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NEW IDENTITIES FOR THE ARCTAN FUNCTION

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Abstract. We consider particular algebraic identities and by the use of the Beta function we derive an infinite set of binomial series for the arctangent function. Setting specific parameter values we obtain a number of new identities for π.

1. Introduction

In 1671 James Geogory, a Scottish mathematician discovered the arctangent series, see [19]

$$\arctan z=\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{2n+1};\qquad -1<z\leq 1, \tag{1.1}$$

and when $z=1$ we obtain the Leibniz series of 1674, hence the slow converging series for π. Using his transformation formula [1], in1755 Euler, [8], discovered

$$\arctan z=\sum_{n=0}^{\infty}\frac{4^n z^{2n+1}}{(1+z^2)^{n+1}(n+1)\binom{2n+1}{n}},\qquad z\in\mathbb{R}. \tag{1.2}$$

Euler's works on infinite series are extensive and the paper by Varadarajan, [20] dealing with this subject is worthwhile consulting for the interested reader. There are also some other works dealing with the expansion of the arctan function. Recently Chen, [7], obtained the nice identity

$$\left(\frac{\arctan z}{z}\right)^n=n!\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}\ p_k(n-1)}{2k+n} \tag{1.3}$$

where

$$\begin{aligned} p_k(0) &= 1 \text{ and}\\ p_k(n) &= \sum_{j=0}^{k}\frac{p_j(n-1)}{2j+n},\qquad \text{for } n=1,2,3..., \end{aligned}$$

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and when $n=1$, (1.3) reduces to (1.1). Also, in a similar vein, Adegoke and Layeni, [2] obtained the q^{th} derivative of the $\arctan z$ function, viz:

$$\begin{aligned} \frac{d^q}{dz^q}(\arctan z) &= \frac{(-1)^{q-1}(q-1)!}{(1+z^2)^{\frac{q}{2}}}\sin\left(q\arcsin\left(\frac{1}{\sqrt{1+z^2}}\right)\right), \\ \text{for } q &= 1,2,3..., \end{aligned} \tag{1.4}$$

however Lampret, [12] noted that (1.4) is incorrect due to some errors in the analysis. Choosing specific parameter values in (1.1), (1.2) and (1.3) we can obtain various series representations for π. In this paper we shall present two algebraic identities from which, through integration and the application of the classical Beta function, we obtain new series expansions of the $\arctan z$ function. By appropriate selection of the parameter z we obtain new representations of π. We remind the reader of the following notation which will be useful throughout this paper. The generalized hypergeometric representation ${}_pF_q\ [\cdot,\cdot]$, is defined as, see [13]

$${}_pF_q\left[\begin{array}{c} a_1,a_2,...,a_p \\ b_1,b_2,...,b_q \end{array}\middle| t\right] = \sum_{n=0}^{\infty}\frac{(a_1)_n\ (a_2)_n ... (a_p)_n\ t^n}{(b_1)_n\ (b_2)_n ... (b_q)_n\ n!} \tag{1.5}$$

where for $w\in\mathbb{C}\backslash\mathbb{Z}_0^-$, $(w)_n$ is Pochhammer's symbol defined by

$$(w)_n = \left\{\begin{array}{c} \frac{\Gamma(w+n)}{\Gamma(w)} = w(w+1)....(w+n-1),\ \text{for } n\in\mathbb{N} \\ 1,\ \text{for } n=0 \end{array}\right., \tag{1.6}$$

we also note that

$$\frac{(w)_n}{(w)_m} = \left\{\begin{array}{ll} (w+m)_{n-m}, & \text{for } n>m \\ \frac{1}{(w+n)_{m-n}}, & \text{for } n\le m \end{array}\right.. \tag{1.7}$$

Here $\mathbb{Z}_0^-$ denotes the set of non positive integers and the Gamma and Beta functions are defined respectively as

$$\Gamma(z) = \int_0^{\infty} w^{z-1}e^{-w}dw$$

and

$$B(s,z) = B(z,s) = \int_0^1 w^{s-1}(1-w)^{z-1}dw = \frac{\Gamma(s)\Gamma(z)}{\Gamma(s+z)}$$

forThe numbers p and q are zero or positive integers (interpreting an empty product as 1) and we assume, for simplicity, that the variable t, the numerator parameters $a_1,a_2,...,a_p$ and the denominator parameters $b_1,b_2,...,b_q$ take on complex values $\mathbb{C}$ provided that no zeros appear in the denominator of ${}_pF_q\ [\cdot,\cdot]$, that is $b_j\notin\mathbb{Z}_0^-$; $j=1,2,3,...,q$. Hence if a numerator parameter is zero or a negative integer then the hypergeometric series ${}_pF_q\ [\cdot,\cdot]$ terminates, since, see [18]

$$(-n)_j = \left\{\begin{array}{c} 0,\ j>n \\ \frac{(-1)^j\ n!}{(n-j)!},\ 0\le j\le n;\ n\in\mathbb{N} \end{array}\right..$$

The following Lemmas and Theorems are the main results presented in this paper.

2. THE MAIN RESULTS

The following Lemma will be useful in the proof of the main theorem.

Lemma 2.1. *Let* $m \in \mathbb{N} \cup \{0\}$, $\mathbb{N} := 1, 2, 3...,$ $x \in \mathbb{R}$ *and* $z \in \mathbb{R} \setminus \{-1\}$, *then*

$$(1+xz)\left[\sum_{i=0}^{m-1}(-xz)^i + \frac{(-xz)^m}{1+z}\right] = 1 + \frac{(-z)^{m+1}}{1+z}x^m(1-x). \tag{2.1}$$

Proof. Consider the left hand side of (2.1), then

$$\begin{aligned}(1+xz)\left[\sum_{i=0}^{m-1}(-xz)^i + \frac{(-xz)^m}{1+z}\right] &= (1+xz)\left[\frac{1-(-xz)^m}{1+xz} + \frac{(-xz)^m}{1+z}\right] \\ &= 1-(-xz)^m + \frac{(1+xz)(-xz)^m}{1+z}\end{aligned}$$

and the result follows. □

Theorem 2.2. *Let* $m \in \mathbb{N} \cup \{0\}$, $\mathbb{N} := 1, 2, 3...,$ $x \in \mathbb{R}$ *and* $z \in \mathbb{R}$, *then*

$$\begin{aligned}\frac{\arctan z}{z} = & \sum_{n=0}^{\infty}\frac{(-1)^{nm}\ z^{2n(m+1)}\ n!}{(1+z^2)^n}\left(\sum_{i=0}^{m-1}\frac{(-1)^i\ z^{2i}}{(2nm+2n+2i+1)\left(i+\frac{1}{2}+mn\right)_n}\right) \\ & + \sum_{n=0}^{\infty}\frac{(-1)^{n(m+1)}\ z^{2(nm+n+m)}\ n!}{(1+z^2)^{n+1}\ (2nm+2n+2m+1)\left(m+\frac{1}{2}+mn\right)_n}.\end{aligned} \tag{2.2}$$

Proof. From (2.1) and using Lemma 2.1, we replace the variables $x :\to x^2$, $z :\to z^2$ and rewrite as

$$\frac{1}{1+(xz)^2} = \frac{\sum_{i=0}^{m-1}(-1)^i(xz)^{2i} + \frac{(-1)^m(xz)^{2m}}{1+z^2}}{1+\frac{(-1)^{m+1}(z)^{2m+2}}{1+z^2}x^{2m}(1-x^2)}$$

now we expand the denominator as a geometric series, so that

$$\frac{1}{1+(xz)^2} = \sum_{n=0}^{\infty}\frac{(-1)^{n(m+2)}\ z^{2n(m+1)}\ x^{2mn}\left(1-x^2\right)^n}{(1+z^2)^n}\left[\begin{array}{c}\sum_{i=0}^{m-1}(-1)^i(xz)^{2i} \\ +\frac{(-1)^m(xz)^{2m}}{1+z^2}\end{array}\right].$$

Rearranging and integrating both sides with respect to $x \in [0,1]$ gives us

$$\int_0^1 \frac{dx}{1+(xz)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{nm} \; z^{2n(m+1)}}{(1+z^2)^n} \left[\begin{array}{c} \sum_{i=0}^{m-1} (-1)^i z^{2i} \int\limits_0^1 x^{2(nm+i)} \left(1-x^2\right)^n dx \\ \\ +\frac{(-1)^m z^{2m}}{1+z^2} \int\limits_0^1 x^{2m(n+1)} \left(1-x^2\right)^n dx \end{array} \right],$$

we notice the integrals can be represented as Beta functions, hence

$$\begin{aligned} \frac{\arctan z}{z} &= \sum_{n=0}^{\infty} \frac{(-1)^{nm} \; z^{2n(m+1)}}{(1+z^2)^n} \left[\begin{array}{c} \sum_{i=0}^{m-1} \frac{(-1)^i z^{2i}}{2} B\left(n+1, mn+i+\frac{1}{2}\right) \\ \\ +\frac{(-1)^m z^{2m}}{1+z^2} B\left(n+1, mn+m+\frac{1}{2}\right) \end{array} \right] \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{nm} \; z^{2n(m+1)}}{(1+z^2)^n} \left[\begin{array}{c} \sum_{i=0}^{m-1} \frac{(-1)^i z^{2i} \; n! \; \Gamma\left(mn+i+\frac{1}{2}\right)}{2 \; \Gamma\left(mn+n+i+\frac{3}{2}\right)} \\ \\ +\frac{(-1)^m z^{2m} \; n! \; \Gamma\left(mn+m+\frac{1}{2}\right)}{(1+z^2) \; \Gamma\left(mn+n+m+\frac{3}{2}\right)} \end{array} \right] \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{nm} \; z^{2n(m+1)} \; n!}{(1+z^2)^n} \left[\begin{array}{c} \sum_{i=0}^{m-1} \frac{(-1)^i z^{2i} \; \left(i+\frac{1}{2}\right)_{mn}}{(2i+1) \; \left(i+\frac{3}{2}\right)_{n(m+1)}} \\ \\ +\frac{(-1)^m z^{2m} \; \left(m+\frac{1}{2}\right)_{mn}}{(1+z^2) \; (2m+1) \; \left(m+\frac{3}{2}\right)_{n(m+1)}} \end{array} \right]. \end{aligned}$$

Using the relational properties (1.6) and (1.7) of the Pochhammer function we can display

$$\frac{\arctan z}{z} = \sum_{n=0}^{\infty} \frac{(-1)^{nm} \; z^{2n(m+1)} \; n!}{(1+z^2)^n} \left[\begin{array}{c} \sum_{i=0}^{m-1} \frac{(-1)^i \; z^{2i}}{(2nm+2n+2i+1) \; \left(i+\frac{1}{2}+mn\right)_n} \\ \\ +\frac{(-1)^m z^{2m}}{(1+z^2) \; (2nm+2n+2m+1) \; \left(m+\frac{1}{2}+mn\right)_n} \end{array} \right]$$

and the result (2.2) follows. We can also rewrite (2.2) in more familiar form

$$\frac{\arctan z}{z} = \sum_{i=0}^{m-1} 2\,(-1)^i \; z^{2i} \; \times$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{nm} \; z^{2n(m+1)} \; 4^n \binom{2nm+2i}{nm+i}}{(n+1)\,(1+z^2)^n \binom{2\,(nm+i+n+1)}{nm+i+n+1} \binom{nm+i+n+1}{nm+i}}$$

$$+\sum_{n=0}^{\infty}\frac{2\,(-1)^{n(m+1)}\;z^{2(nm+n+m)}\;4^{n}\binom{2nm+2m}{nm+i}}{(n+1)\,(1+z^2)^{n+1}\binom{(m+1)\,(n+1)}{m\,(n+1)}\binom{2\,(m+1)\,(n+1)}{(m+1)\,(n+1)}}.$$

□

Remark. *For special values of z the series (2.2) does not converge to π faster than the classical Ramanujan series or the BBP- type formulas,* [21]. *However (2.2) is another new interesting representation of the* $\arctan z$ *function.*

We now highlight some examples.

Example 1. For $z=1$, we have the general representation

$$\frac{\pi}{4}=\sum_{n=0}^{\infty}\frac{(-1)^{nm}\;n!}{2^n}\left[\begin{array}{c}\sum_{i=0}^{m-1}\frac{(-1)^i}{(2nm+2n+2i+1)\;\left(i+\frac{1}{2}+mn\right)_n}\\ \\ +\frac{(-1)^m}{2\;(2nm+2n+2m+1)\;\left(m+\frac{1}{2}+mn\right)_n}\end{array}\right]$$

Example 2. For $m=0$ the first term on the right hand side of (2.2) vanishes, therefore

$$\begin{aligned}\frac{\arctan z}{z} &= \sum_{n=0}^{\infty}\frac{z^{2n}\;n!\;\left(\frac{1}{2}\right)_0}{(1+z^2)^{n+1}\;\left(\frac{3}{2}\right)_n},\\ &= \frac{1}{1+z^2}\;{}_2F_1\left[\begin{array}{c}1,1\\ \frac{3}{2}\end{array}\middle|\,\frac{z^2}{1+z^2}\right]\end{aligned}$$

using the properties of the Pochhammer and Gamma functions we recover Euler's identity (1.2). Where ${}_pF_q\;[\cdot,\cdot]$ is the hypergeometric function as defined in (1.5) and follows by the consideration of

$$\frac{T_{n+1}}{T_n}\;\text{where}\;\;T_n=\frac{4^n z^{2n+1}}{(1+z^2)^{n+1}\,(n+1)\binom{2n+1}{n}}.$$

Example 3. For $m=1$

$$\begin{aligned}\frac{\arctan z}{z} &= \sum_{n=0}^{\infty}\frac{(-1)^n\;z^{4n}\;n!}{(1+z^2)^n}\left[\frac{1}{(4n+1)\left(n+\frac{1}{2}\right)_n}-\frac{z^2}{(1+z^2)\,(4n+3)\left(n+\frac{3}{2}\right)_n}\right]\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n\;z^{4n}\;n!}{(1+z^2)^n\;\left(n+\frac{1}{2}\right)_n}\left[\frac{1}{(4n+1)}-\frac{z^2\,(2n+1)}{(1+z^2)\,(4n+3)\,(4n+1)}\right]\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n\;z^{4n}\;4^n\;\left(2z^2\,(n+1)+4n+3\right)}{(1+z^2)^{n+1}\;(4n+1)\,(4n+3)\binom{4n}{2n}}\end{aligned}$$

$$= \frac{\left(2z^2+3\right)}{3\left(1+z^2\right)}\ {}_3F_2\left[\left.\begin{array}{c}\frac{1}{2},1,1\\ \frac{5}{4},\frac{7}{4}\end{array}\right| -\frac{z^4}{4\left(1+z^2\right)}\right]$$

$$-\frac{4z^4\left(z^2+2\right)}{105\left(1+z^2\right)^2}\ {}_3F_2\left[\left.\begin{array}{c}\frac{3}{2},2,2\\ \frac{9}{4},\frac{11}{4}\end{array}\right| -\frac{z^4}{4\left(1+z^2\right)}\right]$$

$$= \frac{\left(2z^2+3\right)}{3\left(1+z^2\right)}\ {}_4F_3\left[\left.\begin{array}{c}\frac{1}{2},1,1,\frac{4z^2+7}{2(z^2+2)}\\ \frac{5}{4},\frac{7}{4},\frac{2z^2+3}{2(z^2+2)}\end{array}\right| -\frac{z^4}{4\left(1+z^2\right)}\right]$$

and when $z=1$,

$$\frac{\pi}{4}=\sum_{n=0}^{\infty}\frac{(-1)^n\ 4^n\ (6n+5)}{2^{n+1}\ (4n+1)(4n+3)\binom{4n}{2n}}.$$

Example 4. For $m=2$

$$\frac{\arctan z}{z}=\sum_{n=0}^{\infty}\frac{z^{6n}\ n!}{\left(1+z^2\right)^n}\left[\begin{array}{c}\frac{1}{(6n+1)\left(2n+\frac{1}{2}\right)_n}-\frac{z^2}{(6n+3)\left(2n+\frac{3}{2}\right)_n}\\ \\ -\frac{z^4}{(1+z^2)(6n+5)\left(2n+\frac{5}{2}\right)_n}\end{array}\right]$$

$$=\sum_{n=0}^{\infty}\frac{2\ z^{6n}\ 4^{n+1}\binom{4n}{2n}A(n,z)}{\left(1+z^2\right)^{n+1}\ (n+3)(n+2)(n+1)\binom{6n+6}{3n+3}\binom{3n+3}{2n}}$$

where

$$A(n,z)=\left(1+z^2\right)(6n+5)(6n+3)-z^2\left(1+z^2\right)(4n+1)(6n+5)+z^4(4n+1)(4n+3).$$

For $z=1$

$$\frac{\pi}{4}=\sum_{n=0}^{\infty}\frac{4^{n+1}\binom{4n}{2n}\left(40n^2+60n+23\right)}{2^n\ (n+3)(n+2)(n+1)\binom{6n+6}{3n+3}\binom{3n+3}{2n}}$$

Example 5. In the case of $m=3$ and $z=\frac{1}{\sqrt{3}}$, from (2.2) we have, omitting the algebraic details,

$$\pi=\frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n\binom{6n}{3n}\left(1640n^3+3148n^2+1934n+381\right)}{3^{3n}\binom{8n}{4n}\binom{4n}{3n}(8n+7)(8n+5)(8n+3)(8n+1)}.$$

There are many interesting representations of π, see for example [3], [4], [5], [6], [9], [10], [11], [14], [15], [16], [17], and the references therein.

Remark. *From any of these five examples or from (2.2), it is possible to obtain the derivatives of the* $\arctan z$ *function. From example 2, for* $q = 1, 2, 3, ...$

$$\frac{d^q}{dz^q}(\arctan z) = \sum_{n=0}^{\infty} \frac{n!}{\left(\frac{3}{2}\right)_n} \frac{d^q}{dz^q} \left(\frac{z^{2n+1}}{(1+z^2)^{n+1}} \right)$$

$$= \sum_{n=0}^{\infty} \frac{n!}{\left(\frac{3}{2}\right)_n} \frac{d^q}{dz^q} \left(\sum_{r=0}^{\infty} (-1)^r \binom{r+n}{r} z^{2r+2n+1} \right)$$

$$= \sum_{n=0}^{\infty} \frac{n!}{\left(\frac{3}{2}\right)_n} \sum_{r=0}^{\infty} (-1)^r \binom{r+n}{r} q! \binom{2r+2n+1}{q} z^{2r+2n+1-q}$$

$$= \sum_{n=0}^{\infty} \frac{n!\ q! z^{2n+1-q}}{\left(\frac{3}{2}\right)_n} \binom{2n+1}{q} {}_3F_2 \left[\begin{matrix} n+1, n+1, n+\frac{3}{2} \\ n+1-\frac{q}{2}, n+\frac{3}{2}-\frac{q}{2} \end{matrix} \middle| -z^2 \right], \text{ for } |z| \le 1.$$

The next Lemma introduces a new generalized algebraic identity, similar to Lemma 2.1, and following the ideas used in Theorem 2.2 allows us to develop some new $\arctan z$ identities.

Lemma 2.3. *Let* $m \in \mathbb{N} \cup \{0\}$, $\mathbb{N} := 1, 2, 3...$, $x \in \mathbb{R}$ *and* $z \in \mathbb{R} \backslash \{-1\}$, *also let* $\{U_i\}_{i=0}^m$ *be a set of positive real numbers, then*

$$(1+xz) \left[\sum_{i=0}^{m} U_i\ z^i + \sum_{r=1}^{m} (-1)^r (xz)^r \sum_{i=0}^{m-r} U_i\ z^i \right] \tag{2.3}$$
$$= \sum_{i=0}^{m} U_i\ z^i + xz^{m+1} \sum_{r=0}^{m} (-1)^r U_{m-r}\ x^r$$

Proof. The proof follows the same idea as used in Lemma 2.1. Expand the left hand side of (2.3) collect terms, hence we are done. □

Theorem 2.4. *Let* $m \in \mathbb{N} \cup \{0\}$, $\mathbb{N} := 1, 2, 3...$, $x \in \mathbb{R}$ *and* $z \in \mathbb{R}$, *also let* $\{U_i\}_{i=0}^m$ *be a set of positive real numbers, then*

$$\frac{\arctan z}{z} = \sum_{n=0}^{\infty} (-1)^n \int_{x=0}^{1} \left(1 + \frac{\sum_{r=1}^{m} (-1)^r (xz)^{2r} \sum_{i=0}^{m-r} U_i\ z^{2i}}{\sum_{i=0}^{m} U_i\ z^{2i}} \right) \times$$
$$\left(\frac{x^{2n} z^{2n(m+1)} \left[\sum_{r=0}^{m} (-1)^r U_{m-r}\ x^{2r} \right]^n}{\left[\sum_{i=0}^{m} U_i\ z^{2i} \right]^n} \right) dx \tag{2.4}$$

Proof. Using Lemma 2.3, we replace the variables $x :\to x^2$, $z :\to z^2$ and rewrite as

$$\frac{1}{1+(xz)^2} = \frac{1 + \frac{\sum_{r=1}^{m} (-1)^r (xz)^{2r} \sum_{i=0}^{m-r} U_i\ z^{2i}}{\sum_{i=0}^{m} U_i\ z^{2i}}}{1 + \frac{x^2 z^{2(m+1)} \sum_{r=0}^{m} (-1)^r U_{m-r}\ x^{2r}}{\sum_{i=0}^{m} U_i\ z^{2i}}}$$

if we now expand the denominator as a geometric series and integrate both sides with respect to $x \in [0,1]$ we obtain the identity (2.4). $\square$

By appropriate choices of the set of real positive numbers $\{U_i\}_{i=0}^{m}$ it is possible to explicitly integrate (2.4) therefore producing a set of new identities for the $\arctan z$ function. Two examples are highlighted.

EXAMPLE 6. For $m=2$, put $U_0 = 1,\ U_1 = 2,\ U_2 = 1$, therefore

$$\frac{\arctan z}{z} = \int\limits_{x=0}^{1} \frac{1 - \frac{(xz)^2\left(1+2z^2\right)}{(1+\ z^2)^2} + \frac{(xz)^4}{(1+\ z^2)^2}}{1 + \frac{x^2 z^6\left(1-x^2\right)^2}{(1+\ z^2)^2}} dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{6n}}{(1+\ z^2)^{2n}} \int\limits_{x=0}^{1} x^{2n}\left(1-x^2\right)^{2n}\left[1 - \frac{(xz)^2\left(1+2z^2\right)}{(1+\ z^2)^2} + \frac{(xz)^4}{(1+\ z^2)^2}\right] dx,$$

the integrals, as in Theorem 2.2, are special cases of the Beta function and therefore may be evaluated to produce

$$\begin{aligned} \frac{\arctan z}{z} &= \sum_{n=0}^{\infty} \frac{(-1)^n\ n!\ z^{6n}}{(1+\ z^2)^{2n}} \left[\begin{array}{c} \frac{1}{\left(n+\frac{1}{2}\right)_{2n+1}} - \frac{z^2\left(1+2z^2\right)}{(1+\ z^2)^2\left(n+\frac{3}{2}\right)_{2n+1}} \\ + \frac{z^4}{(1+\ z^2)^2\left(n+\frac{5}{2}\right)_{2n+1}} \end{array}\right] \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n\ z^{6n} 4^{2n} \binom{2n}{n} Q(n,z)}{(1+\ z^2)^{2n+2}\, 3\,(6n+1)(6n+5)\binom{6n}{3n}\binom{3n}{n}} \end{aligned}$$

where $Q(n,z) = 8z^4(n+1) + 5z^2(6n+5) + 3(6n+1)$. In hypergeometric form

$$\begin{aligned} \frac{\arctan z}{z} &= \frac{z^4}{5(1+\ z^2)^2}\ {}_4F_3\left[\begin{array}{c} \frac{5}{2}, 1, 1, \frac{1}{2} \\ \frac{11}{6}, \frac{9}{6}, \frac{7}{6} \end{array}\middle| -\frac{4z^6}{27(1+\ z^2)^2}\right] \\ &\quad + \frac{z^4+5z^2+3}{3(1+\ z^2)^2}\ {}_3F_2\left[\begin{array}{c} 1, 1, \frac{1}{2} \\ \frac{7}{6}, \frac{5}{6} \end{array}\middle| -\frac{4z^6}{27(1+\ z^2)^2}\right] \end{aligned}$$

where $\left|\frac{4z^6}{27(1+\ z^2)^2}\right| \le 1$. When $z = \frac{1}{\sqrt{3}}$

$$\pi = \frac{1}{4\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}(130n+109)}{3^n\binom{6n}{3n}\binom{3n}{n}(6n+1)(6n+5)}.$$

EXAMPLE 7. For $m=3$, put $U_0 = 1,\ U_1 = U_2 = 3,\ U_3 = 1$, therefore

$$\frac{\arctan z}{z} = \int\limits_{x=0}^{1} \frac{1 - \frac{(xz)^2\left(1+3z^2+3z^4\right)}{(1+\ z^2)^3} + \frac{(xz)^4\left(1+3z^2\right)}{(1+\ z^2)^2} - \frac{(xz)^6}{(1+\ z^2)^3}}{1 + \frac{x^2 z^8\left(1-x^2\right)^3}{(1+\ z^2)^3}} dx.$$

Choosing $z=\frac{1}{\sqrt{3}}$, and omitting the algebraic details, we obtain the identity

$$\begin{aligned}\pi &= \frac{3\sqrt{3}}{8}\sum_{n=0}^{\infty}\frac{(-1)^n\binom{2n}{n}\left(2522n^3+4715n^2+2775n+508\right)}{3^n\binom{8n}{4n}\binom{4n}{n}(8n+1)(8n+3)(8n+5)(8n+7)}\\
&= a\ {}_5F_4\left[\left.\begin{matrix}1,1,\frac{2}{3},\frac{1}{2},\frac{1}{3}\\ \frac{15}{8},\frac{13}{8},\frac{11}{8},\frac{9}{8}\end{matrix}\right|-\frac{3^2}{2^{14}}\right]-b\ {}_5F_4\left[\left.\begin{matrix}2,2,\frac{5}{3},\frac{3}{2},\frac{4}{3}\\ \frac{23}{8},\frac{21}{8},\frac{19}{8},\frac{17}{8}\end{matrix}\right|-\frac{3^2}{2^{14}}\right]\\
&\quad +c\ {}_5F_4\left[\left.\begin{matrix}3,3,\frac{8}{3},\frac{5}{2},\frac{7}{3}\\ \frac{31}{8},\frac{29}{8},\frac{27}{8},\frac{25}{8}\end{matrix}\right|-\frac{3^2}{2^{14}}\right]-d\ {}_5F_4\left[\left.\begin{matrix}4,4,\frac{11}{3},\frac{7}{2},\frac{10}{3}\\ \frac{39}{8},\frac{37}{8},\frac{35}{8},\frac{33}{8}\end{matrix}\right|-\frac{3^2}{2^{14}}\right]\end{aligned}$$

where

$$\begin{aligned}b &= \frac{2503}{2^3\cdot 3^2\cdot 5^2\cdot 7\cdot 11\cdot 13\cdot\sqrt{3}},\ c=\frac{12281}{2^4\cdot 3^2\cdot 5\cdot 7^2\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot\sqrt{3}}\\
d &= \frac{97}{2^3\cdot 3^3\cdot 5^2\cdot 7\cdot 11\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot\sqrt{3}},\ a=\frac{127\sqrt{3}}{70}\end{aligned}$$

Remark. *We wish to point out that from the identity*

$$\frac{1}{1-xz} = \frac{\sum_{i=0}^{m-1}(xz)^i+\frac{(xz)^m}{1-z}}{1+\frac{z^{m+1}}{1-z}x^m(1-x)};\ x\neq 1,\ z\neq 1, \tag{2.5}$$

it may be possible to obtain further identities for the $\arctan z$ *function. Integrating (2.5) as follows*

$$\int_{x=0}^{1}\frac{x^{\alpha-1}(1-x)^{\beta-1}}{1-xz}dx = \int_{x=0}^{1}x^{\alpha-1}(1-x)^{\beta-1}\left(\frac{\sum_{i=0}^{m-1}(xz)^i+\frac{(xz)^m}{1-z}}{1+\frac{z^{m+1}}{1-z}x^m(1-x)}\right)dx \tag{2.6}$$

produces, on the left hand side, for $\mathbb{R}(\alpha)>0$ *and* $\mathbb{R}(\beta)>0$

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\sum_{n=0}^{\infty}\frac{(\alpha)_n\ z^n}{(\alpha+\beta)_n} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\ {}_2F_1\left[\left.\begin{matrix}1,\alpha\\ \alpha+\beta\end{matrix}\right|z\right].$$

Expanding the right hand side of (2.6) in the same way as in Theorem 2.2 allows us to obtain interesting identities of the form

$$\frac{\Gamma\left(\frac{\alpha}{2}+1\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}+\frac{1}{2}\right)} = \sum_{n=0}^{\infty}\frac{(-1)^n(\alpha)_n\left(1-\frac{\alpha}{2}\right)_n(6n+2+3\alpha)}{2^{n+1}\left(\frac{\alpha}{2}+1\right)_{2n+1}}.$$

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