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REDUCTION THEOREMS FOR MONOTONICITY

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Abstract. Given f continuous on a compact interval $[a, b]$, we show that if the derivative Df is positive except possibly on a countable subset of an interval, then f is increasing. This theorem is generalized by the Goldowsky-Tonelli Theorem. We then show that there are no additional functions to which Goldowsky-Tonelli applies but our countable exception theorem does not apply. We then apply our analysis to a theorem of Zahorski.

1. Introduction

Andrew Bruckner [5, 6] and Brian Thompson [23, 24] have raised the problem of unifying our study of monotonicity. The attack has been split into two categories: reduction theorems and abstraction theorems. Abstraction theorems address the underlying structure of what makes a function monotone by invoking the machinery of abstract differentiation theory, e.g., see [23, 24]. Reduction theorems relate more technical results to simplified ones. We can find examples of reduction in Bruckner's encyclopedic monograph *Differentiation of Real Functions* [5]. Chapter 11 contains a collection of theorems discussing how information from a function and its derivative can give us information about the monotonicity of that function. These include theorems by Goldowsky, Tonelli, and Zahorski (see [5], p. 120). Bruckner describes these theorems as rather technical or specialized, but points out that the theorems can arise naturally when one only has certain information about a function. For example, Bruckner shows that a monotonicity theorem that one can prove about continuous functions of bounded variation can be extended to Darboux-Baire Category 1 functions. He then uses this analysis to do a reduction of Zahorski's theorem to Goldowsky-Tonelli. The purpose of this paper is to reduce Goldowsky-Tonelli and to show how this reduction fits with other similar theorems.

We will assume throughout the paper that all of the functions considered in connection to our investigation are continuous. If we do not assume continuity, we can vary behavior at a single point, thus easily changing monotonicity.

1.1. **Derivatives and Derivates.** The simplest result on monotonicity is given by the Mean Value Theorem from calculus. Let f continuous on a compact interval

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$[a,b]$ and differentiable on (a,b) with derivative Df. The Mean Value Theorem gives that for all x, y, $a \leq x < y \leq b$, there exists ξ, $x < \xi < y$ such that

$$f(y) - f(x) = Df(\xi)(y-x)\,.$$

If $Df(\xi) > 0$ for all $\xi \in (a,b)$, then f is strictly monotone increasing on $[a,b]$.

The four *Dini derivates* of a function f always exist (finite or infinite). They generalize the derivative Df.

Definition 1.1. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be finite in a neighborhood of x_0. Then the four* **Dini derivates** *(or numbers) are given by*

$$D^+f(x_0) = \limsup_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h}\,,$$

$$D_+f(x_0) = \liminf_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h}\,,$$

$$D^-f(x_0) = \limsup_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h}\,,$$

$$D_-f(x_0) = \liminf_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h}\,.$$

The Dini derivates satisfy

$$D^+f(x_0) \geq D_+f(x_0)\ ,\ D^-f(x_0) \geq D_-f(x_0)\,.$$

A function is differentiable at x_0 with derviative $D(x_0)$ if and only if all four Dini derivates are identical and are different from $+\infty$ or $-\infty$.

1.2. **Cantor Sets and Counterexamples.** We will use Cantor sets and Cantor-Lebesgue functions frequently in our paper. We can construct the *Cantor middle thirds set* C as follows. Given $[0,1]$, remove the open middle third segment, $(\frac{1}{3}, \frac{2}{3})$. Then, remove the open middle thirds of the remaining two segments, i.e., remove $(\frac{1}{9}, \frac{2}{9})$ from $[0, \frac{1}{3}]$ and $(\frac{7}{9}, \frac{8}{9})$ from $[\frac{2}{3}, 1]$. Repeat. Let C_n denote the intervals left at the n^{th} stage of this process. C_n consists of 2^n segments, each of length $(1/3)^n$. Continue ad infinitum. We let

$$C = \bigcap\nolimits_{n=1}^{\infty} C_n\,.$$

Many other Cantor constructions are possible, e.g., removing the second and fourth sections of an interval divided into equal fifths.

The set C has several interesting properties. Let R denote the union of the removed intervals. Then $C \cap R = \emptyset$, $C \cup R = [0,1]$. The measure of $[0,1]$ is its length, namely 1. The measure of R is its total length, which equals

$$\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{3}\frac{1}{(1-\frac{2}{3})} = 1\,.$$

Therefore, by additivity, C has measure zero. Also, any point of C belongs to an interval in C_n for all n, and therefore is a limit point of the endpoints of the intervals. Thus C is perfect, and therefore uncountable. Additionally, C does not contain any interval. A set which does not contain any interval is called *totally disconnected*. Finally, if $x \in C$ with ternary expansion $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$, then each a_n is either 0 or 2.

A fact we will find quite useful is that every bounded perfect set that contains no interval is homeomorphic to the Cantor set (see Hocking and Young [14], pp. 97–100). In fact, any such set A can be formed by a middle-third-like process, in which the deleted intervals have varying lengths. If the removed intervals are mapped linearly onto those removed from the Cantor set, we get a homeomorphism of an interval containing the set onto [0, 1], carrying the set A onto the Cantor set. We shall call a bounded, perfect, totally disconnected set *a* Cantor set (as opposed to *the* Cantor set.)

Definition 1.2. *A* **Cantor set** *is a bounded, perfect, totally disconnected set.*

Every bounded perfect set contains a Cantor set. This is easy to see. Simply perform a Cantor removal process on any interval in the set.

The *Cantor-Lebesgue function* $\mathcal{C}$ associated with the Cantor middle thirds set is described as follows. (Also see, for example, [8], pp. 40, 175-176, [15], pp. 27–30, 181–189, or [26], p. 35.) Let $R_n = [0,1]-C_n$. Then R_n consists of the 2^n-1 intervals I_m^n (ordered from left to right) removed in the first n stages of the construction of the Cantor set. Let f_n be the continuous, piecewise linear function on $[0,1]$ which satisfies

$$f_n(0) = 0\,,\ f_n(1) = 1\,,\ f_n(x) = m2^{-n} \text{ on } I_m^n\,, m = 1\ \ldots, 2^n - 1$$

and which is linear on each piece of C_n. By construction, each f_n is continuous, piecewise linear, and monotone increasing. Also, $f_{n+1} = f_n$ on $I_m^n\,, m = 1\ \ldots, 2^n - 1$, and $|f_n - f_{n+1}| < 2^{-n}$. Therefore, by the Weierstrass M test, $\sum(f_n - f_{n+1})$ converges uniformly on $[0,1]$. We let

$$\mathcal{C} = \lim_{n\to\infty} f_n\,.$$

Since the sequence $\{f_n\}$ converges uniformly, $\mathcal{C}$ is continuous. The function is also piecewise linear, monotone increasing, and

$$\mathcal{C}(0) = 0\,, \mathcal{C}(1) = 1\,, \text{ and } \mathcal{C} \text{ is constant on every } R_n\,.$$

$\mathcal{C}$ equals 1/2 on the first removed interval, 1/4 and 3/4 on the second and third respectively, then 1/8, 3/8, 5/8, and 7/8 on the next four, and so forth. Since the measure of $R = \bigcup R_n$ equals 1, $\mathcal{C}$ has zero derivative except on the Cantor set, i.e., $D\,\mathcal{C} = 0$ *a.e.*

We can not weaken the hypotheses of the Mean Value Theorem, arbitrarily and still get monotonicity. To see this, we assume, in addition to continuity, only that $Df > 0$ almost everywhere (*a.e.*), i.e., except on a set of Lebesgue measure zero. This condition is too weak. Here is an example. Let $\mathcal{C}$ denote the Cantor-Lebesgue function associated with the Cantor middle-thirds set. Let

$$f(x) = \frac{x}{2} + \mathcal{C}(1-x)\,. \tag{1.1}$$

Then f is a continuous mapping on $[0,1]$ with $Df = \frac{1}{2}$ *a.e.* However, $f(0) = 1 > \frac{1}{2} = f(1)$. The function f is not increasing. Here, the "risers" of the Cantor-Lebesgue function, which occur over the Cantor middle-thirds set, allow the function f to "flow" against the derivative. Measure zero gives too much room, allowing for this flow. We can, of course, produce other such functions, such as, for fixed $\alpha \in (0,1)$,

$$f_\alpha(x) = \alpha x + \mathcal{C}(1-x) \text{ and } g_\alpha(x) = \alpha x - \mathcal{C}(x)\,.$$

These examples can be generalized. A perfect set is a closed non-empty set in which every point is a limit point. The *Cantor-Bendixson Theorem* (see [2], p. 67) gives us that every uncountable set can be written as the union of a perfect set and a countable set. Given that every Cantor set is homeomorphic to the Cantor middle-thirds set, we can construct an associated Cantor-Lebesgue function, which is differentiable with zero derivative on the complement of the set. Use this Cantor-Lebesgue function to construct the example.

2. Countable Exception

If we assume that f is continuous on $[a,b]$ and differentiable on (a,b) *a.e.*, what additional conditions we can impose on Df to guarantee that f is monotone increasing? The first approach is to strengthen continuity.

Let $f : [a,b] \longrightarrow \mathbb{R}$ be finite on $[a,b]$. The function f is *absolutely continuous* if given any $\epsilon > 0$ there exists a $\delta > 0$ such that for any collection $\{[a_i,b_i]\}$ of non-overlapping subintervals of $[a,b]$,

$$\sum_i |f(b_i) - f(a_i)| < \epsilon \text{ whenever } \sum_i |b_i - a_i| < \delta\,.$$

The Cantor-Lebesgue function is an example of a continuous function which is not absolutely continuous. A function f is absolutely continuous in $[a,b]$ if and only if Df exists *a.e.*, is integrable on (a,b) and

$$f(x) - f(a) = \int_a^x Df \text{ for all } a \leq x \leq b\,.$$

(see [26], pp. 115–117.)

Theorem 2.1 ([5], p. 120)**.** *Let f be a absolutely continuous function on $[a,b]$ and suppose that $Df \geq 0$ a.e. on $[a,b]$. Then f is non-decreasing.*

If we assume only that f is continuous, we can formulate the question at the beginning of this section in terms of sets.

Definition 2.2. *Let f be continuous on $[a,b]$ and suppose that $Df > 0$ on $[a,b] \setminus S$. Then S is called the* **exceptional set**.

If we assume that the exceptional set is countable, we have the following. We refer to this as the *Countable Exceptional Set Theorem.*

Theorem 2.3. *Let f be a continuous function on $[a,b]$ and suppose that the derivative Df is such that $Df > 0$ on $[a,b] \setminus S$, where S is countable. Then f is strictly increasing on $[a,b]$.*

Proof. We first show that $f(a) \leq f(b)$. Suppose, by way of contradiction, that $f(a) > f(b)$. Choose λ such that $f(a) > \lambda > f(b)$, with $\lambda \notin f(S)$. This is possible since $f(S)$ is countable, and therefore does not exhaust $(f(b), f(a))$. Let

$$A = \{x \in [a,b] : f(x) > \lambda\}\,.$$

Since $a \in A$, $A \neq \emptyset$. Therefore, A has a supremum. Denote it by c. Then

$$(i.) \text{ if } x > c, f(x) \leq \lambda\,,$$

$$(ii.) \text{ for each } \delta > 0 \text{ there exists } x_\delta \text{ such that } c - \delta < x_\delta \leq c \text{ and } f(x_\delta) > \lambda\,.$$

Now, suppose $f(c) > \lambda$. Then, since f is continuous, $f(c+\epsilon) > \lambda$ for sufficiently small $\epsilon > 0$, contradicting (i). On the other hand, if $f(c) < \lambda$, $f(c-\epsilon) < \lambda$ for sufficiently small $\epsilon > 0$, which contradicts (*ii.*). Therefore, $f(c) = \lambda$. From (*i.*) and (*ii.*), we have that the derivative $Df(c) \leq 0$, contradicting the assumption that $Df > 0$ on $(a, b) \setminus S$. Thus, $f(b) \geq f(a)$.

Now let $x \in (a, b)$, and apply the results derived above on $[a, x]$, $[x, b]$, obtaining $f(a) \leq f(x) \leq f(b)$. If $f(b) = f(a)$, f is constant on $[a, b]$, and so $Df = 0$ on $[a, b]$, contradicting $Df > 0$. Thus $f(b) > f(a)$. To complete the proof, note that since the result is true on any $[\alpha, \beta]$, $a \leq \alpha < \beta \leq b$, f is strictly increasing on $[a, b]$. □

Remarks :
• The proof also works if we only assume that the Dini derivates satisfy $D^+f > 0$, $D_+f > 0$, or $D_-f > 0$. To include the case where $D^-f > 0$ (the upper left Dini derivate), consider $c' = \inf\{x \in [a, b] : f(x) < \lambda\}$. Observe that $D^-f(c') \leq 0$.
• Countable Exception applies to many functions that Mean Value does not, e.g., continuous piecewise linear monotone functions on $[a, b]$ with increasing slopes on each segment. For example, on $[0, 1]$, start with $f(0) = 0$, $f(1) = 1$. Define $f(1/2) = (f(0) + 2f(1))/3$, $f(1/4) = (f(0) + 2f(1/2))/3$, $f(3/4) = (f(1/2) + 2f(1))/3$, and so forth. Connect the points.
• Discussions on the Mean Value Theorem can be found in the papers of Bers [3] and Cohen [9].
• On page 153 of Dieudonné's *Foundations of Modern Analysis* [10] there is a result similar to Theorem 2.3 in that it allows a countable set of exceptions, that is, a countable set of points at which Df is not necessarily positive.

We also get the following as a corollary, allowing us to get monotonicity from information on one of the Dini derivates.

Corollary 2.4. *If f is continuous on $[a, b]$ and one of its derivates is everywhere non-negative on (a, b), then f is non-decreasing on $[a, b]$.*

3. Between Countable and Measure Zero

A reading of Saks' classic treatise *Theory of the Integral* ([22], pp. 206–207) gives that Theorem 2.3 is generalized by the Goldowsky-Tonelli Theorem. (Also see Kannan and Krueger's *Advanced Analysis on the Real Line* [15], pp. 102–103.)

Theorem 3.1 (Goldowsky-Tonelli [5], [11], [15], [22], [25])**.** *Let f be a continuous function on $[a, b]$ and suppose that Df exists (finite or infinite) on $[a, b] \setminus S$, where S is countable. Also suppose that $Df \geq 0$ a.e. on $[a, b]$. Then f is a non-decreasing function on $[a, b]$.*

We show in this section that if the exceptional set S includes a Cantor set, then we can construct a continuous function f for which $Df > 0$ on $[a, b] \setminus S$, but for which $f(a) > f(b)$ (Theorem 3.2). Moreover, this occurs whenever the exceptional set is uncountable (Theorems 3.3 and 3.4). The key item is that whenever the exceptional set is uncountable, it contains a perfect set, and therefore contains a Cantor set. This in turn shows that Goldowsky-Tonelli is a vacuous extension of the Countable Exceptional Set Theorem, in that there are no additional functions to which Goldowsky-Tonelli applies but Countable Exception does not apply. Goldowsky-Tonelli, however, requires less information. Given a function satisfying the hypotheses of Goldowsky-Tonelli, one needs Goldowsky-Tonelli to show that it satisfies the hypotheses of Countable Exception.

There are generalizations of Goldowsky-Tonelli. By introducing generalizations of continuity and differentiability, a variety of theorems can be derived. Several of these can be found in Chapter 11 of Bruckner's monograph (see [5], pp. 120-128).

We note that if our exceptional set contains a Cantor set, then we can construct a continuous non-monotone function g such that $Dg > 0$ *a.e.* Simply compose the homeomorphism from the Cantor set onto the middle-thirds set with the function f given in Equation 1.1.

We show below that if the exceptional set S includes a Cantor set, then we can construct a continuous function f for which $Df > 0$ on $[a, b] \setminus S$, but for which $f(a) > f(b)$ (Theorem 3.2). Moreover, this occurs whenever the exceptional set is uncountable (Theorems 3.3 and 3.4). The key item is that whenever the exceptional set is uncountable, it contains a perfect set, and therefore contains a Cantor set. If the exceptional set contains a Cantor set, then we can construct a continuous non-monotone function g such that $Dg > 0$ *a.e.* Simply compose the homeomorphism from the Cantor set onto the middle-thirds set with the function f given in Equation 1.1.

Theorem 3.2. *If g is continuous on $[a, b]$ and the exceptional set S of points where the derivative Dg is not positive (or fails to exist) does not contain a perfect set, then g is non-decreasing. However, for any exceptional set that does contain a perfect set, there are counterexamples.*

Proof. Let g be a continuous function on $[a, b]$ with exceptional set S. Assume that $g(a) > g(b)$. We shall show that S contains a perfect set. Define the function h on $[a, b]$ by

$$h(x) = \sup\{g(t) : x \leq t \leq b\}\,.$$

For any x with derivative $Dg(x) > 0$, $g(x) < h(x)$, so such an x lies in one of the intervals where h is constant. Therefore, we have a closed set, which must be uncountable because h maps it onto $[g(a), g(b)]$. Since we have a bounded uncountable closed set, any such set contains a bounded perfect set, which in turn contains a Cantor set. Homeomorphically map this set onto the Cantor middle-thirds set. Composing with the function f of Equation 1.1 gives us the function we need. □

Remark :

- For any x with $Dg(x) > 0$, $g(x) < h(x)$, and so such an x lies in one of the intervals where h is constant. If we remove the interiors of all these intervals, we are performing a Cantor procedure, and so are left with a perfect set.

The class of *Borel sets* is the smallest family containing the intervals that is closed under the operations of countable union and countable intersection.

Theorem 3.3. *The set of points where a continuous function fails to have a positive derivative is a Borel set.*

Proof. The condition that the derivative $Dg(x)$ exist and be positive can be stated as follows. For any integer $n > 0$ there is an integer $m > 0$ and there are rational numbers $0 < r < s$ with $(s - r) < 1/n$ such that for any rational $t \neq 0$ with $|t| < 1/m$, $r < (g(x + t) - g(x))/t < s$. Thus, for any positive rational numbers $r < s$, and any rational $t \neq 0$, we let $U_{rst} = \{x : r < (g(x + t) - g(x))/t < s\}$. The set in question is

$$\bigcap\nolimits_n \bigcup\nolimits_m \bigcup\nolimits_{((s-r)<1/n)} \bigcap\nolimits_{(|t|<1/m)} U_{rst}\,. \tag{3.1}$$

□

Theorem 3.4 (Alexandroff [1], Hausdorff [13]). *Any uncountable Borel set contains a perfect set.*

Proof. See section 32 (pp. 203–206) of Hausdorff's *Set Theory* [12]. □

Thus, if the exceptional set is countable, the function is increasing. However, if it is uncountable, we can construct f such that $Df > 0$ *a.e.*, but for which $f(a) > f(b)$. There is no condition that we can place on S between countable and measure zero that guarantees monotonicity. This in turn shows there are no additional functions to which Goldowsky-Tonelli applies but Countable Exception does not apply. Goldowsky-Tonelli, however, requires less information. Given a function satisfying the hypotheses of Goldowsky-Tonelli, one needs Goldowsky-Tonelli to show that it satisfies the hypotheses of Countable Exception.

4. Further Generalizations

There are generalizations of Goldowsky-Tonelli. In particular, we cite the theorem of Zahorski (see [5], p. 120).

We say that a function has the *intermediate value property* on an interval $[a, b]$ if given any $x_1, x_2 \in [a, b]$ and y in between $f(x_1)$ and $f(x_2)$, there exists x_3 in between x_1 and x_2 such that $y = f(x_3)$. Functions having the intermediate value property are call *Darboux functions*.

Zahorski generalized Goldowsky-Tonelli in 1950.

Theorem 4.1 (Zahorski [5], p. 120). *Let f be a Darboux function on $[a, b]$ and suppose that the derivative Df exists (finite or infinite) on $[a, b] \setminus S$, where S is countable. Also suppose that $Df \geq 0$ a.e. on $[a, b]$. Then f is a non-decreasing continuous function on $[a, b]$.*

Bruckner shows that a monotonicity theorem that one can prove about continuous functions of bounded variation can be extended to Darboux-Baire Category 1 functions. Bruckner uses this analysis to prove that Zahorski's Theorem follows from Goldowsky-Tonelli (see [5], pp. 121-125). But then, Zahorski follows from Goldowsky-Tonelli which in turn follows from Countable Exception.

Tolstoff generalized Goldowsky-Tonelli in a different direction in 1939. This involved f being approximately continuous (see [5], pp. 15-31) and getting information from the approximate derivative Df_{ap}.

Theorem 4.2 (Tolstoff [5], p. 120). *Let f be an approximately continuous function on $[a, b]$ and suppose that the approximate derivative Df_{ap} exists (finite or infinite) on $[a, b] \setminus S$, where S is countable. Also suppose that $Df_{ap} \geq 0$ a.e. on $[a, b]$. Then f is a non-decreasing continuous function on $[a, b]$.*

Tolstoff and other generalizations can be reduced to Goldowsky-Tonelli (see [5], pp. 121-125). By introducing generalizations of continuity and differentiability, a variety of theorems can be derived. Many of the theorems in Chapter 11 of Bruckner's monograph have the condition that the derivative Df exists "except perhaps on a denumerable set" (see [5], pp. 120-128).

5. Monotone Functions

We close by asking the extent to which monotonicity implies positive derivative. The reader may also be interested in the paper of Katznelson and Stromberg [16]. We start a theorem of Lebesgue.

Theorem 5.1 (Lebesgue)**.** *A monotone function on an interval is differentiable except on a set of measure zero.*

There are different ways to prove this. A proof using the Vitali Covering Lemma may be found in Wheeden and Zygmund [26], pp. 111–113. Proofs based on F. Riesz's Rising Sun Lemma may be found in Riesz and Nagy [18], pp. 5–9, and Chae [8], pp. 162–170.

Theorem 5.2 (Riesz)**.** *There is a continuous strictly increasing function that has zero derivative almost everywhere.*

Proof. See section 24 (pp. 48–49) of Riesz and Nagy [18]. Start with $f(0) = 0$, $f(1) = 1$. Define $f(1/2) = (f(0)+2f(1))/3$, $f(1/4) = (f(0)+2f(1/2))/3$, $f(3/4) = (f(1/2) + 2f(1))/3$, and so forth, defining f at every dyadic fraction as a weighted sum. The function extends continuously to $[0, 1]$. You can picture this as a linear function which is successively distorted by raising the graph at the midpoints of successively smaller dyadic intervals. The derivative at any point, if it exists, is the limit of a product of factors, each of which is either $2/3$ or $4/3$. But the only number that can be represented as such a limit is 0. Since Lebesgue's theorem says the derivative must exist except on a set of measure 0, it must therefore be 0 except on a set of measure 0. □

If we require differentiability on the open interval (a, b), the situation is different. A differentiable function on (a, b) with zero derivative except on a set of measure 0 is constant. On the other hand, if we just want the set where the derivative is non-zero to be small, we get a very different result.

Theorem 5.3. *For any $0 < \epsilon < (b - a)$, there is a C^∞ function on $[a, b]$ that is strictly increasing, but has zero derivative except on a set of measure at most ϵ.*

Proof. Choose a sequence $\{a_n\}$ of positive numbers summing to ϵ. Now perform a "middle-third" construction, removing an open interval of length a_1 from the interior of $[a, b]$, then an interval from the interior of each of the two remaining intervals with lengths adding up to a_2, then intervals from the four remaining ones with lengths adding up to a_3, etc., being sure to choose the intervals we remove so that the remaining intervals have lengths that approach 0 (e.g., choose each interval to be removed so that the two remaining intervals have equal length). Now define a function f which is 0 on the remaining set, which clearly has measure $1 - \epsilon$, and which satisfies $f(x) = \exp(-1/(x-c)^2 - 1/(x-d)^2)$ on each removed interval (c, d). This function is C^∞. Because the lengths of the intervals "left behind" approach 0, f is not identically 0 on any interval within $[a, b]$. An indefinite integral of f is the function we seek. □

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