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GENERAL EXTENSION RESULTS FOR ABSOLUTE SUMMABILITY

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Abstract. New general results concerning absolute summability of an infinite series are presented. Other special cases are also deduced.

1. Introduction

Let T be a lower triangular matrix, (S_n) a sequence of the nth partial sums of $\sum a_n$, and

$$T_n := \sum_{v=0}^{n} t_{nv} S_v. \tag{1}$$

A series $\sum a_n$ is said to be summable $\left|T, \delta\right|_k$, $\delta \geq 0$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{\delta k+k-1} \left|\Delta T_{n-1}\right|^k < \infty. \tag{2}$$

Summability $\left|T, \delta\right|_k$ reduces to summability $\left|T\right|_k$ whenever $\delta = 0$.

Given any lower triangular matrix T one can associate the matrices $\overline{T}$ and $\hat{T}$, with entries defined by

$$\bar{t}_{nv} = \sum_{i=v}^{n} t_{ni}, \quad n, i = 0, 1, 2, \ldots, \qquad \widehat{t}_{nv} = \bar{t}_{nv} - \bar{t}_{n-1,v}$$

respectively. With $s_n = \sum_{i=0}^{n} a_i \lambda_i$,

$$t_n = \sum_{v=0}^{n} t_{nv} s_v = \sum_{v=0}^{n} t_{nv} \sum_{i=0}^{v} a_i \lambda_i = \sum_{i=0}^{n} a_i \lambda_i \sum_{v=i}^{n} t_{nv} = \sum_{i=0}^{n} \widehat{t}_{ni} a_i \lambda_i \tag{3}$$

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$$\begin{aligned} Y_n \quad &: \quad = t_n - t_{n-1} = \sum_{i=0}^{n} \bar{t}_{ni} a_i \lambda_i - \sum_{i=0}^{n-1} \bar{t}_{n-1,i} a_i \lambda_i \\ &= \sum_{i=0}^{n} \widehat{t}_{ni} a_i \lambda_i \text{ as } \bar{t}_{n-1,n} = 0. \end{aligned} \tag{4}$$

$$X_n := u_n - u_{n-1} = \sum_{i=0}^{n} \widehat{u}_{ni} a_i \mu_i, \text{ where } u_n = \sum_{i=0}^{n} u_{ni} \mu_i a_i \tag{5}$$

We call T a triangle if T is lower triangular and $t_{nn} \neq 0$ for all n. A triangle A is called factorable if its nonzero entries a_{mn} can be written in the form $b_m c_n$ for each m and n. We also assume that $U = (u_{ij})$ is a triangle. (p_n), (q_n) are assumed to be positive sequences of numbers such that

$$P_n = p_0 + p_1 + \ldots + p_n \to \infty, \text{ as } n \to \infty \ ,$$

$$Q_n = q_0 + q_1 + \ldots + q_n \to \infty, \text{ as } n \to \infty \ .$$

BK-space is a sequence space endowed with a suitable norm to turn it into a Banach Space. All BK-spaces are normable FK-spaces.

The series $\sum a_n$ is said to be summable $\left|R, p_n, \delta\right|_k$, $k \geq 1$, $\delta \geq 0$, if

$$\sum_{n=1}^{\infty} n^{\delta k + k - 1} \left|\Delta z_{n-1}\right|^k < \infty \quad ,$$

where

$$z_n = \sum_{i=0}^{n} p_i s_i.$$

When $\delta = 0$, summability $\left|R, p_n, \delta\right|_k$ reduces to summability $\left|R, p_n\right|_k$.

In [4], Rhoades and Savas established a general summability factors theorem involving two lower triangular matrices A and B. In their results they are representing for the first time two arbitrary triangles and obtained sufficient (Necessary) conditions for the series $\sum a_n \lambda_n$ to be $|B|_k$ −summable whenever $\sum a_n$ is $|A|_k$ −summable. In fact they have proved the following two results (see [4]).

Theorem 1.1. *Let (λ_n) be a sequence of constants, A and B triangles satisfying*

$$(i) \quad \frac{|b_{nn}|}{|a_{nn}|} = O\left(\frac{1}{|\lambda_n|}\right),$$

$$(ii) \quad |a_{nn} - a_{n+1,n}| = O\left(|a_{nn}a_{n+1,n+1}|\right),$$

$$(iii) \quad \sum_{v=0}^{n-1} \left|\Delta_v \left(\widehat{b}_{nv}\lambda_v\right)\right| = O\left(|b_{nn}\lambda_n|\right),$$

$$(iv) \quad \sum_{n=v+1}^{\infty} \left(n\,|b_{nn}\lambda_n|\right)^{k-1} \left|\Delta_v \left(\widehat{b}_{nv}\lambda_v\right)\right| = O\left(v^{k-1}\,|b_{vv}\lambda_v|^k\right),$$

$$(v) \quad \sum_{v=0}^{n-1} |b_{vv}| \left|\widehat{b}_{n,v+1}\lambda_{v+1}\right| = O\left(|b_{nn}\lambda_{n+1}|\right),$$

$$(vi) \quad \sum_{n=v+1}^{\infty} \left(n\,|b_{nn}\lambda_{n+1}|\right)^{k-1} \left|\widehat{b}_{n,v+1}\right| = O\left(\left(v\,|b_{vv}\lambda_{v+1}|\right)^{k-1}\right),$$

$$(vii) \quad \sum_{v=1}^{\infty} v^{k-1}\,|\lambda_{v+1}X_v|^k = O(1),$$

$$(viii) \quad \sum_{n=1}^{\infty} n^{k-1} \left|\sum_{v=2}^{n} \widehat{b}_{nv}\lambda_v \sum_{i=0}^{v-2} \widehat{a}'_{vi}X_i\right|^k = O(1),$$

where $X_n = x_n - x_{n-1} = \sum_{v=0}^{n} \widehat{a}_{nv}a_v$, x_n *denotes the n-th term of the A-transform of the series* $\sum a_n$, *and* $a_n = \sum_{v=0}^{n} \widehat{a}'_{nv}X_v$. *Then the series* $\sum a_n\lambda_n$ *is summable* $|B|_k$ *whenever* $\sum a_n$ *is summable* $|A|_k$.

Theorem 1.2. *Let A and B be two lower triangular matrices with A satisfying*

$$(i) \quad \sum_{n=v+1}^{\infty} n^{k-1}\,|\Delta_v\widehat{a}_{nv}|^k = O(|a_{vv}|^k).$$

Then necessary conditions for the series $\sum a_n\lambda_n$ *to be summable* $|B|_k$ *whenever* $\sum a_n$ *is summable* $|A|_k$ *are*

$$(ii) \quad |b_{vv}\lambda_v| = O(|a_{vv}|),$$

$$(iii) \quad \left(\sum_{n=v+1}^{\infty} n^{k-1} \left|\Delta_v\widehat{b}_{nv}\lambda_v\right|^k\right)^{1/k} = O(v^{1-1/k}\,|a_{vv}|),$$

$$(iv) \quad \sum_{n=v+1}^{\infty} n^{k-1} \left|\widehat{b}_{n,v+1}\lambda_{v+1}\right|^k = O\Big(\sum_{n=v+1}^{\infty} n^{k-1}\,|\widehat{a}_{n,v+1}|^k\Big).$$

Our aim in this paper is to present new results concerning more general cases as well as via simpler conditions. We state and prove the following

2. Results

The coming two results are main result in this paper. The object of our results are to move in the same direction as theorems 1.1 and 1.2 but via factorable matrices which gives us easier ways to obtain the result

Theorem 2.1. *Let $1 < k \leq s < \infty$, $(\lambda_n), (\mu_n)$ be sequences of constants. Let T and U be triangles with bounded entries such that $\hat{U}$ is factorable, that is $\hat{u}_{nv}$ can be written as $\hat{u}_{nv} = \phi_n \varphi_v$, and they satisfy the following:*

$$(i) \quad t_{vv} = O\left(|\phi_v \varphi_v|\right),$$

$$(ii) \quad n^{\delta s - \gamma k + s - k} \left|X_n\right|^{s-k} = O(1),$$

$$(iii) \quad \sum_{v=1}^{n-1} \left|\Delta_v\left(\widehat{t}_{nv}\right)\right| = O\left(|t_{nn}|\right),$$

$$(iv) \quad \sum_{n=v+1}^{\infty} n^{\delta s + s - 1} \left|t_{nn}\right|^{s-1} \left|\Delta_v\left(\widehat{t}_{nv}\right)\right| = O\left(v^{\delta s + s - 1} \left|t_{vv}\right|^{s}\right),$$

$$(v) \quad \sum_{v=1}^{n-1} |t_{vv}| \left|\widehat{t}_{n,v+1}\right| = O\left(|t_{nn}|\right),$$

$$(vi) \quad \sum_{n=v+1}^{\infty} n^{\delta s + s - 1} \left|t_{nn}\right|^{s-1} \left|\widehat{t}_{n,v+1}\right| = O\left(v^{\delta s + s - 1} \left|t_{vv}\right|^{s-1}\right),$$

$$(vii) \quad \lambda_v = O(|\mu_v|),$$

$$(viii) \quad \Delta\left(\frac{\lambda_v}{\mu_v}\right) = O\left(|\phi_v| \left|\varphi_{v+1}\right|\right), \textit{and}$$

$$(ix) \quad \Delta\left(\varphi_v^{-1}\right) = O\left(|\phi_v|\right).$$

Then the series $\sum a_n \lambda_n$ is summable $\left|T, \delta\right|_s$ whenever $\sum a_n \mu_n$ is summable $\left|U, \gamma\right|_k$, $\delta \leq \gamma$.

Proof. By Abel's transformation we have

$$
\begin{aligned}
Y_n &= \sum_{v=1}^{n} \varphi_v a_v \mu_v \frac{\widehat{t}_{nv}\lambda_v}{\varphi_v \mu_v} \\
&= \sum_{v=1}^{n-1} \left(\sum_{r=1}^{v} \varphi_r a_r \mu_r \right) \Delta_v \left(\frac{\widehat{t}_{nv}\lambda_v}{\varphi_v \mu_v} \right) + \left(\sum_{v=1}^{n} \varphi_v a_v \mu_v \right) \frac{\widehat{t}_{nn}\lambda_n}{\varphi_n \mu_n} \\
&= \sum_{v=1}^{n-1} -\frac{X_v}{\phi_v} \left(\frac{\Delta_v \left(\widehat{t}_{nv}\right) \lambda_v}{\varphi_v \mu_v} + \widehat{t}_{n,v+1} \Delta \left(\varphi_v^{-1}\right) \frac{\lambda_v}{\mu_v} + \frac{\widehat{t}_{n,v+1}}{\varphi_{v+1}} \Delta \left(\frac{\lambda_v}{\mu_v} \right) \right) - \frac{X_n \widehat{t}_{nn} \lambda_n}{\phi_n \varphi_n \mu_n} \\
&= Y_{n1} + Y_{n2} + Y_{n3} + Y_{n4}.
\end{aligned}
$$

To complete the proof, by Minkowski's inequality, it is sufficient to show that

$$
\sum_{n=1}^{\infty} n^{\delta s+s-1} \left| Y_{nj} \right|^s < \infty, \qquad j = 1, 2, 3, 4.
$$

Now applying Hölder's inequality, we have

$$
\begin{aligned}
\sum_{n=1}^{\infty} n^{\delta s+s-1} \left| Y_{n1} \right|^s &= \sum_{n=1}^{\infty} n^{\delta s+s-1} \left| \sum_{v=1}^{n-1} \frac{\Delta_v \left(\widehat{t}_{nv}\right) \lambda_v}{\phi_v \varphi_v \mu_v} X_v \right|^s \\
&\le \sum_{n=1}^{\infty} n^{\delta s+s-1} \sum_{v=1}^{n-1} \frac{\left|\Delta_v \left(\widehat{t}_{nv}\right)\right| \left|X_v\right|^s \left|\lambda_v\right|^s}{\left|\phi_v\right|^s \left|\varphi_v\right|^s \left|\mu_v\right|^s} \left(\sum_{v=1}^{n-1} \left|\Delta_v \left(\widehat{t}_{nv}\right)\right| \right)^{s-1} \\
&= O(1) \sum_{n=1}^{\infty} n^{\delta s+s-1} \left|t_{nn}\right|^{s-1} \sum_{v=1}^{n-1} \frac{\left|\Delta_v \left(\widehat{t}_{nv}\right)\right| \left|X_v\right|^s \left|\lambda_v\right|^s}{\left|\phi_v\right|^s \left|\varphi_v\right|^s \left|\mu_v\right|^s} \\
&= O(1) \sum_{v=1}^{\infty} \frac{\left|X_v\right|^s \left|\lambda_v\right|^s}{\left|\phi_v\right|^s \left|\varphi_v\right|^s \left|\mu_v\right|^s} \sum_{n=v+1}^{\infty} n^{\delta s+s-1} \left|t_{nn}\right|^{s-1} \left|\Delta_v \left(\widehat{t}_{nv}\right)\right| \\
&= O(1) \sum_{v=1}^{\infty} \frac{v^{\delta s+s-1} \left|t_{vv}\right|^s \left|\lambda_v\right|^s \left|X_v\right|^s}{\left|\phi_v\right|^s \left|\varphi_v\right|^s \left|\mu_v\right|^s} \\
&= O(1) \sum_{v=1}^{\infty} v^{\delta s+s-s/k} \left|X_v\right|^s \\
&= O(1) \sum_{v=1}^{\infty} v^{\gamma k+k-1} \left|X_v\right|^k \left(v^{\delta s+s-s/k-k-\gamma k+1} \left|X_v\right|^{s-k} \right) \\
&= O(1) \sum_{v=1}^{\infty} v^{\gamma k+k-1} \left|X_v\right|^k = O(1).
\end{aligned}
$$

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{\delta s+s-1}\left|Y_{n2}\right|^{s} &= \sum_{n=1}^{\infty} n^{\delta s+s-1}\left|\sum_{v=1}^{n-1}\Delta\left(\varphi_{v}^{-1}\right)\frac{X_{v}\widehat{t}_{n,v+1}\lambda_{v}}{\phi_{v}\mu_{v}}\right|^{s} \\
&\leq \sum_{n=1}^{\infty} n^{\delta s+s-1}\sum_{v=1}^{n-1}\left|\Delta\left(\varphi_{v}^{-1}\right)\right|^{s}\frac{\left|\widehat{t}_{n,v+1}\right|\left|\lambda_{v}\right|^{s}\left|X_{v}\right|^{s}\left|t_{vv}\right|^{1-s}}{\left|\phi_{v}\right|^{s}\left|\mu_{v}\right|^{s}}\left(\sum_{v=0}^{n-1}\left|t_{vv}\right|\left|\widehat{t}_{n,v+1}\right|\right)^{s-1} \\
&= O(1)\sum_{n=1}^{\infty} n^{\delta s+s-1}\left|t_{nn}\right|^{s-1}\sum_{v=1}^{n-1}\left|\Delta\left(\varphi_{v}^{-1}\right)\right|^{s}\frac{\left|\lambda_{v}\right|^{s}\left|X_{v}\right|^{s}\left|t_{vv}\right|^{1-s}\left|\widehat{t}_{n,v+1}\right|}{\left|\phi_{v}\right|^{s}\left|\mu_{v}\right|^{s}} \\
&= O(1)\sum_{v=1}^{\infty}\left|\Delta\left(\varphi_{v}^{-1}\right)\right|^{s}\frac{\left|\lambda_{v}\right|^{s}\left|X_{v}\right|^{s}\left|t_{vv}\right|^{1-s}}{\left|\phi_{v}\right|^{s}\left|\mu_{v}\right|^{s}}\sum_{n=v+1}^{\infty} n^{\delta s+s-1}\left|t_{nn}\right|^{s-1}\left|\widehat{t}_{n,v+1}\right| \\
&= O(1)\sum_{v=1}^{\infty}\left|\Delta\left(\varphi_{v}^{-1}\right)\right|^{s}\frac{\left|\lambda_{v}\right|^{s}\left|X_{v}\right|^{s}\left|t_{vv}\right|^{1-s}}{\left|\phi_{v}\right|^{s}\left|\mu_{v}\right|^{s}}v^{\delta s+s-1}\left|t_{vv}\right|^{s-1} \\
&= O(1)\sum_{v=1}^{\infty} v^{\delta s+s-1}\left|X_{v}\right|^{s} \\
&= O(1)\sum_{v=1}^{\infty} v^{\gamma k+k-1}\left|X_{v}\right|^{k} v^{\delta s-\gamma k+s-k}\left|X_{v}\right|^{s-k} \\
&= O(1)\sum_{v=1}^{\infty} v^{\gamma k+k-1}\left|X_{v}\right|^{k} = O(1).
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{\delta s+s-1}\left|Y_{n3}\right|^{s} &= \sum_{n=1}^{\infty} n^{\delta s+s-1}\left|\sum_{v=1}^{n-1}\frac{\widehat{t}_{n,v+1}X_{v}}{\phi_{v}\varphi_{v+1}}\Delta\left(\frac{\lambda_{v}}{\mu_{v}}\right)\right|^{s} \\
&\leq \sum_{n=1}^{\infty} n^{\delta s+s-1}\sum_{v=1}^{n-1}\frac{\left|t_{vv}\right|^{1-s}\left|\widehat{t}_{n,v+1}\right|\left|X_{v}\right|^{s}}{\left|\phi_{v}\right|^{s}\left|\varphi_{v+1}\right|^{s}}\left|\Delta\left(\frac{\lambda_{v}}{\mu_{v}}\right)\right|^{s}\left(\sum_{v=1}^{n-1}\left|t_{vv}\right|\left|\widehat{t}_{n,v+1}\right|\right)^{s-1} \\
&= O(1)\sum_{n=1}^{\infty} n^{\delta s+s-1}\left|t_{nn}\right|^{s-1}\sum_{v=1}^{n-1}\frac{\left|t_{vv}\right|^{1-s}\left|\widehat{t}_{n,v+1}\right|\left|X_{v}\right|^{s}}{\left|\phi_{v}\right|^{s}\left|\varphi_{v+1}\right|^{s}}\left|\Delta\left(\frac{\lambda_{v}}{\mu_{v}}\right)\right|^{s} \\
&= O(1)\sum_{v=1}^{\infty}\frac{\left|t_{vv}\right|^{1-s}\left|X_{v}\right|^{s}}{\left|\phi_{v}\right|^{s}\left|\varphi_{v+1}\right|^{s}}\left|\Delta\left(\frac{\lambda_{v}}{\mu_{v}}\right)\right|^{s}\sum_{n=v+1}^{\infty} n^{\delta s+s-1}\left|t_{nn}\right|^{s-1}\left|\widehat{t}_{n,v+1}\right| \\
&= O(1)\sum_{v=1}^{\infty}\frac{v^{\delta s+s-1}\left|t_{vv}\right|^{1-s}\left|X_{v}\right|^{s}\left|t_{vv}\right|^{s-1}}{\left|\phi_{v}\right|^{s}\left|\varphi_{v+1}\right|^{s}}\left|\Delta\left(\frac{\lambda_{v}}{\mu_{v}}\right)\right|^{s} \\
&= O(1)\sum_{n=1}^{\infty} n^{\delta s+s-1}\left|X_{n}\right|^{s} \\
&= O(1)\sum_{v=1}^{\infty} v^{\gamma k+k-1}\left|X_{v}\right|^{k} = O(1).
\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\delta s+s-1}\left|Y_{n4}\right|^{s} &= \sum_{n=1}^{\infty} n^{\delta s+s-1}\left|\frac{\widehat{t}_{nn}\lambda_n X_n}{\phi_n\varphi_n\mu_n}\right|^{s} \\ &= O(1)\sum_{n=1}^{\infty} n^{\delta s+s-1}\frac{\left|\widehat{t}_{nn}\right|^{s}\left|\lambda_n\right|^{s}\left|X_n\right|^{s}}{\left|\phi_n\right|^{s}\left|\varphi_n\right|^{s}\left|\mu_n\right|^{s}} \\ &= O(1)\sum_{n=1}^{\infty} n^{\delta s+s-1}\left|X_n\right|^{s} \\ &= O(1)\sum_{n=1}^{\infty} v^{\gamma k+k-1}\left|X_v\right|^{k} = O(1).\end{aligned}$$

□

Theorem 2.2. *Let $1<k\le s<\infty$. Let T,U be lower triangular matrices such that U satisfying*

$$\sum_{n=v+1}^{\infty} n^{\gamma k+k-1}\left|\Delta_v\left(\widehat{u}_{nv}\mu_v\right)\right|^{k} = O\left(v^{\gamma k+k-1}\left|u_{vv}\right|^{k}\left|\mu_v\right|^{k}\right). \tag{6}$$

Then the necessary conditions for $\sum a_n\mu_n$ summable $|U,\gamma|_k$ to imply $\sum a_n\lambda_n$ is summable $|T,\delta|_s$ are

(*i*) $\left|t_{vv}\right|\left|\lambda_v\right| = O\left(v^{1/s-1/k}\left|u_{vv}\right|\left|\mu_v\right|\right)$,

(*ii*) $\left|\lambda_v\right|^{s}\sum_{n=v+1}^{\infty} n^{\delta s+s-1}\left|\Delta_v\widehat{t}_{nv}\right|^{s} = O\left(v^{\delta s+s-s/k}\left|u_{vv}\right|^{s}\left|\mu_v\right|^{s}\right)$,

(*iii*) $\left|\Delta\lambda_v\right|^{s}\sum_{n=v+1}^{\infty} n^{\delta s+s-1}\left|\widehat{t}_{n,v+1}\right|^{s} = O\left(v^{\delta s+s-s/k}\left|u_{vv}\right|^{s}\left|\mu_v\right|^{s}\right)$,

(*iv*) $\left|\lambda_{v+1}\right|^{s}\sum_{n=v+1}^{\infty} n^{\delta s+s-1}\left|\widehat{t}_{n,v+1}\right|^{s} = O\left(\sum_{n=v+1}^{\infty} n^{\gamma k+k-1}\left|\widehat{u}_{n,v+1}\right|^{k}\left|\mu_{v+1}\right|^{k}\right)^{s/k}$.

Proof. We are given that

$$\sum_{n=1}^{\infty} n^{\delta s+s-1}\left|Y_n\right|^{s}<\infty, \tag{7}$$

whenever

$$\sum_{n=1}^{\infty} n^{\gamma k+k-1}\left|X_n\right|^{k}<\infty. \tag{8}$$

The space of sequences (X_n) satisfying (2.3) is a Banach space if normed by

$$\|X\| = \left(\left|X_0\right|^{k}+\sum_{n=1}^{\infty} n^{\gamma k+k-1}\left|X_n\right|^{k}\right)^{1/k}, \tag{9}$$

and the space of sequences (Y_n) satisfying (2.2) BK-space with respect to the norm

$$\|Y\| = \left(\left|Y_0\right|^{s}+\sum_{n=1}^{\infty} n^{\delta s+s-1}\left|Y_n\right|^{s}\right)^{1/s}, \tag{10}$$

We observe that (1.4) and (1.5) transform the space of sequences satisfying (2.3) into the space of sequences satisfying (2.2) . By the Banach-Steinhaus theorem, there exists a constant $K > 0$ such that

$$\|Y\| \leq K \|X\| \tag{11}$$

Applying (1.4) and (1.5) to $a_v = \Delta e_v$, where e_v is the $v-$th coordinate vector, we have

$$X_n = \begin{cases} 0, & if \quad n < v \\ \widehat{u}_{nv}\mu_v, & if \quad n = v \\ \Delta_v\left(\widehat{u}_{nv}\mu_v\right), & if \quad n > v, \end{cases} \tag{12}$$

$$Y_n = \begin{cases} 0, & if \quad n < v \\ \widehat{t}_{nv}\lambda_v, & if \quad n = v \\ \Delta_v\left(\widehat{t}_{nv}\lambda_v\right), & if \quad n > v. \end{cases} \tag{13}$$

By (2.4) and (2.5) if follows that

$$\|X\| = \left(v^{\gamma k+kk-1}\left|u_{vv}\right|^k\left|\mu_v\right|^k + \sum_{n=v+1}^{\infty} n^{\gamma k+k-1}\left|\Delta_v\left(\widehat{u}_{nv}\mu_v\right)\right|^k\right)^{1/k}, \tag{14}$$

$$\|Y\| = \left(v^{\delta s+s-1}\left|t_{vv}\right|^s\left|\lambda_v\right|^s + \sum_{n=v+1}^{\infty} n^{\delta s+s-1}\left|\Delta_v\left(\widehat{t}_{nv}\lambda_v\right)\right|^s\right)^{1/s}. \tag{15}$$

Now, using (2.9) and (2.10) in (2.6), along with (2.1), we have

$$\begin{aligned} & v^{\delta s+s-1}\left|t_{vv}\right|^s\left|\lambda_v\right|^s + \sum_{n=v+1}^{\infty} n^{\delta s+s-1}\left|\Delta_v\left(\widehat{t}_{nv}\right)\lambda_v + \widehat{t}_{n,v+1}\Delta\lambda_v\right|^s \\ \leq \quad & K^s\left(v^{\gamma k+k-1}\left|u_{vv}\right|^k\left|\mu_v\right|^k + \sum_{n=v+1}^{\infty} n^{\gamma k+k-1}\left|\Delta_v\left(\widehat{u}_{nv}\mu_v\right)\right|^k\right)^{s/k} \\ = \quad & O\left(1\right)\left(v^{\gamma k+k-1}\left|u_{vv}\right|^k\left|\mu_v\right|^k + v^{\gamma k+k-1}\left|u_{vv}\right|^k\left|\mu_v\right|^k\right) \\ = \quad & O\left(v^{\gamma k+k-1}\left|u_{vv}\right|^k\left|\mu_v\right|^k\right)^{s/k} \\ = \quad & O\left(v^{\gamma s+s-s/k}\left|u_{vv}\right|^s\left|\mu_v\right|^s\right), \end{aligned} \tag{16}$$

by (2.1), inequality (2.11) is true iff each term of the L.H.S. is

$$O(v^{\gamma s+s-s/k}\left|u_{vv}\right|^s).$$

On taking the first term, we have

$$v^{\delta s+s-1}\left|t_{vv}\right|^s\left|\lambda_v\right|^s = O\left(v^{\gamma s+s-s/k}\left|u_{vv}\right|^s\left|\mu_v\right|^s\right),$$

which implies(i).

Concerning the second term, as λ_v and $\Delta\lambda_v$ are linearly independent, it follows that each of the terms

$$\sum_{n=v+1}^{\infty} n^{\delta s+s-1}\left|\Delta_v\left(\widehat{t}_{nv}\right)\lambda_v\right|^s, \quad \sum_{n=v+1}^{\infty} n^{\delta s+s-1}\left|\widehat{t}_{n,v+1}\Delta\lambda_v\right|^s$$

is $O(v^{\gamma s+s-s/k}\left|u_{vv}\right|^{s}\left|\mu_{v}\right|^{s})$, which implies (ii) and (iii). If we now apply (1.4) and (1.5) to $a_v = e_{v+1}$, we have

$$X_n = \begin{cases} 0, & if \quad n \le v \\ \widehat{u}_{n,v+1}\mu_{v+1}, & if \quad n > v, \end{cases} \tag{17}$$

$$Y_n = \begin{cases} 0, & if \quad n \le v \\ \widehat{t}_{n,v+1}\lambda_{v+1}, & if \quad n > v \end{cases}$$

The corresponding norms are

$$\|X\| = \left(\sum_{n=v+1}^{\infty} n^{k-1}\left|\widehat{u}_{n,v+1}\mu_{v+1}\right|^{k}\right)^{1/k}, \tag{18}$$

$$\|Y\| = \left(\sum_{n=v+1}^{\infty} n^{s-1}\left|\widehat{t}_{n,v+1}\lambda_{v+1}\right|^{s}\right)^{1/s}. \tag{19}$$

Applying (2.6) and (2.1), we obtain (iv). □

3. Applications

As an application to our result, we putting $\mu_n = 1$ (Corollaries 3.1 and 3.2) in order to get equivalent results to Theorem 1.1 and 1.2. While Corollary 3.3 dealing with special kinds of summability such as $\left|R, q_n, \delta\right|_k$ and $\left|R, p_n\right|_k$.

Corollary 3.1. *Let $1 < k \le s < \infty$, (λ_n) be a sequence of constants. Let T and U be triangles with bounded entries such that $\hat{U}$ is factorable, that is $\hat{u}_{nv}$ can be written as $\hat{u}_{nv} = \phi_n\varphi_v$, and they satisfy the following:*

(i) $t_{vv} = O\left(\left|\phi_v\varphi_v\right|\right),$

(ii) $n^{\gamma s-\gamma k+s-k}\left|X_n\right|^{s-k} = O(1),$

(iii) $\sum_{v=1}^{n-1}\left|\Delta_v\left(\widehat{t}_{nv}\right)\right| = O\left(\left|t_{nn}\right|\right),$

(iv) $\sum_{n=v+1}^{\infty} n^{\delta s+s-1}\left|t_{nn}\right|^{s-1}\left|\Delta_v\left(\widehat{t}_{nv}\right)\right| = O\left(v^{\delta s+s-1}\left|t_{vv}\right|^{s}\right),$

(v) $\sum_{v=1}^{n-1}\left|t_{vv}\right|\left|\widehat{t}_{n,v+1}\right| = O\left(\left|t_{nn}\right|\right),$

(vi) $\sum_{n=v+1}^{\infty} n^{\delta s+s-1}\left|t_{nn}\right|^{s-1}\left|\widehat{t}_{n,v+1}\right| = O\left(v^{\delta s+s-1}\left|t_{vv}\right|^{s-1}\right),$

(vii) $\lambda_v = O(1),$

$(viii)$ $\Delta\left(\lambda_v\right) = O\left(\left|\phi_v\right|\left|\varphi_{v+1}\right|\right)$, *and*

(ix) $\Delta\left(\varphi_v^{-1}\right) = O\left(\left|\phi_v\right|\right).$

Then the series $\sum a_n\lambda_n$ *is summable* $|T,\delta|_s$ *whenever* $\sum a_n$ *is summable* $|U,\gamma|_k$, $\delta\le\gamma$.

Proof. Follows from Theorem 2.1 by putting $\mu_n=1$. □

Corollary 3.2. *Let* $1<k\le s<\infty$. *Let* T,U *be lower triangular matrices such that* U *satisfying*

$$\sum_{n=v+1}^{\infty} n^{\gamma k+k-1}\left|\Delta_v\widehat{u}_{nv}\right|^k = O\left(v^{\gamma k+k-1}\left|u_{vv}\right|^k\right). \tag{20}$$

Then the necessary conditions for $\sum a_n\mu_v$ *summable* $|U,\gamma|_k$ *to imply* $\sum a_n\lambda_n$ *is summable* $|T,\delta|_s$ *are*

(i) $\quad |t_{vv}|\,|\lambda_v| = O\left(v^{1/s-1/k}\,|u_{vv}|\right)$,

(ii) $\quad |\lambda_v|^s\displaystyle\sum_{n=v+1}^{\infty} n^{\delta s+s-1}\left|\Delta_v\widehat{t}_{nv}\right|^s = O\left(v^{\delta s+s-s/k}\,|u_{vv}|^s\right)$,

(iii) $\quad |\Delta\lambda_v|^s\displaystyle\sum_{n=v+1}^{\infty} n^{\delta s+s-1}\left|\widehat{t}_{n,v+1}\right|^s = O\left(v^{\delta s+s-s/k}\,|u_{vv}|^s\right)$,

(iv) $\quad |\lambda_{v+1}|^s\displaystyle\sum_{n=v+1}^{\infty} n^{\delta s+s-1}\left|\widehat{t}_{n,v+1}\right|^s = O\left(\sum_{n=v+1}^{\infty} n^{\gamma k+k-1}\left|\widehat{u}_{n,v+1}\right|^k\right)^{s/k}$.

Proof. Follows from Theorem 2.2 by putting $\mu_n=1$. □

Corollary 3.3. *Sufficient conditions for the series* $\sum a_n\lambda_n$ *is summable* $|R,q_n,\delta|_k$, *whenever* $\sum a_n$ *is summable* $|R,p_n|_k$, $k\ge 1$, *are*

(i) $\quad \displaystyle\sum_{n=v+1}^{\infty}\frac{n^{\delta k+k-1}q_n^k}{Q_n^kQ_{n-1}} = O\left(\frac{v^{\delta k+k-1}q_v^{k-1}}{Q_v^k}\right)$,

(ii) $\quad q_nP_n = O(p_nQ_n)$,

(iii) $\quad \lambda_n = O(1)$,

(iv) $\quad \Delta\lambda_n = O\left(\dfrac{p_n}{P_{n-1}}\right)$.

Proof. The result can be obtained from Corollary 3.1 by putting $s=k$ as follows:

For a weighted matrices means, $U=(R,p_n)$, $T=(R,q_n)$, we have

$$\overline{u}_{nv} = \sum_{i=v}^{n}\frac{p_i}{P_n} = \frac{P_n-P_{v-1}}{P_n} = 1-\frac{P_{v-1}}{P_n},$$

$$\widehat{u}_{nv} = u_{n,v}-u_{n-1,v} = \frac{p_nP_{v-1}}{P_nP_{n-1}}.$$

We have to take

$$\phi_n = \frac{p_n}{P_nP_{n-1}},\ \varphi_v = P_{v-1}$$

and also we have

$$\widehat{t}_{nv} = \frac{q_n Q_{v-1}}{Q_n Q_{n-1}}.$$

The following steps shows that the conditions of Corollary 3.1 are all satisfied:

$$(i) \qquad \sum_{v=1}^{n-1} |t_{vv}| \left|\widehat{t}_{n,v+1}\right| = \sum_{v=1}^{n-1} \frac{q_v}{Q_v} \frac{q_n Q_v}{Q_n Q_{n-1}} = \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} q_v = \frac{q_n}{Q_n Q_{n-1}} (Q_n - q_0)$$

$$= O(\frac{q_n}{Q_n}) = O(|t_{nn}|).$$

$$(ii) \qquad \sum_{v=1}^{n-1} \left|\Delta_v \widehat{t}_{nv}\right| = \sum_{v=1}^{n-1} \frac{q_n q_v}{Q_n Q_{n-1}} = O(|t_{nn}|).$$

$$(iii) \qquad \sum_{n=v+1}^{\infty} n^{\delta k+k-1} |t_{nn}|^{k-1} \left|\Delta_v \widehat{t}_{nv}\right| = \sum_{n=v+1}^{\infty} n^{\delta k+k-1} \left(\frac{q_n}{Q_n}\right)^{k-1} \frac{q_n q_v}{Q_n Q_{n-1}} = q_v \sum_{n=v+1}^{\infty} \frac{n^{\delta k+k-1} q_n^k}{Q_n^k Q_{n-1}}$$

$$= q_v O\left(v^{\delta k+k-1} \frac{q_v^{k-1}}{Q_v^k}\right) = O\left(\frac{v^{\delta k+k-1} q_v^k}{Q_v^k}\right) = O\left(\left(v^{\delta k+k-1} |t_{vv}|\right)^k\right)$$

$$(iv) \qquad \sum_{n=v+1}^{\infty} n^{\delta k+k-1} |t_{nn}|^{k-1} \left|\widehat{t}_{n,v+1}\right| = \sum_{n=v+1}^{\infty} n^{\delta k+k-1} \left(\frac{q_n}{Q_n}\right)^{k-1} \frac{q_n q_v}{Q_n Q_{n-1}} = Q_v \sum_{n=v+1}^{\infty} \frac{n^{\delta k+k-1} q_n^k}{Q_n^k Q_{n-1}}$$

$$= O\left(v^{\delta k+k-1} \left(\frac{q_v}{Q_v}\right)^{k-1}\right) = O\left(v^{\delta k+k-1} |t_{vv}|^{k-1}\right).$$

$$(v) \qquad |\Delta\lambda_v|^k = O\left(\left(\frac{p_v}{P_{v-1}}\right)^k\right) = O\left(|\phi_v|^k |\varphi_{v+1}|^k\right).$$

$$(vi) \qquad t_{vv} = \frac{q_v}{Q_v} = O\left(\frac{p_v}{P_v}\right) = O\left(|\phi_v| |\varphi_v|\right).$$

$$(vii) \qquad \Delta\varphi_v^{-1} = \Delta\left(\frac{1}{P_{v-1}}\right) = \frac{p_v}{P_v P_{v-1}} = O\left(|\phi_v|\right).$$

□

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