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INEQUALITIES BETWEEN LOGARITHMIC, HARMONIC, ARITHMETIC AND CENTROIDAL MEANS

YU-MING CHU, SHOU-WEI HOU, WEI-MING GONG

Abstract. For all $a, b > 0$ with $a \neq b$ we prove that $L(a,b) < \alpha H(a,b) + \beta A(a,b) + (1-\alpha-\beta)C(a,b)$ if and only if $4\alpha+\beta \leq 2$ and $L(a,b) > \alpha H(a,b) + \beta A(a,b) + (1-\alpha-\beta)C(a,b)$ if and only if $4\alpha+\beta \geq 4$, where $L(a,b) = \frac{a-b}{\log a - \log b}$, $H(a,b) = 2ab/(a+b)$, $A(a,b) = (a+b)/2$ and $C(a,b) = 2(a^2+ab+b^2)/[3(a+b)]$ are the logarithmic, harmonic, arithmetic, and centroidal means of a and b, respectively.

1. Introduction

The classical logarithmic mean $L(a,b)$ of two positive real numbers a and b with $a \neq b$ is defined by

$$L(a,b) = \frac{a-b}{\log a - \log b}.$$

In the recent past, the bivariate means have been the subject of intensively research. In particular, many remarkable inequalities for $L(a,b)$ can be found in the literature [1, 2, 5, 6, 10, 11, 12, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 27, 28, 30]. It might be surprising that the logarithmic mean has applications in physics, economics, and even in meteorology [9, 13, 14]. In [9] the authors study a variant of Jensen's functional equation involving the logarithmic mean, which appears in a heat conduction problem. A representation of $L(a,b)$ as an infinite product and an iterative algorithm for computing the logarithmic mean as the common limit of two sequences of special geometric and arithmetic means are given in [6]. In [7, 8] it is shown that $L(a,b)$ can be expressed in terms of Gauss's hypergeometric function ${}_2F_1$. And, in [8] the authors prove that the reciprocal of the logarithmic mean is strictly totally positive, that is, every $n \times n$ determinant with elements $1/L(a_i, b_i)$, where $0 < a_1 < a_2 < \cdots < a_n$ and $0 < b_1 < b_2 < \cdots < b_n$, is positive for all $n \geq 1$.

Let $G(a,b) = \sqrt{ab}$, $H(a,b) = \frac{2ab}{a+b}$, $I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$, $A(a,b) = \frac{1}{2}(a+b)$, $C(a,b) = 2(a^2+ab+b^2)/[3(a+b)]$, $M_p(a,b) = ((a^p+b^p)/2)^{1/p} (p \neq 0)$ and $M_0(a,b) =$

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$\sqrt{ab}$, and $L_p(a,b) = (a^{p+1}+b^{p+1})/(a^p+b^p)$ be the geometric, harmonic, identric, arithmetic, p-th power, and p-th Lehmer means of two positive numbers a and b , respectively. Then it is well known that both $M_p(a,b)$ and $L_p(a,b)$ are strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a,b>0$ with $a \neq b$, and inequalities

$$\begin{aligned}&\min\{a,b\} < H(a,b) = M_{-1}(a,b) = L_{-1}(a,b) < G(a,b) = M_0(a,b) = L_{-1/2}(a,b)\\ &< L(a,b) < I(a,b) < A(a,b) = M_1(a,b) = L_0(a,b) < C(a,b) < \max\{a,b\}\end{aligned}$$

holds for $a,b>0$ with $a \neq b$.

Carlson [6] proved that the double inequality

$$\sqrt{G(a,b)(A(a,b)+G(a,b))/2} < L(a,b) < \frac{1}{2}(A(a,b)+G(a,b))$$

holds for $a,b>0$ with $a \neq b$.

In [10], Lin found the best possible upper and lower power bounds for the logarithmic mean as follows:

$$M_0(a,b) < L(a,b) < M_{1/3}(a,b)$$

holds for $a,b>0$ with $a \neq b$.

Sándor [16] established that

$$\sqrt{G(a,b)I(a,b)} < L(a,b) < A(a,b)+G(a,b)-I(a,b)$$

for all $a,b>0$ with $a \neq b$.

In [3], Alzer gave the optimal Lehmer mean bounds for L, $(LI)^{1/2}$, and $(L+I)/2$ as follows:

$$L_{-1/3}(a,b) < L(a,b) < L_0(a,b),$$

$$L_{-1/4}(a,b) < \sqrt{L(a,b)I(a,b)} < L_0(a,b)$$

and

$$L_{-1/4}(a,b) < \frac{1}{2}(L(a,b)+I(a,b)) < L_0(a,b)$$

for all $a,b>0$ with $a \neq b$.

The following sharp bounds for $(LI)^{1/2}$ and $(L+I)/2$ in terms of power mean were presented in [4]:

$$M_0(a,b) < \sqrt{L(a,b)I(a,b)} < M_{1/2}(a,b)$$

and

$$M_{\log 2/(1+\log 2)}(a,b) < \frac{1}{2}(L(a,b)+I(a,b)) < M_{1/2}(a,b)$$

for all $a,b>0$ with $a \neq b$.

For any $\alpha \in (0,1)$, in [26, 29] the authors obtained the sharp bounds for the products $A^\alpha(a,b)L^{1-\alpha}(a,b)$ and $G^\alpha(a,b)L^{1-\alpha}(a,b)$, and the sum $\alpha A(a,b)+(1-\alpha)L(a,b)$ in terms of power mean as follows:

$$M_0(a,b) < A^\alpha(a,b)L^{1-\alpha}(a,b) < M_{(1+2\alpha)/3}(a,b),$$

$$M_0(a,b) < G^\alpha(a,b)L^{1-\alpha}(a,b) < M_{(1-\alpha)/3}(a,b)$$

and

$$M_{\log 2/(\log 2-\log \alpha)}(a,b) < \alpha A(a,b)+(1-\alpha)L(a,b) < M_{(1+2\alpha)/3}(a,b)$$

for all $a,b>0$ with $a \neq b$.

It is the aim of this paper to answer the question: what are the values of α and β such that the inequlity $L(a,b) < \alpha H(a,b) + \beta A(a,b) + (1-\alpha-\beta)C(a,b)$ or $L(a,b) > \alpha H(a,b) + \beta A(a,b) + (1-\alpha-\beta)C(a,b)$ holds for all $a, b > 0$ with $a \neq b$?

2. Main Results

Theorem 2.1. *For all $a, b > 0$ with $a \neq b$ we have*

(1) $L(a,b) < \alpha H(a,b) + \beta A(a,b) + (1-\alpha-\beta)C(a,b)$ if and only if $4\alpha + \beta \leq 2$;

(2) $L(a,b) > \alpha H(a,b) + \beta A(a,b) + (1-\alpha-\beta)C(a,b)$ if and only if $4\alpha + \beta \geq 4$.

Proof. Without loss of generality, we assume $a > b$. Let $t = a/b > 1$, then it is not difficult to verify that

$$\begin{aligned}&\alpha H(a,b) + \beta A(a,b) + (1-\alpha-\beta)C(a,b) - L(a,b)\\ =&\frac{b}{6(t+1)\log t}\{[(4-4\alpha-\beta)t^2 + 2(2+4\alpha+\beta)t + (4-4\alpha-\beta)]\log t - 6t^2 + 6\}.\end{aligned} \tag{2.1}$$

Let

$$f(t) = [(4-4\alpha-\beta)t^2 + 2(2+4\alpha+\beta)t + (4-4\alpha-\beta)]\log t - 6t^2 + 6. \tag{2.2}$$

Then simple computations lead to

$$f(1) = 0, \tag{2.3}$$

$$\begin{aligned}f'(t) =&[2(4-4\alpha-\beta)t + 2(2+4\alpha+\beta)]\log t - (8+4\alpha+\beta)t\\ &+ \frac{4-4\alpha-\beta}{t} + 2(2+4\alpha+\beta),\end{aligned} \tag{2.4}$$

$$f'(1) = 0, \tag{2.5}$$

$$f''(t) = 2(4-4\alpha-\beta)\log t + \frac{2(2+4\alpha+\beta)}{t} - \frac{4-4\alpha-\beta}{t^2} - 3(4\alpha+\beta), \tag{2.6}$$

$$f''(1) = 0, \tag{2.7}$$

$$f'''(t) = \frac{2[(4-4\alpha-\beta)t^2 - (2+4\alpha+\beta)t + (4-4\alpha-\beta)]}{t^2}. \tag{2.8}$$

We divide the proof into three cases.

Case 1: $4\alpha + \beta \leq 2$. Then equation (2.8) leads to

$$f'''(t) = \frac{2(4-4\alpha-\beta)}{t^2}\left[\left(t - \frac{2+4\alpha+\beta}{2(4-4\alpha-\beta)}\right)^2 + \frac{3(2-4\alpha-\beta)(10-4\alpha-\beta)}{4(4-4\alpha-\beta)^2}\right] > 0 \tag{2.9}$$

for $t > 1$.

From equations (2.1)-(2.3), (2.5) and (2.7) together with inequality (2.9) we clearly see that $L(a,b) < \alpha H(a,b) + \beta A(a,b) + (1-\alpha-\beta)C(a,b)$.

Case 2: $4\alpha + \beta \geq 4$. Then equation (2.8) leads to

$$f'''(t) = -\frac{2[(4\alpha+\beta-4)t^2 + (2+4\alpha+\beta)t + (4\alpha+\beta-4)]}{t^2} < 0 \tag{2.10}$$

for $t > 1$.

Therefore, inequality $L(a,b) > \alpha H(a,b) + \beta A(a,b) + (1-\alpha-\beta)C(a,b)$ follows from equations (2.1)-(2.3), (2.5) and (2.7) together with inequality (2.10).

Case 3: $2 < 4\alpha + \beta < 4$. Then from equations (2.2), (2.4) and (2.6) we get

$$\lim_{t\to+\infty} f(t) = +\infty, \tag{2.11}$$

$$\lim_{t\to+\infty} f'(t) = +\infty \tag{2.12}$$

and

$$\lim_{t\to+\infty} f''(t) = +\infty. \tag{2.13}$$

Let

$$g(t) = (4 - 4\alpha - \beta)t^2 - (2 + 4\alpha + \beta)t + (4 - 4\alpha - \beta). \tag{2.14}$$

Then

$$g(1) = -3(4\alpha + \beta - 2) < 0, \tag{2.15}$$

$$\lim_{t\to+\infty} g(t) = +\infty, \tag{2.16}$$

$$g'(t) = 2(4 - 4\alpha - \beta)t - (2 + 4\alpha + \beta), \tag{2.17}$$

$$g'(1) = -3(4\alpha + \beta - 2) < 0 \tag{2.18}$$

and

$$\lim_{t\to+\infty} g'(t) = +\infty. \tag{2.19}$$

From equation (2.17) we know that $g'(t)$ is strictly increasing in $[1, +\infty)$. Then inequality (2.18) and equation (2.19) lead to the conclusion that there exists $\lambda_1 > 1$ such that $g'(t) < 0$ for $t \in [1, \lambda_1)$ and $g'(t) > 0$ for $t \in (\lambda_1, +\infty)$. Therefore, $g(t)$ is strictly decreasing in $[1, \lambda_1]$ and strictly increasing in $[\lambda_1, +\infty)$.

It follows from inequality (2.15) and equation (2.16) together with the piecewise monotonicity of $g(t)$ that there exists $\lambda_2 > \lambda_1 > 1$ such that $g(t) < 0$ for $t \in [1, \lambda_2)$ and $g(t) > 0$ for $t \in (\lambda_2, +\infty)$. Then equations (2.8) and (2.14) lead to the conclusion that $f''(t)$ is strictly decreasing in $[1, \lambda_2]$ and strictly increasing in $[\lambda_2, +\infty)$.

From equations (2.7) and (2.13) together with the piecewise monotonicity of $f''(t)$ we conclude that there exists $\lambda_3 > \lambda_2 > 1$ such that $f'(t)$ is strictly decreasing in $[1, \lambda_3]$ and strictly increasing in $[\lambda_3, +\infty)$. Then equations (2.5) and (2.12) lead to the conclusion that there exists $\lambda_4 > \lambda_3 > 1$ such that $f(t)$ is strictly decreasing in $[1, \lambda_4]$ and strictly increasing in $[\lambda_4, +\infty)$.

It follows from equations (2.3) and (2.11) together with the piecewise monotonicity of $f(t)$ that there exixts $\lambda > \lambda_4 > 1$ such that $f(t) < 0$ for $t \in (1, \lambda)$ and $f(t) > 0$ for $t \in (\lambda, +\infty)$. Then equations (2.1) and (2.2) lead to the conclusion that $L(a, b) > \alpha H(a, b) + \beta A(a, b) + (1 - \alpha - \beta)C(a, b)$ for $a/b \in (1, \lambda)$ and $L(a, b) < \alpha H(a, b) + \beta A(a, b) + (1 - \alpha - \beta)C(a, b)$ for $a/b \in (\lambda, +\infty)$. □

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Yu-Ming Chu
Department of Mathematics, Huzhou Teachers College, Huzhou 313000, China
E-mail address: chuyuming@hutc.zj.cn

Shou-Wei Hou
Department of Mathematics, Hangzhou Normal University, Hangzhou 310012, China
E-mail address: houshouwei2008@163.com

Wei-Ming Gong
Department of Mathematics, Hunan City University, Yiyang 413000, China
E-mail address: gwming1963@163.com