\phi}(0) = \phi^1,\ \psi(0) = \psi^0,\ \dot{\psi}(0) = \psi^1.$$

Let us verify the boundedness (3.36), of the transfer function associed to our system (3.32). For this, let $k \in \mathbb{R},\ Re\lambda > 0$ and the system

$$\lambda^2 u(x) - \frac{d^2u}{dx^2}(x) + \gamma\lambda\frac{dv}{dx}(x) = 0, \quad x \in (0,1), \tag{3.38}$$

$$\lambda^2 v(x) - \frac{d^2v}{dx^2}(x) + \gamma\lambda\frac{du}{dx}(x) = 0, \quad x \in (0,1), \tag{3.39}$$

$$\frac{du}{dx}(0) = k, \quad u(1) = 0, \tag{3.40}$$

$$v(0) = 0, \quad \frac{dv}{dx}(1) = 0. \tag{3.41}$$

Then, the transfer function is given by

$$H(\lambda) = \lambda u(0).$$

To solve the system (3.38)-(3.41), one will take $w = u + v$ and $\widetilde{w} = u - v$, then we have

$$\lambda^2 w(x) - \frac{d^2w}{dx^2}(x) + \gamma\lambda\frac{dw}{dx}(x) = 0, \quad x \in (0,1), \tag{3.42}$$

$$\lambda^2 \widetilde{w}(x) - \frac{d^2\widetilde{w}}{dx^2}(x) + \gamma\lambda\frac{d\widetilde{w}}{dx}(x) = 0, \quad x \in (0,1), \tag{3.43}$$

$$\frac{d(w + \widetilde{w})}{dx}(0) = k, \quad (w + \widetilde{w})(1) = 0, \tag{3.44}$$

$$(w - \widetilde{w})(0) = 0, \quad \frac{d(w - \widetilde{w})}{dx}(1) = 0. \tag{3.45}$$

Let $r_1,\ r_2$ be the roots of the equation $r^2 - \gamma\lambda r + \lambda^2 = 0$, given by $\frac{\gamma\lambda \pm \lambda\sqrt{\gamma^2+4}}{2}$. Then the solutions of the equation (3.42) and (3.43) are given by

$$w(x) = A_1 e^{r_1 x} + B_1 e^{r_2 x} \quad \text{and} \quad \widetilde{w}(x) = A_2 e^{-r_1 x} + B_2 e^{-r_2 x}.$$

Therefore, the boundary conditions yield

$$w(x) = \frac{k}{r_1 - r_2}\left(\frac{e^{r_1 x}}{1 + e^{2r_1}} - \frac{e^{r_2 x}}{1 + e^{2r_2}}\right),$$

$$\widetilde{w}(x) = \frac{k}{r_2 - r_1}\left(\frac{e^{2r_1}}{1 + e^{2r_1}} e^{-r_1 x} - \frac{e^{2r_2}}{1 + e^{2r_2}} e^{-r_2 x}\right).$$

After the calculation of $H(\lambda)$, one has

$$\sup_{Re\lambda = 2\delta} |H(\lambda)| \leq \frac{1}{\sqrt{\gamma^2+4}} \frac{\sinh(2\delta\sqrt{\gamma^2+4})}{\sinh[\delta(\gamma + \sqrt{\gamma^2+4})]\sinh[\delta(-\gamma + \sqrt{\gamma^2+4})]}.$$

Thus, we obtain the inequality (3.36) on the axis $Re\lambda = 2\delta,\ \delta > 0$.
Then, we consider the conservative adjoint system

$$\frac{\partial^2\phi}{\partial t^2}(x,t) - \frac{\partial^2\phi}{\partial x^2}(x,t) + \gamma\frac{\partial^2\psi}{\partial x\partial t}(x,t) = 0, \quad (x,t) \in (0,1) \times (0,+\infty),$$

$$\frac{\partial^2\psi}{\partial t^2}(x,t) - \frac{\partial^2\psi}{\partial x^2}(x,t) + \gamma\frac{\partial^2\phi}{\partial x\partial t}(x,t) = 0, \quad (x,t) \in (0,1)\times(0,+\infty),$$

$$\frac{\partial\phi}{\partial x}(0,t) = \phi(1,t) = 0, \quad t \in (0,+\infty),$$

$$\psi(0,t) = \frac{\partial\psi}{\partial x}(1,t) = 0, \quad t \in (0,+\infty),$$

$$\phi(x,0) = \phi^0(x),\ \frac{\partial\phi}{\partial t}(x,0) = \phi^1(x),\ \psi(x,0) = \psi^0(x),\ \frac{\partial\psi}{\partial t}(x,0) = \psi^1(x).$$

The operator which generates the evolution of this system is

$$A = \begin{pmatrix} 0 & I & 0 & 0 \\ \partial_x^2 & 0 & 0 & -\gamma\partial_x \\ 0 & 0 & 0 & I \\ 0 & -\gamma\partial_x & \partial_x^2 & 0 \end{pmatrix}. \tag{3.46}$$

By a study of the spectral problem

$$AU_n = i\lambda_n U_n, \tag{3.47}$$

where $U_n = (u_n, \phi_n, v_n, \psi_n)^T \in D(A)$ and $(\lambda_n) \subset \mathbb{R}$, we can write the initial conditions as follows

$$\phi^0(x) = \sum_{n\in\mathbb{Z}} a_n \cos((2n+1)\frac{\pi}{2}x), \qquad \phi^1(x) = \sum_{n\in\mathbb{Z}} i\lambda_n a_n \cos((2n+1)\frac{\pi}{2}x),$$

$$\psi^0(x) = \sum_{n\in\mathbb{Z}} a_n \sin((2n+1)\frac{\pi}{2}x), \qquad \psi^1(x) = \sum_{n\in\mathbb{Z}} i\lambda_n a_n \sin((2n+1)\frac{\pi}{2}x),$$

with $\lambda_n = \frac{(2n+1)\pi}{\gamma+\sqrt{\gamma^2+4}}$, $n \in \mathbb{Z}$, and $(\lambda_n a_n) \in l^2(\mathbb{R})$. Hence, by standard technics, we obtain

$$\phi(x,t) = \sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t} \cos((2n+1)\frac{\pi}{2}x).$$

Now, by the Ingham's inequality see [14], for any T large enough there is $C_{T,\gamma} > 0$ such that

$$\int_0^T \left|\frac{\partial\phi}{\partial t}(0,t)\right|^2 dt \geq C_{T,\gamma} \sum_{n\in\mathbb{Z}} |\lambda_n|^2 |a_n|^2.$$

Finally, this implies that the system given by its boundary dissipation is exponentially stable in the associated energy space $\mathcal{H}_2$. $\square$

Remark. *The systems of coupled wave equations* (1.1) *and* (1.3) *can also be stabilized, in the same manner, by replacing the damping applied on the first part u by a pointwise control $u_t(t,\xi)\delta_\xi$, $\xi \in (0,1)$. Thus, for the first system, we obtain the* *following system*

$$\begin{cases} u_{tt}(x,t) - \partial_x^2 u(x,t) + u_t(\xi,t)\delta_\xi + \gamma\partial_{xt}^2 v(x,t) = 0, \ (0,1)\times\mathbb{R}_+, \\ v_{tt}(x,t) - \partial_x^2 v(x,t) + \gamma\partial_{xt}^2 u(x,t) = 0, \ (0,1)\times\mathbb{R}_+, \\ \partial_x u(0,t) = \partial_x u(1,t) = v(0,t) = v(1,t) = 0, \ t>0, \\ u(x,0) = u^0(x), \ u_t(x,0) = u^1(x), \ x\in(0,1) \\ v(x,0) = v^0(x), \ v_t(x,0) = v^1(x), \ x\in (0,1), \end{cases} \tag{3.48}$$

this case is studied in [3] *for a system with delay. Then, as in* [3]*, we have for* $T > \frac{\gamma+\sqrt{\gamma^2+4}}{2}$ *that, there exists* $C_{T,\xi,\gamma} > 0$ *such that*

$$\int_0^T \left|\frac{\partial\phi}{\partial t}(\xi,t)\right|^2 dt \geq C_{T,\gamma,\xi}\sum_{n\in\mathbb{Z}} |\lambda_n|^2|a_n|^2\cos^2(n\pi\xi), \tag{3.49}$$

where ϕ *is the first component of the solution of the associated conservative system. This inequality, as in* [1, 6] *for the only string equation, implies that the system is exponentially stable in the energy space if and only if* ξ *is a rational number with coprime factorisation* $\xi = \frac{p}{q}$*, where* p *is odd.*
By the same method as above and as in [3, 9]*, we can, according to* (3.49)*, establish a polynomial stability result under some diophantine conditions on the point* ξ*.*

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