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I-PRE-CAUCHY SEQUENCES AND ORLICZ FUNCTIONS

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Abstract. Let $x = (x_k)$ be a sequence and let M be a bounded Orlicz function. We prove that x is I-pre- Cauchy if and only if

$$I - \lim_k \frac{1}{k^2} \sum_{i,j\leq k} M\left(\frac{|x_i - x_j|}{\rho}\right) = 0.$$

This implies a theorem due to Connor,Fridy and Klin[4],Vakeel.A.Khan and Q.M.Danish Lohani[21].

1. INTRODUCTION

The concept of statistical convergence was first introduced by Fast[8] and also independently by Buck [1] and Schoenberg [16] for real and complex sequences.Further this concept was studied by Salat [14],Fridy[9],Connor[2]and many others. Statistical convergence is a generalization of the usual notation of convergence that parallels the usual theory of convergence.

A sequence $x = (x_k)$ is said to be Statistically convergent to L if for a given $\varepsilon > 0$

$$\lim_k \frac{1}{k}|\{i : |x_i - L| \geq \varepsilon, i \leq k\}| = 0,$$

and Statistically pre-Cauchy if

$$\lim_k \frac{1}{k^2}|\{(j, i) : |x_i - x_j| \geq \varepsilon, j, i \leq k\}| = 0.$$

Connor,Fridy and Klin[4] proved that Statistically convergent sequences are Statistically pre-Cauchy and any bounded Statistically pre-Cauchy sequence with a nowhere dense set of limit points is Statistically convergent. They also gave an example showing Statistically pre-Cauchy sequences are not necessarily Statistically convergent.

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An Orlicz function is a function M :[0,∞)→[0,∞), which is continuous, non-decreasing and convex with M(0)=0, M(x)>0 for x>0 and M(x)→ ∞ as x→ ∞. If convexity of M is replaced by subaddivity then this function is called a modulus function.(See Maddox[12]).

Lindenstrauss and Tzafriri[11] used the idea of Orlicz functions to construct the sequence space

$$\ell_M = \{x \in \omega : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) < \infty, \text{for some } \rho > 0\}$$

The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) \leq 1\}$$

The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with M(x)= x^p for $1 \leq p < \infty$.

An Orlicz function M is said to satisfy $\triangle_2$ condition for all values of x if there exists a constant $K > 0$ such that $M(Lx) \leq KLM(x)$ for all values of $L > 1$. The study of Orlicz sequence spaces have been made recently by various authors. ([6],[7],[13],[17],[22]).

In [4] Connor,Fridy and Klin proved that a bounded sequence $x = (x_k)$ is Statistically pre-Cauchy if and only if

$$\lim_k \frac{1}{k^2} \sum_{i,j \leq k} (|x_i - x_j|) = 0.$$

The notion of I-convergence is a generalization of the statistical convergence. At the initial stage it was studied by Kostyrko,Salat,Wilezynski[10]. Later on it was studied by Salat,Tripathy,Ziman[15] and Demirci[5].

Recently further it was studied by Tripathy and Hazarika[18,19,20].Here we give some preliminaries about the notion of I-convergence. Let X be a non empty set. Then a family of sets I⊆ 2^X(power set of X)is said to be an ideal if I is additive i.e A,B∈I ⇒A∪ B∈I and hereditary i.e A∈I, B⊆A⇒B∈I.

A non-empty family of sets $\pounds \subseteq 2^X$ is said to be filter on X if and only if $\phi \notin \pounds$,for A,B∈ $\pounds$ we have A∩B∈ $\pounds$ and for each A∈ $\pounds$ and A⊆B implies B∈ $\pounds$. An Ideal I⊆ 2^X is called non-trivial if I≠ 2^X.

A non-trivial ideal I⊆ 2^X is called admissible if $\{\{x\} : x \in X\}$ ⊆I.
A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal J≠I containing I as a subset.
For each ideal I, there is a filter $\pounds$(I) corresponding to I.
i.e $\pounds$(I) $= \{K \subseteq N : K^c \in I\}$,where $K^c = N \setminus K$.

2. MAIN RESULTS

In this article we establish the following criterion for arbitrary sequences to be I-pre-Cauchy.

Theorem 2.1. Let $x = (x_k)$ be a sequence and let M be Orlicz function then x is I-pre-Cauchy if and only if

$$I - \lim_k \frac{1}{k^2} \sum_{i,j \le k} M(\frac{|x_i - x_j|}{\rho}) = 0 \text{ for some } \rho > 0.$$

Proof: Suppose that

$$I - \lim_k \frac{1}{k^2} \sum_{i,j \le k} M(\frac{|x_i - x_j|}{\rho}) = 0 \quad \text{for some} \quad \rho > 0.$$

For each $\varepsilon > 0$,$\rho > 0$ and $k \in \mathbb{N}$ we have that

$$A_k = \{k \in \mathbb{N} : M(\frac{|x_i - x_j|}{\rho} \ge \frac{\varepsilon}{2k}, i, j \le k\} \in I, \ (1)$$

$$A_k^c = \{k \in \mathbb{N} : M(\frac{|x_i - x_j|}{\rho} < \frac{\varepsilon}{2k}, i, j \le k\} \in I. \ (2)$$

$$\lim_k \frac{1}{k^2} \sum_{i,j \le k} M(\frac{|x_i - x_j|}{\rho}) = \lim_k \frac{1}{k^2} \sum_{|x_i - x_j| < \frac{\varepsilon}{2k}} M(\frac{|x_i - x_j|}{\rho}) + \lim_k \frac{1}{k^2} \sum_{|x_i - x_j| \ge \frac{\varepsilon}{2k}} M(\frac{|x_i - x_j|}{\rho})$$

$$\ge \lim_k \frac{1}{k^2} \sum_{|x_i - x_j| \ge \frac{\varepsilon}{2k}} M(\frac{|x_i - x_j|}{\rho})$$

Now by (1) and (2) we have

$$\{k \in \mathbb{N} : \lim_k \frac{1}{k^2} \sum_{i,j \le k} M(\frac{|x_i - x_j|}{\rho}) \ge \varepsilon, i, j \le k\} \subset A_k \cup A_k^c \in I.$$

thus x is I-pre-Cauchy.

Now conversely suppose that x is I-pre-Cauchy, and that ε has been given. Then we have

$$\{k \in \mathbb{N} : \lim_k \frac{1}{k^2} \sum_{i,j \le k} M(\frac{|x_i - x_j|}{\rho}) \ge \varepsilon, i, j \le k\} \subset A_k \cup A_k^c \in I.$$

where

$$A_k = \{k \in \mathbb{N} : M(\frac{|x_i - x_j|}{\rho} \ge \frac{\varepsilon}{2k}, i, j \le k\} \in I,$$

$$A_k^c = \{k \in \mathbb{N} : M(\frac{|x_i - x_j|}{\rho} < \frac{\varepsilon}{2k}, i, j \le k\} \in I.$$

Let $\delta > 0$ be such that $M(\delta) < \frac{\varepsilon}{2}$. Since M is an Orlicz function there exists an integer B such that $M(x) < \frac{B}{2}$ for all $x \ge 0$.Note that for each $k \in \mathbb{N}$,

$$\lim_k \frac{1}{k^2} \sum_{i,j \le k} M(\frac{|x_i - x_j|}{\rho}) = \lim_k \frac{1}{k^2} \sum_{|x_i - x_j| < \frac{\varepsilon}{2k}} M(\frac{|x_i - x_j|}{\rho}) + \lim_k \frac{1}{k^2} \sum_{|x_i - x_j| \ge \frac{\varepsilon}{2k}} M(\frac{|x_i - x_j|}{\rho})$$

$$\le M(\delta) + \lim_k \frac{1}{k^2} \sum_{i,j \le k} M(\frac{|x_i - x_j|}{\rho})$$

$$\leq \frac{\varepsilon}{2} + \frac{B}{2}(\frac{1}{k^2}|\{(i,j) : |x_i - x_j| \geq \varepsilon, i,j \leq k\}|)$$

$$\leq \varepsilon + B(\frac{1}{k^2}|\{(i,j) : |x_i - x_j| \geq \varepsilon, i,j \leq k\}|). \ (3)$$

Since x is I-pre-Cauchy, there is an $\mathbb{N}$ such that the right hand side of (3) is less than ε for all $k \in \mathbb{N}$.Hence

$$I - \lim_k \frac{1}{k^2} \sum_{i,j \leq k} M(\frac{|x_i - x_j|}{\rho}) = 0$$

Theorem 2.2. Let $x = (x_k)$ be a sequence and let M be Orlicz function then x is I-convergent to L if and only if

$$I - \lim_k \frac{1}{k} \sum_{i=1}^{k} M(\frac{|x_i - L|}{\rho}) = 0 \text{ for some } \rho > 0.$$

Proof: Suppose that

$$I - \lim_k \frac{1}{k} \sum_{i=1}^{k} M(\frac{|x_i - L|}{\rho}) = 0 \text{ for some } \rho > 0,$$

with an Orlicz function M, then x is I-convergent to L.(See[4])

Conversely suppose that x is I-convergent to L.We can prove that in the similar manner to Theorem 2.1 that

$$I - \lim_k \frac{1}{k} \sum_{i=1}^{k} M(\frac{|x_i - L|}{\rho}) = 0 \text{ for some } \rho > 0.$$

using that M is an Orlicz function.

Corollary 2.3.A sequence $x = (x_k)$ is I-convergent if and only if

$$I - \lim_k \frac{1}{k^2} \sum_{i,j \leq k} |x_i - x_j| = 0$$

Proof: Let $M(x) = x$. Then

$$M(\frac{|x_i - x_j|}{\rho}) \leq |x_i - x_j| \text{ for all } i,j \leq k \in \mathbb{N},$$

$$B_k = \{k \in \mathbb{N} : M(\frac{|x_i - x_j|}{\rho} < \varepsilon, i,j \leq k\} \in I, \ (4)$$

and let

$$B_k^c = \{k \in \mathbb{N} : M(\frac{|x_i - x_j|}{\rho} \geq \varepsilon, i,j \leq k\} \in I. \ (5)$$

Therefore from (4) and (5) we have

$$\{k \in \mathbb{N} : M(\frac{|x_i - x_j|}{\rho} \geq \varepsilon, i,j \leq k\} \subset B_k \cup B_k^c \in I.$$

Hence

$$I - \lim_k \frac{1}{k^2} \sum_{i,j \leq k} |x_i - x_j| = 0 \text{ if and only if } I - \lim_k \frac{1}{k^2} \sum_{i,j \leq k} M(\frac{|x_i - x_j|}{\rho}) = 0$$

and an immediate application of Theorem 2.1 completes the proof.

Corollary 2.4. A sequence $x = (x_k)$ is I-convergent to L if and only if

$$I - \lim_k \frac{1}{k} \sum_{i=1}^{k} |x_i - L| = 0$$

Proof:Let $M(x) = x$.
We can prove in the similar manner as in the proof of Corollary 2.3.

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References

[1] Buck,R.C.:*Generalized Asymptotic Density,*Amer.J.Math.75:335-346(1953)
[2] Connor,J.S.:*The statistical and strong P-Cesaro convergence of sequences,*Analysis.8(1988),47-63.
[3] Connor,J.S.:*On strong matrix summability with respect to a modulus and statistical convergence,*Cnad.Math.Bull.32(1989),194-198.
[4] Connor,J.,Fridy,J.A. and Kline,J.*Statistically Pre-Cauchy sequence,*Analysis.14(1994),311-317.
[5] Demirci,K.:I-limit superior and limit inferior.*Math. Commun.,*6: 165-172(2001).
[6] Esi,A.and Et,M.Some new sequence spaces defined by a sequence of Orlicz functions.*Indian J. Pure Appl. Math.,*31(8): 967-972(2000).
[7] Et.M:On some new Orlicz sequence spaces,*J.Analysis.*9(2001),21-28.
[8] Fast,H.:*Sur la convergence statistique*,Colloq.Math.2(1951),241-244.
[9] Fridy,J.A.:*On statistical convergence*,Analysis.5(1985).301-313.
[10] Kostyrko,P.,Salat,T.,Wilczynski,W.I-convergence.:*Real Analysis Exchange,*26(2): 669-686(2000).
[11] Lindenstrauss,J.,Tzafriri,L. On Orlicz sequence spaces.*Israel J. Math.,*101: 379-390(1971).
[12] Maddox,I.J.*Elements of Functional Analysis*,Cambridge University Press.(1970)
[13] Parashar,S.D.,Choudhary,B. Sequence spaces defined by Orlicz function.*Indian J. Pure Appl. Math.,*25: 419-428(1994).
[14] Salat,T.:*On Statistically Convergent Sequences Of Real Numbers.*Math Slovaca.30: 139-150(1980)
[15] Salat,T.,Tripathy,B.C.,Ziman,M.: On some properties of I-convergence.*Tatra Mt. Math. Publ.,*28: 279-286(2004).
[16] Schoenberg,I.J.:The integrability of certain functions and related summability methods,*Amer.Math.Monthly.*66: 361-375(1959).
[17] Tripathy,B.C. and Mahantas.:On a class of sequences related to the l^p space defined by the Orlicz functions*Soochow.J.Math.*29(4): 379-391(2003).
[18] Tripathy,B.C. and Hazarika,B.:*Paranorm I-Convergent sequence spaces.*,Math Slovaca.59(4)485-494.(2009)
[19] Tripathy,B.C. and Hazarika,B.:*Some I-Convergent sequence spaces defined by Orlicz function.*,Acta Mathematica Applicatae Sinica.27(1)149-154.(2011)
[20] Tripathy,B.C. and Hazarika,B.:I-Monotonic and I-Convergent Sequences.Kyungpook Math.J.51. 233-239.(2011)
[21] Vakeel,A.K.,Danish Lohani,Q.M.: Statistically Pre-Cauchy sequences and Orlicz Functions.*Southeast Asian Bulletin of Mathematics.*31:,1107-1112,(2007).
[22] Yan,Y.:An interpolation inequality in Orlicz spaces.*Southeast Asian Bulletin of Mathematics.*28:,931-936,(2004).

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