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SOME NEW DOUBLE SEQUENCE SPACES IN n-NORMED SPACES DEFINED BY A SEQUENCE OF ORLICZ FUNCTION

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Abstract. In the present paper we introduce some new double sequence spaces in n-normed spaces defined by a sequence of Orlicz function $\mathcal{M} = (M_{k,l})$ and also examined some properties of the resulting sequence spaces.

1. Introduction and Preliminary

The initial works on double sequences is found in Bromwich [2]. Later on it was studied by Hardy [9], Moricz [12], Moricz and Rhoades [13], Tripathy ([22], [23]), Basarir and Sonalcan [1] and many others. Hardy[9] introduced the notion of regular convergence for double sequences. The concept of paranormed sequences was studied by Nakano [15] and Simmons [21] at the initial stage. The concept of 2-normed spaces was initially developed by Gähler [5] in the mid of 1960's while that of n-normed spaces one can see in Misiak [14]. Since, then many others have studied this concept and obtained various results, see Gunawan ([6], [7]) and Gunawan and Mashadi [8]. By the convergence of a double sequences we mean the convergence in the Pringsheim sense i.e. a double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $n \in N$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > n$, see [16]. We shall write more briefly as P-convergent. The double sequence $x = (x_{k,l})$ is bounded if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l. Let l''_∞ the space of all bounded double sequences such that $||x_{k,l}||_{\infty,2} = \sup_{k,l} |x_{k,l}| < \infty$.

The idea of difference sequence spaces were introduced by Kizmaz [10] and he defined the sequence spaces

$$X(\Delta) = \Big\{ x = (x_k) : (\Delta x_k) \in X \Big\}$$

for $X = l_\infty$, c or c_0, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$ for all $k \in \mathbb{N}$.
Later, these difference sequence spaces were generalized by Et and Colak see [3]. In [3] Et and Colak generalized the above sequence spaces to the sequence spaces as follows :

$$X(\Delta^m) = \Big\{ x = (x_k) : (\Delta^m x_k) \in X \Big\}$$

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for $X = l_\infty,\ c$ or c_0, where $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$,

$$\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}) \quad \text{for all} \ \ k \in \mathbb{N}.$$

The generalized difference has the following binomial representation,

$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \begin{pmatrix} m \\ v \end{pmatrix} x_{k+v}$$

for all $k \in \mathbb{N}$.

An orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. Lindenstrauss and Tzafriri [11] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \Big\{ x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty \Big\}$$

which is called as an Orlicz sequence space. Also ℓ_M is a Banach space with the norm

$$||x|| = \inf \Big\{ \rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1 \Big\}.$$

Also, it was shown in [11] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2- condition is equivalent to $M(Lx) \leq LM(x)$, for all L with $0 < L < 1$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t) dt$$

where η is known as the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

A double sequence space E is said to be solid if $(\alpha_{k,l} x_{k,l}) \in E$ whenever $(x_{k,l}) \in E$ and for all double sequences $(\alpha_{k,l})$ of scalars with $|\alpha_{k,l}| \leq 1$, for all $k, l \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

(1) $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;
(2) $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;
(3) $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \ \ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{K}$, and
(4) $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called a n-norm on X and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{K}$.

For example, we may take $X = \mathbb{R}^n$ being equipped with the n-norm $||x_1, x_2, \cdots, x_n||_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.
A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be Cauchy if

$$\lim_{k,p\to\infty} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.
Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called a paranorm, if

(1) $p(x) \geq 0$, for all $x \in X$;
(2) $p(-x) = p(x)$, for all $x \in X$;
(3) $p(x+y) \leq p(x) + p(y)$, for all $x, y \in X$;
(4) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda \ as \ n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0 \ as \ n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0 \ as \ n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [24], Theorem 10.4.2, P-183). For more details about sequence spaces see ([4], [17], [18], [19]).
Let $(X, ||\cdot, \cdots, \cdot||)$ be any n-normed space and let $S''(n-X)$ denote X-valued sequence spaces. Clearly $S''(n-X)$ is a linear space under addition and scalar multiplication.

Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz function and $(X, ||\cdot, \cdots, \cdot||)$ an n-normed space. Let $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be any sequence of strictly positive real numbers. In the present paper, we define the following classes of double sequences:
$l''(\mathcal{M}, \Delta^m, p, u, ||\cdot, \cdots, \cdot||) =$

$$\Big\{x \in S''(n-X) : \sum_{k,\, l=1}^{\infty,\, \infty} \Big[u_{k,l} M_{k,l}\Big(||\frac{\Delta^m x_{k,l}}{\rho}, z_1, z_2, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}} < \infty,$$

$$\text{for each } z_1, z_2, \cdots, z_{n-1} \in X \text{ and } \rho > 0\Big\}.$$

If we take $\mathcal{M}(x) = x$, we get

$l''(\Delta^m, p, u, ||\cdot, \cdots, \cdot||) =$

$$\Big\{x \in S''(n-X) : \sum_{k,\, l=1}^{\infty,\, \infty} \Big[u_{k,l}\Big(||\frac{\Delta^m x_{k,l}}{\rho}, z_1, z_2, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}} < \infty,$$

$$\text{for each } z_1, z_2, \cdots, z_{n-1} \in X \text{ and } \rho > 0\Big\}.$$

If we take $p = (p_{k,l}) = 1$, we have

$l''(\mathcal{M}, \Delta^m, u, ||\cdot, \cdots, \cdot||) =$

$$\Big\{x \in S''(n-X) : \sum_{k,\, l=1}^{\infty,\, \infty} \Big[u_{k,l} M_{k,l}\Big(||\frac{\Delta^m x_{k,l}}{\rho}, z_1, z_2, \cdots, z_{n-1}||\Big)\Big] < \infty,$$

$$\text{for each } \ z_1, z_2, \cdots, z_{n-1} \in X \ \text{ and } \ \rho > 0\Big\}.$$

If we take $m = 0$, $u = u_{k,l} = 1$ and $M_{k,l} = M$, we get the spaces which were defined and studied by E. Savas [20]. The work of this paper is motivated by the work of E. Savas [20], M. Basarir and O. Sonalcan [1], B. C. Tripathy [22], A. Esi [4] and H. Gunawan and M. Mashadi [8].

The following inequality will be used throughout the paper. Let $p = (p_{k,l})$ be a double sequence of positive real numbers with $0 < p_{k,l} \le \sup_{k,l} = H$ and let $K = \max\{1, 2^{H-1}\}$. Then for the factorable sequences $\{a_{k,l}\}$ and $\{b_{k,l}\}$ in the complex plane, we have

$$|a_{k,l} + b_{k,l}|^{p_{k,l}} \le K(|a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}}). \tag{1.1}$$

The aim of this paper is to introduce some new double sequence spaces in n-normed spaces defined by a sequence of Orlicz function $\mathcal{M} = (M_{k,l})$ and to establish some topological properties and some inclusion relation between above defined sequence spaces.

2. Main Results

Theorem 2.1. *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz function, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be any sequence of strictly positive real numbers. Then $l''(\mathcal{M}, \Delta^m, p, u, ||\cdot, \cdots, \cdot||)$ is a linear space over the field of complex number $\mathbb{C}$.*

Proof. Let $x = (x_{k,l}),\ y = (y_{k,l}) \in l''(\mathcal{M}, \Delta^m, p, u, ||\cdot, \cdots, \cdot||)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\sum_{k,\, l=1}^{\infty,\, \infty} \Big[u_{k,l} M_{k,l}\Big(||\frac{\Delta^m x_{k,l}}{\rho_1}, z_1, z_2, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}} < \infty, \ \text{ for some } \ \rho_1 > 0,$$

$$\sum_{k,\, l=1}^{\infty,\, \infty} \Big[u_{k,l} M_{k,l}\Big(||\frac{\Delta^m y_{k,l}}{\rho_2}, z_1, z_2, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}} < \infty, \ \text{ for some } \ \rho_2 > 0$$

Define $\rho_3 = \max(|\alpha|\rho_1, |\beta|\rho_2)$. Since $||\cdot, \cdots, \cdot||$ is a n-norm on X and $\mathcal{M} = (M_{k,l})$ is a sequence of Orlicz function, we get

$$\sum_{k,\,l=1}^{\infty,\,\infty}\Big[u_{k,l}M_{k,l}\Big(||\frac{\Delta^m(\alpha x_{k,l}+\beta y_{k,l})}{\rho_3},z_1,z_2,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}$$

$$=\sum_{k,\,l=1}^{\infty,\,\infty}\Big[u_{k,l}M_{k,l}\Big(||\frac{\Delta^m(\alpha x_{k,l}+\beta y_{k,l})}{\max(|\alpha|\rho_1,|\beta|\rho_2)},z_1,z_2,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}$$

$$\leq K\sum_{k,\,l=1}^{\infty,\,\infty}\Big[\frac{|\alpha|}{(|\alpha|\rho_1+|\beta|\rho_2)}u_{k,l}M_{k,l}\Big(||\frac{\Delta^m(\Delta^m\alpha x_{k,l}+\beta y_{k,l})}{\rho_1},z_1,z_2,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}$$

$$+K\sum_{k,\,l=1}^{\infty,\,\infty}\Big[\frac{|\beta|}{(|\alpha|\rho_1+|\beta|\rho_2)}u_{k,l}M_{k,l}\Big(||\frac{\Delta^m(\alpha x_{k,l}+\beta y_{k,l})}{\rho_2},z_1,z_2,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}$$

$$\leq KF\sum_{k,\,l=1}^{\infty,\,\infty}\Big[u_{k,l}M_{k,l}\Big(||\frac{\Delta^m x_{k,l}}{\rho_1},z_1,z_2,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}$$

$$+KF\sum_{k,\,l=1}^{\infty,\,\infty}\Big[u_{k,l}M_{k,l}\Big(||\frac{\Delta^m y_{k,l}}{\rho_2},z_1,z_2,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}},$$

where

$$F=\max\Big[1,\Big(\frac{|\alpha|}{(|\alpha|\rho_1+|\beta|\rho_2)}\Big)^H,\Big(\frac{|\beta|}{(|\alpha|\rho_1+|\beta|\rho_2)}\Big)^H\Big],$$

and this completes the proof of the theorem. □

Theorem 2.2. *Let $\mathcal{M}=(M_{k,l})$ be a sequence of Orlicz function, $p=(p_{k,l})$ be a bounded sequence of positive real numbers and $u=(u_{k,l})$ be any sequence of strictly positive real numbers. Then $l''(\mathcal{M},\Delta^m,p,u,||\cdot,\cdots,\cdot||)$ is a paranormed space with the paranorm defined by*

$$g(x)=\inf\Big\{\rho^{\frac{p_{k,l}}{H}}:\Big(\sum_{k,\,l=1}^{\infty,\,\infty}\Big[u_{k,l}M_{k,l}\Big(||\frac{\Delta^m x_{k,l}}{\rho},z_1,z_2,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{G}}<\infty\Big\},$$

where $0<p_{k,l}\leq\sup p_{k,l}=H$, $G=\max(1,H)$.

Proof. Clearly, $g(\theta)=0$, where $\theta=(0,0,\cdots,0)$ is the zero sequence and $g(-x)=g(x)$. Let $x=(x_{k,l}),\ y=(y_{k,l})\in l''(\mathcal{M},\Delta^m,p,u,||\cdot,\cdots,\cdot||)$ and $\alpha,\beta\in\mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\sum_{k,\,l=1}^{\infty,\,\infty}\Big[u_{k,l}M_{k,l}\Big(||\frac{\Delta^m x_{k,l}}{\rho_1},z_1,z_2,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}<\infty,\ \text{ for some }\ \rho_1>0$$

and

$$\sum_{k,\,l=1}^{\infty,\,\infty}\Big[u_{k,l}M_{k,l}\Big(||\frac{\Delta^m y_{k,l}}{\rho_2},z_1,z_2,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}<\infty,\ \text{ for some }\ \rho_2>0$$

Let $\rho=\rho_1+\rho_2$, we have

$$
\begin{aligned}
&u_{k,l}M_{k,l}\Big(||\tfrac{\Delta^m(x_{k,l}+y_{k,l})}{\rho}, z_1, z_2, \cdots z_{n-1}||\Big)\\
&\quad = u_{k,l}M_{k,l}\Big(||\frac{\Delta^m(x_{k,l}+y_{k,l})}{\rho_1+\rho_2}, z_1, z_2, \cdots z_{n-1}||\Big)\\
&\quad \leq u_{k,l}M_{k,l}\Big(||\frac{\Delta^m x_{k,l}}{\rho_1+\rho_2}, z_1, z_2, \cdots z_{n-1}|| + ||\frac{\Delta^m y_{k,l}}{\rho_1+\rho_2}, z_1, z_2, \cdots z_{n-1}||\Big)\\
&\quad \leq \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)u_{k,l}M_{k,l}\Big(||\frac{\Delta^m x_{k,l}}{\rho_1}, z_1, z_2, \cdots z_{n-1}||\Big)\\
&\quad + \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)u_{k,l}M_{k,l}\Big(||\frac{\Delta^m y_{k,l}}{\rho_2}, z_1, z_2, \cdots z_{n-1}||\Big),
\end{aligned}
$$

and thus
$g(x+y)$

$$
\begin{aligned}
&= \inf\Big\{(\rho_1+\rho_2)^{\frac{p_{k,l}}{H}} : \Big(\sum_{k,\, l=1}^{\infty,\, \infty}\Big[u_{k,l}M_{k,l}\Big(||\frac{\Delta^m(x_{k,l}+y_{k,l})}{\rho_1+\rho_2}, z_1, z_2, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{G}}\Big\}\\
&\leq \inf\Big\{(\rho_1)^{\frac{p_{k,l}}{H}} : \Big(\sum_{k,\, l=1}^{\infty,\, \infty}\Big[u_{k,l}M_{k,l}\Big(||\frac{\Delta^m x_{k,l}}{\rho_1}, z_1, z_2, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{G}}\Big\}\\
&+ \inf\Big\{(\rho_2)^{\frac{p_{k,l}}{H}} : \Big(\sum_{k,\, l=1}^{\infty,\, \infty}\Big[u_{k,l}M_{k,l}\Big(||\frac{\Delta^m y_{k,l}}{\rho_2}, z_1, z_2, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{G}}\Big\}.
\end{aligned}
$$

Now, let $\lambda \to 0$ and $g(x^n - x) \to 0$ as $n \to \infty$. Since

$$
g(\lambda x) = \inf\Big\{(\rho)^{\frac{p_{k,l}}{H}} : \Big(\sum_{k,\, l=1}^{\infty,\, \infty}\Big[u_{k,l}M_{k,l}\Big(||\frac{\lambda\Delta^m x_{k,l}}{\rho}, z_1, z_2, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{G}} < \infty\Big\}.
$$

This gives us $g(\lambda x^n) \to 0$ as $n \to \infty$. □

Theorem 2.3. *If $0 < p_{k,l} < q_{k,l} < \infty$ for each k and l, then*

$$
l''(\mathcal{M}, \Delta^m, p, u, ||\cdot, \cdots, \cdot||) \subseteq l''(\mathcal{M}, \Delta^m, q, u, ||\cdot, \cdots, \cdot||).
$$

Proof. If $x \in l''(\mathcal{M}, \Delta^m, p, u, ||\cdot, \cdots, \cdot||)$, then there exists some $\rho > 0$ such that

$$
\sum_{k,\, l=1}^{\infty,\, \infty}\Big[u_{k,l}M_{k,l}\Big(||\frac{\Delta^m x_{k,l}}{\rho}, z_1, z_2, \cdots z_{n-1}||\Big)\Big]^{p_{k,l}} < \infty.
$$

This implies that

$$
u_{k,l}M_{k,l}\Big(||\frac{\Delta^m x_{k,l}}{\rho}, z_1, z_2, \cdots, z_{n-1}||\Big) < 1,
$$

for sufficiently large value of k and l. Since $\mathcal{M} = (M_{k,l})$ is non-decreasing, we have

$$
\begin{aligned}
&\sum_{k,l=1}^{\infty,\, \infty}\Big[u_{k,l}M_{k,l}\Big(||\frac{\Delta^m x_{k,l}}{\rho}, z_1, z_1, \cdots, z_{n-1}||\Big)\Big]^{q_{k,l}}\\
&\quad \leq \sum_{k,l=1}^{\infty,\, \infty}\Big[u_{k,l}M_{k,l}\Big(||\frac{\Delta^m x_{k,l}}{\rho}, z_1, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}\\
&\quad < \infty.
\end{aligned}
$$

Thus $x \in l''(\mathcal{M}, \Delta^m, q, u, ||\cdot, \cdots, \cdot||)$. This completes the proof. □

Theorem 2.4. *(i) If $0 < p_{k,l} < 1$ for each k and l, then*

$$l''(\mathcal{M}, \Delta^m, p, u, ||\cdot, \cdots, \cdot||) \subseteq l''(\mathcal{M}, \Delta^m, u, ||\cdot, \cdots, \cdot||),$$

(ii) If $p_{k,l} \geq 1$ for each k and l, then

$$l''(\mathcal{M}, \Delta^m, u, ||\cdot, \cdots, \cdot||) \subseteq l''(\mathcal{M}, \Delta^m, p, u, ||\cdot, \cdots, \cdot||).$$

Proof. (i) Let $x = (x_{k,l}) \in l''(\mathcal{M}, \Delta^m, p, u, ||\cdot, \cdots, \cdot||)$. Since $0 < \inf p_{k,l} < 1$, we have

$$\sum_{k,l=1}^{\infty,\infty} \Big[u_{k,l} M_{k,l}\Big(||\frac{\Delta^m x_{k,l}}{\rho}, z_1, z_1, \cdots, z_{n-1}||\Big)\Big]$$
$$\leq \sum_{k,l=1}^{\infty,\infty} \Big[u_{k,l} M_{k,l}\Big(||\frac{\Delta^m x_{k,l}}{\rho}, z_1, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}$$

and hence $x = (x_{k,l}) \in l''(\mathcal{M}, \Delta^m, p, u, ||\cdot, \cdots, \cdot||)$.

(ii) Let $p_{k,l}$ for each (k, l) and $\sup_{k,l} p_{k,l} < \infty$. Let $x = (x_{k,l}) \in l''(\mathcal{M}, \Delta^m, u, ||\cdot, \cdots, \cdot||)$. Then, for each $0 < \epsilon < 1$, there exists a positive integer $\mathbb{N}$ such that

$$\sum_{k,l=1}^{\infty,\infty} \Big[u_{k,l} M_{k,l}\Big(||\frac{\Delta^m x_{k,l}}{\rho}, z_1, z_1, \cdots, z_{n-1}||\Big)\Big] \leq \epsilon < 1,$$

for all $k, l \in \mathbb{N}$. This implies that

$$\sum_{k,l=1}^{\infty,\infty} \Big[u_{k,l} M_{k,l}\Big(||\frac{\Delta^m x_{k,l}}{\rho}, z_1, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}$$
$$\leq \sum_{k,l=1}^{\infty,\infty} \Big[u_{k,l} M_{k,l}\Big(||\frac{\Delta^m x_{k,l}}{\rho}, z_1, z_1, \cdots, z_{n-1}||\Big)\Big].$$

Thus $x = (x_{k,l}) \in l''(\mathcal{M}, \Delta^m, p, u, ||\cdot, \cdots, \cdot||)$ and this completes the proof. □

Theorem 2.5. *Let $\mathcal{M}' = (M'_{k,l})$ and $\mathcal{M}'' = (M''_{k,l})$ be sequences of Orlicz function. Then*

$$l''(\mathcal{M}', \Delta^m, p, u, ||\cdot, \cdots, \cdot||) \cap l''(\mathcal{M}'', \Delta^m, p, u, ||\cdot, \cdots, \cdot||) \subseteq l''(\mathcal{M}' + \mathcal{M}'', \Delta^m, p, u, ||\cdot, \cdots, \cdot||).$$

Proof. We have

$$\Big[u_{k,l}(M'_{k,l} + M''_{k,l})\Big(||\tfrac{\Delta^m x_{k,l}}{\rho}, z_1, z_2, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}$$
$$= \Big[u_{k,l} M'_{k,l}\Big(||\frac{\Delta^m x_{k,l}}{\rho}, z_1, z_2, \cdots, z_{n-1}||\Big) + u_{k,l} M''_{k,l}\Big(||\frac{\Delta^m x_{k,l}}{\rho}, z_1, z_2, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}$$
$$\leq K\Big[u_{k,l} M'_{k,l}\Big(||\frac{\Delta^m x_{k,l}}{\rho}, z_1, z_2, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}$$
$$+ K\Big[u_{k,l} M''_{k,l}\Big(||\frac{\Delta^m x_{k,l}}{\rho}, z_1, z_2, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}.$$

Let $x = (x_{k,l}) \in l''(\mathcal{M}', \Delta^m, p, u, ||\cdot, \cdots, \cdot||) \cap l''(\mathcal{M}'', \Delta^m, p, u, ||\cdot, \cdots, \cdot||)$, when adding the above inequality from $k,\ l = 0, 0$ to ∞, ∞ we get $x = (x_{k,l}) \in l''(\mathcal{M}' + \mathcal{M}'', \Delta^m, p, u, ||\cdot, \cdots, \cdot||)$ and this completes the proof. □

Theorem 2.6. *The sequence space $l''(\mathcal{M}, \Delta^m, p, u, ||\cdot, \cdots, \cdot||)$ is solid.*

Proof. Let $x = (x_{k,l}) \in l''(\mathcal{M}, \Delta^m, p, u, ||\cdot, \cdots, \cdot||)$, i.e.

$$\sum_{k.\ l=1}^{\infty,\ \infty} \Big[u_{k,l} M_{k,l}\Big(||\frac{\Delta^m x_{k,l}}{\rho}, z_1, z_2, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}} < \infty.$$

Let $(\alpha_{k,l})$ be double sequence of scalars such that $|\alpha_{k,l}| \leq 1$ for all $k, l \in \mathbb{N} \times \mathbb{N}$. Then, the result follows from the following inequality

$$\sum_{k,\ l=1}^{\infty,\ \infty} \Big[u_{k,l} M_{k,l}\Big(||\frac{\Delta^m \alpha_{k,l} x_{k,l}}{\rho}, z_1, z_2, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}$$
$$\leq \sum_{k,\ l=1}^{\infty,\ \infty} \Big[u_{k,l} M_{k,l}\Big(||\frac{\Delta^m x_{k,l}}{\rho}, z_1, z_2, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}},$$

and this completes the proof. □

Theorem 2.7. *The sequence space $l''(\mathcal{M}, \Delta^m, p, u, ||\cdot, \cdots, \cdot||)$ is monotone.*

Proof. It is obvious. □

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