# GROWTH AND OSCILLATION THEORY OF [P,Q]-ORDER ANALYTIC SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS IN THE UNIT DISC 

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#### Abstract

In this paper, we study the growth and the oscillation of analytic solutions of homogeneous linear differential equations with analytic coefficients of $[p, q]$-order in the unit disc. We also consider the nonhomogeneous linear differential equations.


## 1. Introduction and main results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's theory in the unit disc $\Delta=$ $\{z \in \mathbb{C}:|z|<1\}$ (see [11, 12, 18, 21]). Recently, there has been an increasing interest in studying the growth of analytic solutions of linear differential equations in the unit disc by making use of Nevanlinna theory (see [2, 3, 4, 6, 7, 8, 9, 10, 12, 13, 18, 19).

Consider for $k \geqslant 2$ the linear differential equations

$$
\begin{gather*}
f^{(k)}+A_{k-1}(z) f^{(k-1)}+\ldots+A_{1}(z) f^{\prime}+A_{0}(z) f=0,  \tag{1.1}\\
f^{(k)}+A_{k-1}(z) f^{(k-1)}+\ldots+A_{1}(z) f^{\prime}+A_{0}(z) f=F(z), \tag{1.2}
\end{gather*}
$$

where $A_{0}(z), \ldots, A_{k-1}(z), F(z)$ are analytic functions in the unit disc $\Delta=\{z \in$ $\mathbb{C}:|z|<1\}$. It is well-known that all solutions of equation (1.1) and (1.2) are analytic functions in $\Delta$ and that there are exactly $k$ linearly independent solutions of (1.1) (see [12]). In [14, 15], Juneja, Kapoor and Bajpai have investigated some properties of entire functions of $[p, q]$-order and obtained some results. In [20], by using the concept of $[p, q]$-order Liu, Tu and Shi have considered equations (1.1), (1.2) with entire coefficients and obtained different results concerning the growth of its solutions. In this paper, we continue to consider this subject and investigate the complex linear differential equations 1.1 and $(1.2)$ when the coefficients $A_{0}, A_{1}, \ldots, A_{k-1}, F$ are analytic functions of $[p, q]$-order in $\Delta$.

Before, we state our results we need to give some definitions and discussions. Firstly, let us give definition about the degree of small growth order of functions in

[^0]$\Delta$ as polynomials on the complex plane $\mathbb{C}$. There are many types of definitions of small growth order of functions in $\Delta$ (i.e., see [8, 9]).
Definition 1.1. For a meromorphic function $f$ in $\Delta$ let
$$
D(f):=\limsup _{r \rightarrow 1^{-}} \frac{T(r, f)}{\log \frac{1}{1-r}}
$$
where $T(r, f)$ is the characteristic function of Nevanlinna of $f$. If $D(f)<\infty$, we say that $f$ is of finite degree $D(f)$ (or is non-admissible); if $D(f)=\infty$, we say that $f$ is of infinite degree (or is admissible). If $f$ is an analytic function in $\Delta$, and
$$
D_{M}(f):=\limsup _{r \rightarrow 1^{-}} \frac{\log ^{+} M(r, f)}{\log \frac{1}{1-r}}
$$
in which $M(r, f)=\max _{|z|=r}|f(z)|$ is the maximum modulus function, then we say that $f$ is a function of finite degree $D_{M}(f)$ if $D_{M}(f)<\infty$; otherwise, $f$ is of infinite degree.

Now, we give the definitions of iterated order and growth index to classify generally the functions of fast growth in $\Delta$ as those in $\mathbb{C}$ (see [5, 16, 17]). Let us define inductively, for $r \in[0,1)$, $\exp _{1} r:=e^{r}$ and $\exp _{p+1} r:=\exp \left(\exp _{p} r\right)$, $p \in \mathbb{N}$. We also define for all $r$ sufficiently large in $(0,1), \log _{1} r:=\log r$ and $\log _{p+1} r:=\log \left(\log _{p} r\right), p \in \mathbb{N}$. Moreover, we denote by $\exp _{0} r:=r, \log _{0} r:=r$, $\log _{-1} r:=\exp _{1} r$ and $\exp _{-1} r:=\log _{1} r$.

Definition 1.2. [6, 7, 18] Let $f$ be a meromorphic function in $\Delta$. Then the iterated $p$-order of $f$ is defined by

$$
\rho_{p}(f)=\limsup _{r \rightarrow 1^{-}} \frac{\log _{p}^{+} T(r, f)}{\log \frac{1}{1-r}} \quad(p \geqslant 1 \quad \text { is an integer }),
$$

where $\log _{1}^{+} x=\log ^{+} x=\max \{\log x, 0\}, \log _{p+1}^{+} x=\log ^{+} \log _{p}^{+} x$. For $p=1$, this notation is called order and for $p=2$ hyper-order [12, [19]. If $f$ is analytic in $\Delta$, then the iterated $p$-order of $f$ is defined by

$$
\rho_{M, p}(f)=\limsup _{r \rightarrow 1^{-}} \frac{\log _{p+1}^{+} M(r, f)}{\log \frac{1}{1-r}} \quad(p \geqslant 1 \text { is an integer }) .
$$

Remark 1.3. It follows by M. Tsuji (21], p. 205) that if $f$ is an analytic function in $\Delta$, then we have the inequalities

$$
\rho_{1}(f) \leqslant \rho_{M, 1}(f) \leqslant \rho_{1}(f)+1
$$

which are the best possible in the sense that there are analytic functions $g$ and $h$ such that $\rho_{M, 1}(g)=\rho_{1}(g)$ and $\rho_{M, 1}(h)=\rho_{1}(h)+1$, see [9. However, it follows by Proposition 2.2.2 in [17] that $\rho_{M, p}(f)=\rho_{p}(f)$ for $p \geqslant 2$.

Definition 1.4. (see [6]) The growth index of the iterated order of a meromorphic function $f(z)$ in $\Delta$ is defined by

$$
i(f)=\left\{\begin{array}{cc}
0, & \text { if } f \text { is non-admissible, } \\
\min \left\{j \in \mathbb{N}: \rho_{j}(f)<+\infty\right\} \quad \text { if } f \text { is admissible } \\
+\infty, & \text { if } \rho_{j}(f)=+\infty \text { for all } j \in \mathbb{N}
\end{array}\right.
$$

For an analytic function $f$ in $\Delta$, we also define

$$
i_{M}(f)=\left\{\begin{array}{cc}
0, & \text { if } f \text { is non-admissible } \\
\min \left\{j \in \mathbb{N}: \rho_{M, j}(f)<+\infty\right\} & \text { if } f \text { is admissible } \\
+\infty, & \text { if } \rho_{M, j}(f)=+\infty \text { for all } j \in \mathbb{N}
\end{array}\right.
$$

Remark 1.5. If $\rho_{p}(f)<\infty$ or $i(f) \leqslant p$, then we say that $f$ is of finite iterated $p$-order; if $\rho_{p}(f)=\infty$ or $i(f)>p$, then we say that $f$ is of infinite iterated $p$-order. In particular, we say that $f$ is of finite order if $\rho_{1}(f)<\infty$ or $i(f) \leqslant 1$; $f$ is of infinite order if $\rho_{1}(f)=\infty$ or $i(f)>1$.

Definition 1.6. (see [2, 7]) Let $f$ be a meromorphic function in $\Delta$. Then the iterated exponent of convergence of the sequence of zeros of $f(z)$ is defined by

$$
\lambda_{p}(f)=\limsup _{r \rightarrow 1^{-}} \frac{\log _{p}^{+} N\left(r, \frac{1}{f}\right)}{\log \frac{1}{1-r}} \quad(p \geqslant 1 \text { is an integer }),
$$

where $N\left(r, \frac{1}{f}\right)$ is the counting function of zeros of $f(z)$ in $\{z:|z|<r\}$. For $p=1$, this notation is called exponent of convergence of the sequence of zeros and for $p=2$ hyper-exponent of convergence of the sequence of zeros. Similarly, the iterated exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$
\bar{\lambda}_{p}(f)=\limsup _{r \rightarrow 1^{-}} \frac{\log _{p}^{+} \bar{N}\left(r, \frac{1}{f}\right)}{\log \frac{1}{1-r}} \quad(p \geqslant 1 \text { is an integer })
$$

where $\bar{N}\left(r, \frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{z:|z|<r\}$. For $p=1$, this notation is called exponent of convergence of the sequence of distinct zeros and for $p=2$ hyper-exponent of convergence of the sequence of distinct zeros.

Now, we introduce the concept of $[p, q]$-order of meromorphic and analytic functions in the unit disc.

Definition 1.7. (see [4]) Let $p \geqslant q \geqslant 1$ be integers. Let $f$ be meromorphic function in $\Delta$, the $[p, q]$-order of $f(z)$ is defined by

$$
\rho_{[p, q]}(f)=\limsup _{r \rightarrow 1^{-}} \frac{\log _{p}^{+} T(r, f)}{\log _{q} \frac{1}{1-r}} .
$$

For an analytic function $f$ in $\Delta$, we also define

$$
\rho_{M,[p, q]}(f)=\limsup _{r \rightarrow 1^{-}} \frac{\log _{p+1}^{+} M(r, f)}{\log _{q} \frac{1}{1-r}} .
$$

Remark 1.8. It is easy to see that $0 \leqslant \rho_{[p, q]}(f) \leqslant \infty$. If $f(z)$ is non-admissible, then $\rho_{[p, q]}(f)=0$ for any $p \geqslant q \geqslant 1$. By Definition 1.7. we have that $\rho_{[1,1]}(f)=$ $\rho_{1}(f)=\rho(f), \rho_{[2,1]}(f)=\rho_{2}(f)$ and $\rho_{[p+1,1]}(f)=\rho_{p+1}(f)$.
Proposition 1.9. (see [4]) Let $p \geqslant q \geqslant 1$ be integers. Let $f$ be analytic function in $\Delta$ of $[p, q]$-order. The following two statements hold:
(i) If $p=q$, then

$$
\rho_{[p, q]}(f) \leqslant \rho_{M,[p, q]}(f) \leqslant \rho_{[p, q]}(f)+1
$$

(ii) If $p>q$, then

$$
\rho_{[p, q]}(f)=\rho_{M,[p, q]}(f) .
$$

Definition 1.10. The $[p, q]$-exponent of convergence of the zero sequence of $f(z)$ in $\Delta$ is defined by

$$
\lambda_{[p, q]}(f)=\limsup _{r \rightarrow 1^{-}} \frac{\log _{p}^{+} N\left(r, \frac{1}{f}\right)}{\log _{q} \frac{1}{1-r}} .
$$

Similarly, the $[p, q]$-exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$
\bar{\lambda}_{[p, q]}(f)=\limsup _{r \rightarrow 1^{-}} \frac{\log _{p}^{+} \bar{N}\left(r, \frac{1}{f}\right)}{\log _{q} \frac{1}{1-r}}
$$

For $F \subset[0,1)$, the upper and lower densities of $F$ are defined by

$$
\overline{\operatorname{dens}}_{\Delta} F=\limsup _{r \rightarrow 1^{-}} \frac{m(F \cap[0, r))}{m([0, r))} \text { and } \underline{\text { dens }_{\Delta}} F=\liminf _{r \rightarrow 1^{-}} \frac{m(F \cap[0, r))}{m([0, r))}
$$

respectively, where $m(G)=\int_{G} \frac{d t}{1-t}$ for $G \subset[0,1)$. We obtain the following results.
Theorem 1.11. Let $p \geqslant q \geqslant 1$ be integers. Let $H$ be a set of complex numbers satisfying $\overline{\text { dens }}_{\Delta}\{|z|: z \in H \subseteq \Delta\}>0$, and let $A_{0}(z), \ldots, A_{k-1}(z)$ be analytic functions in the unit disc $\Delta$ such that for real constants $\alpha$, $\beta$ where $0 \leqslant \beta<\alpha$, we have

$$
\begin{equation*}
\left|A_{0}(z)\right| \geqslant \exp _{p+1}\left\{\alpha \log _{q}\left(\frac{1}{1-|z|}\right)\right\} \tag{1.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|A_{j}(z)\right| \leqslant \exp _{p+1}\left\{\beta \log _{q}\left(\frac{1}{1-|z|}\right)\right\}(j=1, \ldots, k-1) \tag{1.4}
\end{equation*}
$$

as $|z| \rightarrow 1^{-}$for $z \in H$. Then every solution $f \not \equiv 0$ of equation (1.1) satisfies $\rho_{[p, q]}(f)=\rho_{M,[p, q]}(f)=\infty$ and $\rho_{[p+1, q]}(f)=\rho_{M,[p+1, q]}(f) \geqslant \alpha$.

Theorem 1.12. Let $p \geqslant q \geqslant 1$ be integers. Let $H$ be a set of complex numbers satisfying dens ${ }_{\Delta}\{|z|: z \in H \subseteq \Delta\}>0$, and let $A_{0}(z), \ldots, A_{k-1}(z)$ be analytic functions in the unit disc $\Delta$ satisfying $\max \left\{\rho_{M,[p, q]}\left(A_{j}\right): j=1, \ldots, k-1\right\} \leqslant \rho_{M,[p, q]}\left(A_{0}\right)=\rho$. Suppose that there exists a real number $\mu$ satisfying $0 \leqslant \mu<\rho$ such that for any given $\varepsilon(0<\varepsilon<\rho-\mu)$ sufficiently small, we have

$$
\begin{equation*}
\left|A_{0}(z)\right| \geqslant \exp _{p+1}\left\{(\rho-\varepsilon) \log _{q}\left(\frac{1}{1-|z|}\right)\right\} \tag{1.5}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|A_{j}(z)\right| \leqslant \exp _{p+1}\left\{\mu \log _{q}\left(\frac{1}{1-|z|}\right)\right\}(j=1, \ldots, k-1) \tag{1.6}
\end{equation*}
$$

as $|z| \rightarrow 1^{-}$for $z \in H$. Then every solution $f \not \equiv 0$ of equation (1.1) satisfies $\rho_{[p, q]}(f)=\rho_{M,[p, q]}(f)=\infty$ and $\rho_{[p+1, q]}(f)=\rho_{M,[p+1, q]}(f)=\rho_{M,[p, q]}\left(A_{0}\right)=\rho$.
Theorem 1.13. Suppose that the assumptions of Theorem 1.12 are satisfied, and let $F \not \equiv 0$ be analytic function in $\Delta$ of $[p, q]$-order. Then, the following two statements hold:
(i) If $\rho_{[p+1, q]}(F)<\rho_{M,[p, q]}\left(A_{0}\right)$, then every solution $f$ of (1.2) satisfies $\bar{\lambda}_{[p+1, q]}(f)=$ $\lambda_{[p+1, q]}(f)=\rho_{[p+1, q]}(f)=\rho_{M,[p, q]}\left(A_{0}\right)$ with at most one exception $f_{0}$ satisfying $\rho_{[p+1, q]}\left(f_{0}\right)<\rho_{M,[p, q]}\left(A_{0}\right)$.
(ii) If $\rho_{[p+1, q]}(F)>\rho_{M,[p, q]}\left(A_{0}\right)$, then every solution $f$ of (1.2) satisfies $\rho_{[p+1, q]}(f)=$ $\rho_{[p+1, q]}(F)$.

## 2. Preliminary lemmas

In this section we give some lemmas which are used in the proofs of our theorems.

Lemma 2.1. (9], Theorem 3.1) Let $k$ and $j$ be integers satisfying $k>j \geqslant 0$, and let $\varepsilon>0$ and $d \in(0,1)$. If $f$ is a meromorphic in $\Delta$ such that $f^{(j)}$ does not vanish identically, then

$$
\begin{equation*}
\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant\left(\left(\frac{1}{1-|z|}\right)^{2+\varepsilon} \max \left\{\log \frac{1}{1-|z|}, T(s(|z|), f)\right\}\right)^{k-j}\left(|z| \notin E_{1}\right) \tag{2.1}
\end{equation*}
$$

where $E_{1} \subset[0,1)$ is a set with $\int_{E_{1}} \frac{d r}{1-r}<\infty$ and $s(|z|)=1-d(1-|z|)$.
Lemma 2.2. (12) Let $f$ be a meromorphic function in the unit disc $\Delta$, and let $k \geqslant 1$ be an integer. Then

$$
\begin{equation*}
m\left(r, \frac{f^{(k)}}{f}\right)=S(r, f) \tag{2.2}
\end{equation*}
$$

where $S(r, f)=O\left(\log ^{+} T(r, f)+\log \left(\frac{1}{1-r}\right)\right)$, possibly outside a set $E_{2} \subset[0,1)$ with $\int_{E_{2}} \frac{d r}{1-r}<\infty$.

Lemma 2.3. (4) Let $p \geqslant q \geqslant 1$ be integers. Let $f$ be a meromorphic function in the unit disc $\Delta$ such that $\rho_{[p, q]}(f)=\rho<\infty$, and let $k \geqslant 1$ be an integer. Then for any $\varepsilon>0$,

$$
\begin{equation*}
m\left(r, \frac{f^{(k)}}{f}\right)=O\left(\exp _{p-1}\left\{(\rho+\varepsilon) \log _{q}\left(\frac{1}{1-r}\right)\right\}\right) \tag{2.3}
\end{equation*}
$$

holds for all $r$ outside a set $E_{3} \subset[0,1)$ with $\int_{E_{3}} \frac{d r}{1-r}<\infty$.
Lemma 2.4. (1]) Let $g:(0,1) \rightarrow \mathbb{R}$ and $h:(0,1) \rightarrow \mathbb{R}$ be monotone increasing functions such that $g(r) \leqslant h(r)$ holds outside of an exceptional set $E_{4} \subset[0,1)$ for which $\int_{E_{4}} \frac{d r}{1-r}<\infty$. Then there exists a constant $d \in(0,1)$ such that if $s(r)=$ $1-d(1-r)$, then $g(r) \leqslant h(s(r))$ for all $r \in[0,1)$.

Lemma 2.5. (4]) Let $p \geqslant q \geqslant 1$ be integers. If $A_{0}(z), \ldots, A_{k-1}(z)$ are analytic functions of $[p, q]$-order in the unit disc $\Delta$, then every solution $f \not \equiv 0$ of 1.1) satisfies

$$
\begin{equation*}
\rho_{[p+1, q]}(f)=\rho_{M,[p+1, q]}(f) \leqslant \max \left\{\rho_{M,[p, q]}\left(A_{j}\right): j=0,1, \ldots, k-1\right\} \tag{2.4}
\end{equation*}
$$

Lemma 2.6. Let $A_{0}, A_{1}, \ldots, A_{k-1}, F \not \equiv 0$ be finite $[p, q]$-order analytic functions in the unit disc $\Delta$. If $f$ is a solution with $\rho_{[p, q]}(f)=\infty$ and $\rho_{[p+1, q]}(f)=\rho<\infty$ of equation (1.2), then $\bar{\lambda}_{[p, q]}(f)=\lambda_{[p, q]}(f)=\rho_{[p, q]}(f)=\infty$ and $\bar{\lambda}_{[p+1, q]}(f)=$ $\lambda_{[p+1, q]}(f)=\rho_{[p+1, q]}(f)=\rho$.
Proof. Since $A_{0}, A_{1}, \ldots, A_{k-1}, F \not \equiv 0$ are analytic in $\Delta$, then all solutions of 1.2 are analytic in $\Delta$ (see [12]). By 1.2 , we can write

$$
\begin{equation*}
\frac{1}{f}=\frac{1}{F}\left(\frac{f^{(k)}}{f}+A_{k-1} \frac{f^{(k-1)}}{f}+\ldots+A_{1} \frac{f^{\prime}}{f}+A_{0}\right) \tag{2.5}
\end{equation*}
$$

If $f$ has a zero at $z_{0} \in \Delta$ of order $\gamma(>k)$, then $F$ must have a zero at $z_{0}$ of order at least $\gamma-k$. Hence,

$$
\begin{equation*}
N\left(r, \frac{1}{f}\right) \leqslant k \bar{N}\left(r, \frac{1}{f}\right)+N\left(r, \frac{1}{F}\right) . \tag{2.6}
\end{equation*}
$$

By (2.5), we have

$$
\begin{equation*}
m\left(r, \frac{1}{f}\right) \leqslant \sum_{j=1}^{k} m\left(r, \frac{f^{(j)}}{f}\right)+\sum_{j=0}^{k-1} m\left(r, A_{j}\right)+m\left(r, \frac{1}{F}\right)+O(1) \tag{2.7}
\end{equation*}
$$

Applying the Lemma 2.3, we have

$$
\begin{equation*}
m\left(r, \frac{f^{(j)}}{f}\right)=O\left(\exp _{p}\left\{(\rho+\varepsilon) \log _{q}\left(\frac{1}{1-r}\right)\right\}\right) \quad(j=1, \ldots, k) \tag{2.8}
\end{equation*}
$$

where $\rho_{[p+1, q]}(f)=\rho<\infty$, holds for all $r$ outside a set $E_{3} \subset[0,1)$ with $\int_{E_{3}} \frac{d r}{1-r}<$ $\infty$. By (2.6)-2.8), we get

$$
\begin{align*}
T(r, f) & =T\left(r, \frac{1}{f}\right)+O(1) \leqslant k \bar{N}\left(r, \frac{1}{f}\right)+\sum_{j=0}^{k-1} T\left(r, A_{j}\right)+T(r, F) \\
& +O\left(\exp _{p}\left\{(\rho+\varepsilon) \log _{q}\left(\frac{1}{1-r}\right)\right\}\right)\left(|z|=r \notin E_{3}\right) \tag{2.9}
\end{align*}
$$

Set

$$
\mu=\max \left\{\rho_{p}\left(A_{j}\right) \quad(j=0, \ldots, k-1), \rho_{p}(F)\right\}
$$

Then for $r \rightarrow 1^{-}$, we have

$$
\begin{equation*}
T\left(r, A_{0}\right)+\ldots+T\left(r, A_{k-1}\right)+T(r, F) \leqslant(k+1) \exp _{p}\left\{(\mu+\varepsilon) \log _{q}\left(\frac{1}{1-r}\right)\right\} \tag{2.10}
\end{equation*}
$$

Thus, by 2.9 and 2.10 , we have for $r \rightarrow 1^{-}$

$$
\begin{align*}
T(r, f) & \leqslant k \bar{N}\left(r, \frac{1}{f}\right)+(k+1) \exp _{p}\left\{(\mu+\varepsilon) \log _{q}\left(\frac{1}{1-r}\right)\right\} \\
+ & O\left(\exp _{p}\left\{(\rho+\varepsilon) \log _{q}\left(\frac{1}{1-r}\right)\right\}\right)=k \bar{N}\left(r, \frac{1}{f}\right) \\
& +O\left(\exp _{p}\left\{\eta \log _{q}\left(\frac{1}{1-r}\right)\right\}\right),\left(|z|=r \notin E_{3}\right) \tag{2.11}
\end{align*}
$$

where $\eta<\infty$. Hence for any $f$ with $\rho_{[p, q]}(f)=\infty$ and $\rho_{[p+1, q]}(f)=\rho$, by Lemma 2.4 and (2.11), we have

$$
\lambda_{[p, q]}(f) \geqslant \bar{\lambda}_{[p, q]}(f) \geqslant \rho_{[p, q]}(f)=\infty
$$

and $\lambda_{[p+1, q]}(f) \geqslant \bar{\lambda}_{[p+1, q]}(f) \geqslant \rho_{[p+1, q]}(f)$. Since $\bar{\lambda}_{[p+1, q]}(f) \leqslant \lambda_{[p+1, q]}(f) \leqslant$ $\rho_{[p+1, q]}(f)$, we have $\bar{\lambda}_{[p+1, q]}(f)=\lambda_{[p+1, q]}(f)=\rho_{[p+1, q]}(f)=\rho$.
Lemma 2.7. Let $p \geqslant q \geqslant 1$ be integers, and let $f$ and $g$ be meromorphic functions of $[p, q]$-order in $\Delta$. Then we have

$$
\rho_{[p, q]}(f+g) \leqslant \max \left\{\rho_{[p, q]}(f), \rho_{[p, q]}(g)\right\}
$$

and

$$
\rho_{[p, q]}(f g) \leqslant \max \left\{\rho_{[p, q]}(f), \rho_{[p, q]}(g)\right\}
$$

Furthermore, if $\rho_{[p, q]}(f)>\rho_{[p, q]}(g)$, then we obtain

$$
\rho_{[p, q]}(f+g)=\rho_{[p, q]}(f g)=\rho_{[p, q]}(f)
$$

Proof. Set $\rho_{[p, q]}(f)=\rho_{1}$ and $\rho_{[p, q]}(g)=\rho_{2}$. For any given $\varepsilon>0$, we have

$$
\begin{gather*}
T(r, f+g) \leqslant T(r, f)+T(r, g)+O(1) \leqslant \exp _{p}\left\{\left(\rho_{1}+\varepsilon\right) \log _{q}\left(\frac{1}{1-r}\right)\right\} \\
+\exp _{p}\left\{\left(\rho_{2}+\varepsilon\right) \log _{q}\left(\frac{1}{1-r}\right)\right\}+O(1) \\
\leqslant 2 \exp _{p}\left\{\left(\max \left\{\rho_{1}, \rho_{2}\right\}+\varepsilon\right) \log _{q}\left(\frac{1}{1-r}\right)\right\}+O(1) \tag{2.12}
\end{gather*}
$$

and

$$
\begin{equation*}
T(r, f g) \leqslant T(r, f)+T(r, g) \leqslant 2 \exp _{p}\left\{\left(\max \left\{\rho_{1}, \rho_{2}\right\}+\varepsilon\right) \log _{q}\left(\frac{1}{1-r}\right)\right\} \tag{2.13}
\end{equation*}
$$

for all $r$ sufficiently large. Since $\varepsilon>0$ is arbitrary, from 2.12 and 2.13, we easily obtain

$$
\begin{equation*}
\rho_{[p, q]}(f+g) \leqslant \max \left\{\rho_{[p, q]}(f), \rho_{[p, q]}(g)\right\} \tag{2.14}
\end{equation*}
$$

and

$$
\begin{equation*}
\rho_{[p, q]}(f g) \leqslant \max \left\{\rho_{[p, q]}(f), \rho_{[p, q]}(g)\right\} \tag{2.15}
\end{equation*}
$$

Suppose now that $\rho_{[p, q]}(f)>\rho_{[p, q]}(g)$. Considering that

$$
\begin{equation*}
T(r, f)=T(r, f+g-g) \leqslant T(r, f+g)+T(r, g)+O(1) \tag{2.16}
\end{equation*}
$$

and

$$
\begin{align*}
T(r, f) & =T\left(r, \frac{f g}{g}\right) \leqslant T(r, f g)+T\left(r, \frac{1}{g}\right) \\
& =T(r, f g)+T(r, g)+O(1) \tag{2.17}
\end{align*}
$$

By (2.16) and (2.17), by the same method as above we obtain that

$$
\begin{gather*}
\rho_{[p, q]}(f) \leqslant \max \left\{\rho_{[p, q]}(f+g), \rho_{[p, q]}(g)\right\}=\rho_{[p, q]}(f+g),  \tag{2.18}\\
\rho_{[p, q]}(f) \leqslant \max \left\{\rho_{[p, q]}(f g), \rho_{[p, q]}(g)\right\}=\rho_{[p, q]}(f g) . \tag{2.19}
\end{gather*}
$$

By using 2.14 and 2.18) we obtain $\rho_{[p, q]}(f+g)=\rho_{[p, q]}(f)$ and by 2.15 and 2.19, we get $\rho_{[p, q]}(f g)=\rho_{[p, q]}(f)$.

Lemma 2.8. Let $p \geqslant q \geqslant 1$ be integers, and let $f$ be a meromorphic function of $[p, q]$-order in $\Delta$. Then $\rho_{[p, q]}\left(f^{\prime}\right)=\rho_{[p, q]}(f)$.
Proof. Let $f$ be a meromorphic function of $[p, q]$-order in $\Delta$. By ( $[6$, p. 281) we have for $r \rightarrow 1^{-}$

$$
\begin{equation*}
T(r, f)<O\left(T\left(\frac{r+3}{4}, f^{\prime}\right)+\ln \frac{1}{1-r}\right) \tag{2.20}
\end{equation*}
$$

On the other hand,

$$
\begin{align*}
T\left(r, f^{\prime}\right)=m\left(r, f^{\prime}\right)+ & N\left(r, f^{\prime}\right) \leqslant m(r, f)+m\left(r, \frac{f^{\prime}}{f}\right)+2 N(r, f) \\
& \leqslant 2 T(r, f)+m\left(r, \frac{f^{\prime}}{f}\right) \tag{2.21}
\end{align*}
$$

Hence, by using 2.20 and 2.21 we obtain $\rho_{[p, q]}\left(f^{\prime}\right)=\rho_{[p, q]}(f)$.

Proof of Theorem 1.11. Suppose that $f \not \equiv 0$ is a solution of 1.1. From the conditions of Theorem 1.11. there is a set $H$ of complex numbers satisfying $\overline{d e n s}_{\Delta}\{|z|$ : $z \in H \subseteq \Delta\}>0$ such that for $z \in H$, we have (1.2) and (1.3) as $|z| \rightarrow 1^{-}$. Set $H_{1}=\{r=|z|: z \in H \subseteq \Delta\}$, since $\overline{\operatorname{dens}}_{\Delta}\{|z|: z \in H \subseteq \Delta\}>0$, then $H_{1}$ is a set with $\int_{H_{1}} \frac{d r}{1-r}=\infty$. By Lemma 2.1, there exist $s(|z|)=1-d(1-|z|)$ and a set $E_{1} \subset[0,1)$ with $\int_{E_{1}} \frac{d r}{1-r}<\infty$ such that for $r=|z| \notin E_{1}$, we have

$$
\begin{equation*}
\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant\left(\left(\frac{1}{1-|z|}\right)^{2+\varepsilon} \max \left\{\log \frac{1}{1-|z|}, T(s(|z|), f)\right\}\right)^{j}(j=1, \ldots, k) \tag{2.22}
\end{equation*}
$$

By (1.1), we can write

$$
\begin{equation*}
\left|A_{0}(z)\right| \leqslant\left|\frac{f^{(k)}}{f}\right|+\left|A_{k-1}(z)\right|\left|\frac{f^{(k-1)}}{f}\right|+\ldots+\left|A_{0}(z)\right|\left|\frac{f^{\prime}}{f}\right| \tag{2.23}
\end{equation*}
$$

It follows by (1.3), (1.4), 2.22 and 2.23 that

$$
\begin{align*}
& \exp _{p+1}\left\{\alpha \log _{q}\left(\frac{1}{1-|z|}\right)\right\} \leqslant\left|A_{0}(z)\right| \leqslant k \exp _{p+1}\left\{\beta \log _{q}\left(\frac{1}{1-|z|}\right)\right\} \\
& \times\left(\left(\frac{1}{1-|z|}\right)^{2+\varepsilon} \max \left\{\log \frac{1}{1-|z|}, T(s(|z|), f)\right\}\right)^{k} \tag{2.24}
\end{align*}
$$

holds for all $z$ satisfying $|z|=r \in H_{1} \backslash E_{1}$ as $|z| \rightarrow 1^{-}$, where $E_{1} \subset[0,1)$ is a set with $\int_{E_{1}} \frac{d r}{1-r}<\infty$. Noting that $\alpha>\beta \geqslant 0$, by 2.24 we have

$$
\begin{equation*}
(1-o(1)) \exp _{p+1}\left\{\alpha \log _{q}\left(\frac{1}{1-|z|}\right)\right\} \leqslant\left(\frac{1}{1-|z|}\right)^{k(2+\varepsilon)} T^{k}(s(|z|), f) \tag{2.25}
\end{equation*}
$$

for all $z$ satisfying $|z|=r \in H_{1} \backslash E_{1}$ as $|z| \rightarrow 1^{-}$. Hence by Lemma 2.4 and 2.25, we obtain $\rho_{[p, q]}(f)=\rho_{M,[p, q]}(f)=\infty$ and

$$
\rho_{[p+1, q]}(f)=\rho_{M,[p+1, q]}(f)=\limsup _{r \rightarrow 1^{-}} \frac{\log _{p+1}^{+} T(r, f)}{\log _{q} \frac{1}{1-r}} \geqslant \alpha
$$

Proof of Theorem 1.12. Suppose that $f \not \equiv 0$ is a solution of 1.1. Then for any given $\varepsilon>0$, by the results of Theorem 1.11, we have $\rho_{[p, q]}(f)=\rho_{M,[p, q]}(f)=\infty$ and

$$
\begin{equation*}
\rho_{[p+1, q]}(f)=\rho_{M,[p+1, q]}(f) \geqslant \rho-\varepsilon . \tag{2.26}
\end{equation*}
$$

Since $\varepsilon>0$ is arbitrary we get from 2.26 that $\rho_{[p+1, q]}(f)=\rho_{M,[p+1, q]}(f) \geqslant \rho$. On the other hand, by Lemma 2.5, we have

$$
\begin{gather*}
\rho_{[p+1, q]}(f)=\rho_{M,[p+1, q]}(f) \\
\leqslant \max \left\{\rho_{M,[p, q]}\left(A_{j}\right): j=0,1, \ldots, k-1\right\}=\rho_{M,[p, q]}\left(A_{0}\right)=\rho \tag{2.27}
\end{gather*}
$$

It yields $\rho_{[p+1, q]}(f)=\rho_{M,[p+1, q]}(f)=\rho_{M,[p, q]}\left(A_{0}\right)=\rho$.

Proof of Theorem 1.13. (i) Suppose that $\rho_{[p+1, q]}(F)<\rho_{M,[p, q]}\left(A_{0}\right)$. We assume that $f$ is a solution of $(1.2)$ and $\left\{f_{1}, f_{2}, \ldots, f_{k}\right\}$ is a solution base of the corresponding homogeneous equation (1.1) of 1.2 . By Theorem 1.12 , we know that $\rho_{[p, q]}\left(f_{j}\right)=$ $\infty$ and $\rho_{[p+1, q]}\left(f_{j}\right)=\rho_{M,[p, q]}\left(A_{0}\right)(j=1,2, \ldots, k)$. Then $f$ can be expressed in the form

$$
\begin{equation*}
f(z)=B_{1}(z) f_{1}(z)+B_{2}(z) f_{2}(z)+\ldots+B_{k}(z) f_{k}(z) \tag{2.28}
\end{equation*}
$$

where $B_{1}(z), \ldots, B_{k}(z)$ are suitable analytic functions determined by

$$
\left.\begin{array}{c}
B_{1}^{\prime}(z) f_{1}(z)+B_{2}^{\prime}(z) f_{2}(z)+\ldots+B_{k}^{\prime}(z) f_{k}(z)=0 \\
B_{1}^{\prime}(z) f_{1}^{\prime}(z)+B_{2}^{\prime}(z) f_{2}^{\prime}(z)+\ldots+B_{k}^{\prime}(z) f_{k}^{\prime}(z)=0  \tag{2.29}\\
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots
\end{array}\right\}
$$

Since the Wronskian $W\left(f_{1}, f_{2}, \ldots, f_{k}\right)$ is a differential polynomial in $f_{1}, f_{2}, \ldots, f_{k}$ with constant coefficients, it is easy by using Theorem 1.12 to deduce that

$$
\begin{equation*}
\rho_{[p+1, q]}(W) \leqslant \max \left\{\rho_{[p+1, q]}\left(f_{j}\right): j=1,2, \ldots, k\right\}=\rho_{M,[p, q]}\left(A_{0}\right) \tag{2.30}
\end{equation*}
$$

From 2.29, we get

$$
\begin{equation*}
B_{j}^{\prime}=F \cdot G_{j}\left(f_{1}, f_{2}, \ldots, f_{k}\right) \cdot\left(W\left(f_{1}, f_{2}, \ldots, f_{k}\right)\right)^{-1} \quad(j=1,2, \ldots, k), \tag{2.31}
\end{equation*}
$$

where $G_{j}\left(f_{1}, f_{2}, \ldots, f_{k}\right)$ are differential polynomials in $f_{1}, f_{2}, \ldots, f_{k}$ with constant coefficients. Thus

$$
\begin{gather*}
\rho_{[p+1, q]}\left(G_{j}\right) \leqslant \max \left\{\rho_{[p+1, q]}\left(f_{j}\right): j=1,2, \ldots, k\right\} \\
=\rho_{M,[p, q]}\left(A_{0}\right) \quad(j=1,2, \ldots, k) . \tag{2.32}
\end{gather*}
$$

Since $\rho_{[p+1, q]}(F)<\rho_{M,[p, q]}\left(A_{0}\right)$, then by using Lemma 2.7. Lemma 2.8, 2.30 and (2.32, we have from 2.31) for $j=1,2, \ldots, k$

$$
\begin{equation*}
\rho_{[p+1, q]}\left(B_{j}\right)=\rho_{[p+1, q]}\left(B_{j}^{\prime}\right) \leqslant \max \left\{\rho_{[p+1, q]}(F), \rho_{M,[p, q]}\left(A_{0}\right)\right\}=\rho_{M,[p, q]}\left(A_{0}\right) \tag{2.33}
\end{equation*}
$$

Then, by (2.33) and Lemma 2.7, we get from (2.28)

$$
\begin{gather*}
\rho_{[p+1, q]}(f) \leqslant \max \left\{\rho_{[p+1, q]}\left(f_{j}\right), \rho_{[p+1, q]}\left(B_{j}\right): j=1,2, \ldots, k\right\} \\
=\rho_{M,[p, q]}\left(A_{0}\right) . \tag{2.34}
\end{gather*}
$$

Now, we assert that every solution $f$ of 1.2 satisfies $\rho_{[p+1, q]}(f)=\rho_{M,[p, q]}\left(A_{0}\right)$ with at most one exceptional solution $f_{0}$ satisfying $\rho_{[p+1, q]}\left(f_{0}\right)<\rho_{M,[p, q]}\left(A_{0}\right)$. In fact, if $f^{*}$ is another solution with $\rho_{[p+1, q]}\left(f^{*}\right)<\rho_{M,[p, q]}\left(A_{0}\right)$ of equation (1.2), then $\rho_{[p+1, q]}\left(f_{0}-f^{*}\right)<\rho_{M,[p, q]}\left(A_{0}\right)$. But $f_{0}-f^{*}$ is a solution of the corresponding homogeneous equation (1.1) of 1.2 . This contradicts Theorem 1.12 . By Lemma 2.6. we know that every solution with $\rho_{[p, q]}(f)=\infty$ and $\rho_{[p+1, q]}(f)=\rho_{M,[p, q]}\left(A_{0}\right)$ satisfies $\bar{\lambda}_{[p+1, q]}(f)=\lambda_{[p+1, q]}(f)=\rho_{[p+1, q]}(f)=\rho_{M,[p, q]}\left(A_{0}\right)$.
(ii) If $\rho_{M,[p, q]}\left(A_{0}\right)<\rho_{[p+1, q]}(F)$, then by using Lemma 2.7, Lemma 2.8, 2.30 and (2.32), we have from (2.31) for $j=1,2, \ldots, k$

$$
\begin{gather*}
\rho_{[p+1, q]}\left(B_{j}\right)=\rho_{[p+1, q]}\left(B_{j}^{\prime}\right) \\
\leqslant \max \left\{\rho_{[p+1, q]}(F), \rho_{[p+1, q]}\left(f_{j}\right): j=1,2, \ldots, k\right\}=\rho_{[p+1, q]}(F) . \tag{2.35}
\end{gather*}
$$

Then from (2.35) and (2.28), we get

$$
\begin{equation*}
\rho_{[p+1, q]}(f) \leqslant \max \left\{\rho_{[p+1, q]}\left(f_{j}\right), \rho_{[p+1, q]}\left(B_{j}\right): j=1,2, \ldots, k\right\}=\rho_{[p+1, q]}(F) \tag{2.36}
\end{equation*}
$$

On the other hand, if $\rho_{M,[p, q]}\left(A_{0}\right)<\rho_{[p+1, q]}(F)$, it follows from equation 1.2 that a simple consideration of $[p, q]$-order implies $\rho_{[p+1, q]}(f) \geqslant \rho_{[p+1, q]}(F)$. By this inequality and 2.36 we obtain $\rho_{[p+1, q]}(f)=\rho_{[p+1, q]}(F)$.

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