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NEW ITERATIVE SCHEMES FOR GENERAL HARMONIC VARIATIONAL INEQUALITIES

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Abstract. Some new classes of general harmonic convex sets and convex functions are introduced and studied in this paper. The optimality criteria of the differentiable general harmonic functions is characterized by the general harmonic variational inequalities. Special cases are also pointed out as applications of the new concepts. Auxiliary principle technique involving an arbitrary operator is applied to suggest and analysis several inertial type methods are suggested. Convergence criteria is investigated of the proposed methods under weaker conditions. The results obtained in this paper may inspire further research along with implementable numerical methods for solving the general harmonic variational inequalities and related optimization problems.

1. Introduction

The optimization theory has been one of an important branch of mathematical sciences for centuries. It is a tool of great power that can be applied to investigate a wide variety of problems in pure and applied sciences. This theory is being applied to interpret the basic principles of mathematical and physical sciences in the form of simplicity and elegance. During this period, this theory has played an important and significant part as a unifying influence in pure and applied sciences and as a guide in the mathematical interpretation of many physical phenomena. In recent years, variational principles have been enriched by the discovery of the variational inequality theory, which is mainly due to Stampacchia [33]. The ideas and techniques of variational inequalities are being applied in a variety of diverse areas of sciences and prove to be productive and innovative. In fact, this theory provides the most natural, direct, simple, unified and efficient framework for a general treatment of a wide class of unrelated linear and nonlinear problems. The variational inequality theory is related to the simple fact that the minimum of a differentiable convex functional on a convex set in a normed space can be characterized by the variational inequality. It is amazing that this theory allows many diversified applications in various directions. We would like to point out that the

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variational inequality theory can be regarded as a natural development of the 19th and 20th problems of Hilbert, in which he formulated his famous Paris lecture in 1900. During the last five decades, in which have elapsed since its discovery, variational inequality theory has produced a tremendous impact in various branches of mathematical and engineering sciences. For the applications, motivation, numerical methods, generalization and other aspects of variational inequalities, see [1, 6, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 27, 28, 29, 30, 31, 33] and the references therein.

Related to variational inequalities, we have convexity theory, which contains a wealth of novel ideas and techniques. This theory had played a leading role in the development of almost all the branches of pure and applied sciences. Several new generalizations and extensions of the convex functions and convex sets have been introduced and studied to tackle unrelated complicated and complex problems in a unified manner. Harmonic functions and harmonic convex sets are important generalizations of the convex functions and convex sets. The harmonic means have novel applications in electrical circuits theory, entropy, game theory, data analysis, machine learning and information theory. Al-Azemi et al.[3] studied the Asian options with harmonic average, which can be viewed a new direction in the study of the risk analysis and financial mathematics. Noor et al.[19] have shown that the minimum of the differentiable harmonic convex function on the harmonic convex set can be characterized by a class of variational inequalities, known as harmonic variational. For the formulation, motivation, numerical methods, generalizations and other aspects of harmonic convex functions and harmonic inequalities, see [1, 2, 3, 5, 7, 10, 19, 20, 21, 22, 23, 24, 25, 26].

It have been observed that the convex sets and convex function may not apply to tackle some problems efficiently due to nonlinear structure and other constraints. To overcome theses deficiency, several new convex sets and convex functions have been considered with respect to arbitrary functions. Noor [18] introduced and studied the general (g-convex) sets and general convex functions, which are known as Noor-convexity. General convex sets and convex functions contain the m-convex sets and m-convex functions considered by Toader [34] as special cases. Cristescu et al. [8] have studied the several applications of Noor-convex sets in optimization problems such as ecologic-economic efficiency, computer aided design, railway transport system,image processing, machine learning and data analysis.

We would like to emphasize that the general variational inequalities and harmonic variational inequalities are two distinct generalizations of the variational inequalities and related optimizations problems. It is natural to study these different problems in a unified framework. This motivated us to introduce and consider some new classes of harmonic variational inequalities.

In this paper, we prove that the minimum of the differentiable general harmonic function is the solution of the general harmonic variational inequality. Several special cases such as harmonic variational inequalities, harmonic complementarity problems and related problems are discussed. The projection method, resolvent method, Wiene-Hopf equations technique and descent methods are not applicable to propose numerical methods for solving general harmonic variational inequalities. One usually apply the auxiliary principle technique, the origin of can be traced back to Lions et al. [11] and Glowinski et al. [9]. Noor [12, 14, 17] and Noor et al. [18, 19, 20, 21, 24, 25, 27, 28, 29] have used this technique to suggest several iterative schemes for solving various classes of variational inequalities and equilibrium problems. In this paper, we use the auxiliary principle technique involving an arbitrary operator to suggest and analyze some new hybrid inertial iterative schemes for solving general harmonic variational inequalities. The inertial type methods were suggested by Polyak [38] to speed up the convergence of iterative methods. We also prove that the convergence of these new methods requires pseudomonotonicity, which is weaker conation than monotonicity. We have indicated several new and known cases, which can be obtained for harmonic variational inequalities, variational inequalities and related optimization problems. It is an open problem to explore the applications of general harmonic convexity and general Bregman distance functions in various branches of mathematical, information technology and engineering sciences.

2. Preliminaries and basic results

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle ., . \rangle$ and $\|.\|$, respectively. First of all, we recall the following concepts and results from convex and nonsmooth analysis [7] to convey the main ideas. For the sake of completeness, we include the relevant details.

Definition 1. A set $C \subseteq H$ is said to be convex set, if

$$u + \lambda(v - u) \in C, \quad \forall u, v \in C.$$

The ideas and techniques of the convexity are being applied in a variety of diverse areas of sciences and prove to be productive and innovative. In many complicated problems, these concepts have to generalize and extend using some novel ideas and techniques. Noor [15] introduced and studied the new convex sets replacing linear structure by the straight-line segment joining two points of a given set by a displaced straight-line segment.

Definition 2. A set $C \subseteq H$s said to be a general (g-convex) convex set with respect to an arbitrary function g, if

$$u + \lambda(g(v) - u) \in C_g, \quad \forall u, v \in C_g.$$

Cristescu et al.[13] called the general convex set as the Noor-convex set. If $g(v) = mv$, where m is a constant, then the general convex set reduces to an m-convex set, the origin of which can be traced back to Toader [34]. The concept of Noor-convex set C_g differs from that of E -convex set introduced by Youness [33]. Cristescu et al. [8] have studied the applications of Noor-convex sets in optimization problems such as ecologic-economic efficiency, railway transport system,image processing, machine learning and data analysis. Cristescu et al. [8] compared these concepts by using the digitization method of the plane R^2 into the grid $\mathtt{Z}^2$.

Definition 3. A functions φ on the convex set C is said to be convex, if

$$\varphi(u + \lambda(v - u)) \leq (1 - \lambda)\varphi(u) + \lambda\varphi(v), \quad \forall u, v \in C, \quad \lambda \in [0, 1].$$

It is known that the minimum $u \in C$ of the differentiable convex function φ is equivalent to fining $u \in$ such that

$$\langle \varphi\prime(u), v - u \rangle \geq 0, \quad \forall v \in C, (1) \tag{2.1}$$

which is called the variational inequality.

Definition 4. A function φ on the convex set C is said to be strongly convex, if there exists a constant $\alpha \geq 0$ such that

$$\varphi(u+\lambda(v-u)) \leq (1-\lambda)\varphi(u)+\lambda\varphi(v) \geq \alpha\|v-u\|^2, \quad \forall u,v \in C.$$

For the differentiable strongly convex functions, we have the following:

Lemma 1. *A differentiable function φ is strongly convex, if and only if,*

$$\varphi(v)-\varphi(u) \geq \langle \varphi'(u), v-u\rangle \geq \alpha\|v-u\|^2, \quad \forall u,v \in C.$$

For the differentiable convex functions, Bregman [4] introduced the distance function

$$\begin{aligned} B(v,u) &= \varphi(v)-\varphi(u) \geq \langle \varphi'(u), v-u\rangle, \quad \forall u,v \in C \\ &= \varphi(v)-\varphi(u) \geq \langle \varphi'(u), v-u\rangle \geq \alpha\|v-u\|^2, \quad \forall u,v \in C. \end{aligned}$$

which is known as the Bregman distance function and has applications in entropy, data analysis, information technology, machine learning and variational inequalities. Applying Lemma 1, we introduce the following new distance function

$$M(v,u) \quad = \quad M(v)-M(u), v-u\rangle, \quad \forall u,v \in C,$$

or, equivalently for strongly operator M with constant $\alpha \geq 0$, as

$$M(v,u) \quad = \quad M(v)-M(u), v-u\rangle \geq \alpha\|v-u\|^2, \quad \forall u,v \in C,$$

which is called the modified M-distance function. It is an interesting open problem to explore the applications of this equivalent M-distance function in information sciences, entropy, machine learning, data analysis and variational inequalities.

Definition 5. [15] A function φ on the general convex set C_g is said to be general (g-convex) convex with respect to an arbitrary function g, if

$$\varphi(u+\lambda(g(v)-u) \leq (1-\lambda)\varphi(u)+\lambda\varphi(g(v)), \quad \forall u,v \in C_g.$$

Clearly, every convex function is a general convex function, but the converse is not true. For $g(v)=mv$, the general homogenous convex function reduces to:

$$\varphi(u+\lambda(mv-u) \leq (1-\lambda)\varphi(u)+\lambda m\varphi(v), \quad \forall u,v \in C_m.$$

which is called the m- convex function as considered by Toader [34].
It has been shown [15] that $u \in C_g$ is the minimum of the differentiable generalized convex function φ, if and only if, $u \in C_g$ satisfies the inequality

$$\langle \varphi'(u), g(v)-u\rangle \geq 0, \quad \forall v \in C_g, \tag{2.2}$$

which is called the general variational inequality. It is worth mentioning that the inequalities (2.1) and (2.2) are quite and distinctly different from each other and have applications in various fields of pure and applied sciences.

Definition 6. A set $C_h \subseteq H$ is said to be a harmonic convex set, if

$$\frac{uv}{v+\lambda(u-v)} \in C_h, \quad \forall u,v \in C_h, \quad \lambda \in [0,1].$$

Definition 7. A function φ on the harmonic convex set C_h is said to be harmonic convex, if, if

$$\varphi(\frac{uv}{v+\lambda(u-v)}) \leq (1-\lambda)\varphi(u)+\lambda\varphi(v) \quad \forall u,v \in C_h, \quad \lambda \in [0,1].$$

The function φ is said to be a harmonic concave function, if $-\varphi$ is harmonic convex function.

We recall that the minimum of a differentiable harmonic convex function on the harmonic convex set C_h can be characterized by the variational inequality. This result is due to Noor and Noor [19].
For the differentiable harmonic convex function φ, $u \in C_h$ is a minimum of φ, if and only if, $u \in C_h$ satisfies the inequality.

$$\langle \varphi'(u), \frac{uv}{u-v} \rangle \geq 0, \quad \forall v \in C_h. \tag{2.3}$$

The inequality of type (2.3) is called the harmonic variational inequality.
From the above discussion, we note that the convex sets and convex functions have been generalized in different way to tackle the complicated problems. All these concepts are different and distinct from each other. It is natural to unify these concepts. We introduce and study some new classes of harmonic convex sets and harmonic convex functions, which is the main aim of this paper.

Definition 8. [5] A set C_{hg} is said to be a harmonic general convex set with respect to the an arbitrary function g, if

$$\frac{ug(v)}{g(v)+\lambda(u-g(v))} \in \mathcal{C}_{hg}, \quad \forall u, v \in C_{hg}, \quad \lambda \in [0,1].$$

Definition 9. A function ϕ on the harmonic convex set C_{hg} is said to be harmonic general convex with respect to the an arbitrary function g, if

$$\varphi(\frac{ug(v)}{g(v)+\lambda(u-g(v))}) \leq (1-\lambda)\varphi(u) + \lambda\varphi(g(v)), \quad \forall u, v \in C_{hg} \quad \lambda \in [0,1].$$

A function φ is said to be harmonic general concave function, if and only if, $-\varphi$ is harmonic general convex function.

Applying the technique of Noor et al. [19], one can show that the minimum of a differentiable harmonic general convex function on the harmonic general convex set C_{hg} can be characterized by the harmonic general variational inequality. For the sake of completeness and to convey the main idea, we include all the details.

Theorem 1. *Let φ be a differentiable general harmonic convex function on the harmonic general convex set C_{hg}. Then $\eta \in C_{hg}$ is a minimum of φ, if and only if, $u \in C_{hg}$ is the solution of the inequality*

$$\langle \varphi'(u), \frac{ug(v)}{u-g(v)} \rangle \geq 0, \quad \forall \xi \in C_{hg}, \tag{2.4}$$

which is called the harmonic general variational inequality.

Proof. Let $u \in C_{hg}$ is a minimum of φ. Then

$$\phi(u) \leq \phi(g(v)), \quad \forall v \in C_{hg}, \tag{2.5}$$

Since C_{hg} is a harmonic general convex se, for all $u, v \in C_{hg}$
$v_t = \frac{ug(v)}{g(v)+\lambda(u-g(v))} \in C_{hg}$. Replacing $g(v)$ by v_t) in (2.5) and diving by λ and taking limit as $\lambda \to 0$, we have

$$0 \leq \frac{\frac{ug(v)}{g(v)+\lambda(u-g(v))}) - \varphi(g(\eta))}{\lambda} = \langle \varphi'(u), \frac{ug(v)}{u-g(v)} \rangle$$

which is the required result (2.4).
Conversely, let the function φ be harmonic general convex function on the harmonic general convex set C_{hg}. Then

$$\varphi(\frac{ug(v)}{g(v)+\lambda(u-g(v)}) \leq \varphi(u)+\lambda(\varphi(g(v))-\varphi(u)),$$

which implies that

$$\begin{aligned}\varphi(g(\xi))-\varphi(g(\eta)) &\geq \lim_{\lambda\to 0}\frac{\varphi(\frac{ug(v)}{g(v)+\lambda(u)-g(v))})-\phi(u)}{\lambda}\\ &= \langle(\varphi)'(u),\frac{ug(v)}{u-g(v)}\rangle \geq 0, \quad \text{using} \quad (2.4).\end{aligned} \tag{2.6}$$

Consequently, it follows that

$$\varphi(u) \leq \varphi(g(v)), \quad \forall v \in C_{hg}.$$

This shows that $u \in C_{hg}$ is the minimum of the differentiability harmonic general convex function. □

One can define the Bregman distance general harmonic function as:

$$B(v,u)=\varphi(g(v))-\varphi(u) \geq \langle \varphi'(u),\frac{ug(v)}{u-g(v)}, \forall u,v \in C_{hg}. \tag{2.7}$$

From (2.7) and (2.4), it follows that $u \in C_{hg}$ is the minimum of the differentiable harmonic convex functions. It is amazing to observe that general harmonic variational inequality and the Bregman distance general harmonic convex function are closely related. We would like to mention that Theorem 1 implies that general harmonic optimization programming problem can be studied via the general harmonic variational inequality (2.4). Using the ideas and techniques of Theorem 1, we can derive the following result.

Theorem 2. *Let φ be a differentiable harmonic general convex functions on the harmonic general convex set C_{hg}. Then*

$$i \qquad \phi(g(v))-\phi(u) \geq \langle(\varphi'(u),\frac{ug(v)}{u-g(v)}\rangle, \quad \forall u,v \in C_{hg}.$$

$$ii \qquad \langle(\varphi'(u)-\varphi'(g(v)),\frac{ug(v)}{g(v)-u}\rangle \geq 0, \quad \forall u,v \in C_{hg}.$$

In many applications, the inequalities of the type (4) may not arise as the minimum of the differentiable general harmonic convex functions. These facts motivated us to consider more general harmonic variational inequality, which contains the inequalities (2.4) as a special case.

For given nonlinear continuous operators $T,g : H \to H$, we consider the problem of finding $u \in C_{hg}$ such that

$$\langle Tu,\frac{ug(v)}{u-g(v)}\rangle \geq 0, \forall v \in C_{hg} \tag{2.8}$$

which is called the general harmonic variational inequality.
We now discuss some new and known classes of variational inequalities and related

optimization problems.
i. If $g = I$, then the problem (2.8) reduces to finding $u \in C_{hg}$ such that

$$\langle Tu, \frac{uv}{u-v}\rangle \geq 0, \quad \forall v \in C_{hg} \tag{2.9}$$

which is called the harmonic variational inequality, see Noor and Noor [19]
ii. If $\langle Tu, \frac{ug(v)}{u-g(v)}\rangle = T(u, \frac{ug(v)}{u-g(v)}$, then problem (2.8) reduces to finding $u \in C_{hg}$ such that

$$T(u, \frac{ug(v)}{u-g(v)}) \geq 0, \quad \forall v \in C_{hg} \tag{2.10}$$

which ic called the directional harmonic variational inequality.
iii. For $\langle Tu, \frac{ug(v)}{u-g(v)} = B(Tu, \frac{ug(v)}{u-g(v)})$, the problem (2.8) reduces to finding $u \in C_{hg}$ such that

$$B(Tu, \frac{ug(v)}{u-g(v)}) \geq 0, \quad \forall v \in C_{hg}, \tag{2.11}$$

which is known as bifunction general harmonic variational inequality.
iv. If $C^*_{hg} = \{u \in H : \langle u, \frac{ug(v)}{u-g(v)}\rangle, \forall v \in C_{hg}$is a polar harmonic convex cone of the harmonic convex set, then the problem (2.8) is equivalent to fining $u \in H$, such that

$$\frac{ug(v)}{u-g(v)} \geq 0, \quad Tu \in C^*_{hg}, \quad \langle Tu, \frac{ug(v)}{u-g(v)} = 0,$$

is called the general harmonic complementarity problem. For the applications, numerical methods and other aspects of complementarity problems, see [6, 17, 30] and the references therein.
v. If $C_{hg} = H$, then the problem (2.8) is equivalent to fining $u \in H$, such that

$$\langle Tu, \frac{ug(v)}{u-g(v)}\rangle = 0, \quad \forall v \in H, \tag{2.12}$$

which is called the weak formulation of the nonlinear harmonic boundary value problem.
vi. For $Tu = T|u|$,, the problem (refeq2.12) reduces to finding u ? H such that

$$\langle T|u|, \frac{ug(v)}{u-g(v)}\rangle = 0, \quad \forall v \in H \tag{2.13}$$

which is called the system of absolute value general harmonic equations. One can easily show that the systems of absolute value equations considered are special cases of the problem (2.8). This shows that the problem (2.8) is quite and unified one.

3. Iterative methods and convergence analysis

It is worth mentioning that the projection, resolvent, descent methods and fixed point techniques can not applied to compute the approximate solutions of the harmonic variational inequalities. In this section, we apply the auxiliary principle technique involving an arbitrary operator to suggest and analyze some inertial iterative methods for solving general harmonic variational inequalities (2.8). This approach is mainly due to Glowinski et al. [9] and Lions et al.[11] as developed in [12, 14, 17, 18, 19, 20, 21, 22, 23, 24, 28, 29, 31].

For a given $u \in C_{hg}$ satisfying (2.8), consider the problem of finding $w \in C_{hg}$ such that

$$\langle T(w+\eta(u-w)), \frac{vg(w)}{v-g(w)}\rangle + \langle A(w)-A(u), v-w\rangle \geq 0, \quad \forall v \in C_{hg}. \qquad (3.1)$$

where $\rho, \eta \in [0,1]$ are constants and $A : H \to H$ is an arbitrary operator. Inequality of type (3.1) is called the auxiliary general harmonic variational inequality involving an arbitrary operator. This approach was introduced by Noor [14] to suggest some iterative methods for solving the variational inequities.

Remark 1. We now discuss three special cases of the auxiliary problem (15), which are being used frequently.

(I) For $A(u) = \Phi'(u)$, the derivative of the differentiable general harmonic function φ, the auxiliary problem (3.1) reduces to:

$$\langle T(w+\eta(u-w)), \frac{vg(w)}{v-g(w)}\rangle + \langle \varphi'(w)-\varphi'(u), v-w\rangle \geq 0, \quad \forall v \in C_{hg}.$$

which is known as the auxiliary principle involving the Bregman distance function problem. In this case, the general Bregman distance harmonic functions can be defined as

$$\begin{aligned} B(g(v),u) &= \Phi(g(v)) - \Phi(u) - \langle \Phi'(u), \frac{ug(v)}{u-g(v)}\rangle, \quad u,v \in Chg \\ &= \Phi(g(v)) - \Phi(u) - \langle \Phi'(u), \frac{ug(v)}{u-g(v)}\rangle \geq \zeta \| \frac{ug(v)}{u-g(v)} \|^2 \end{aligned}$$

using the strongly general harmonic convexity of the function Φ and $\zeta \geq 0$, a constant.
Or equivalently

$$B(g(v),u) = \Phi(g(v)) - \Phi(u) - \langle \Phi'(u), \frac{ug(v)}{u-g(v)}\rangle \geq \zeta \| \frac{ug(v)}{u-g(v)} \|^2, \quad \forall u,v \in C_{hg}.$$

Such type of the distance function was introduced and studied by Bregman [4] for differentiable convex functions.
(II) For any arbitrary operator A; we consider the distance function as:

$$B(g(v),u) = \langle Av - Au - \frac{ug(v)}{u-g(v)}\rangle, \quad u,v \in C_{hg},$$

from which, we can obtain

$$B(g(v),u) = \langle Av - Au - \frac{ug(v)}{u-g(v)}\rangle \geq \zeta \| \frac{u-g(v)}{ug(v)} \|^2, \forall u,v \in C_{hg},$$

if A is strongly monotone with constant $\zeta \geq 0$.
(III) If $A = I$, the identity operator, then the auxiliary problem (3.1) collapses to

$$\langle T(w+\eta(u-w)), \frac{vg(w)}{v-g(w)}\rangle + \langle w-u, v-w\rangle \geq 0, \quad \forall v \in C_{hg},$$

which is another auxiliary problem related to the problem (2.8).
This shows that the auxiliary problem (3.1) is quite flexible and unifying one.

We observe that, if $w = u$, then w is a solution of (2.8). This simple observation enables us to suggest the following iterative method for solving (2.8).

Algorithm 1. For given $u_0 \in C_{hg}$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho T(u_{n+1} + \eta(u_n - u_{n+1})), \frac{vg(u_{n+1})}{v - g(u_{n+1})}\rangle + \langle A(u_{n+1}) - A(u_n), v - u_{n+1}\rangle \geq 0, \quad \forall v \in C_{hg}.$$

Algorithm 1 is called the hybrid proximal point algorithm for solving the general harmonic variational inequalities(2.8).

Special Cases

We now discuss some special cases of Algorithm 1.
For $\eta = 0$, Algorithm 1 reduces to

Algorithm 2. For given $u_0 \in C_{hg}$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho Tu_{n+1}, \frac{vg(u_{n+1})}{v - g(u_{n+1})}\rangle + \langle A(u_{n+1}) - A(u_n), v - u_{n+1}\rangle \geq 0, \quad \forall v \in C_{hg}. \quad (3.2)$$

For $\eta = 1$, Algorithm 1 reduces to

Algorithm 3. For given $u_0 \in C_{hg}$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho Tu_n, \frac{vg(u_{n+1})}{v - g(u_{n+1})}\rangle + \langle A(u_{n+1}) - A(u_n), v - u_{n+1}\rangle \geq 0, \quad \forall v \in C_{hg}.$$

For $\eta = \frac{1}{2}$, Algorithm 1 reduces to

Algorithm 4. For given $u_0 \in C_{hg}$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho T(\frac{u_{n+1} + u_n}{2}), \frac{vg(u_{n+1})}{v - g(u_{n+1})}\rangle + \langle A(u_{n+1}) - A(u_n), v - u_{n+1}\rangle \geq 0, \quad \forall v \in C_{hg}.$$

which is called the mid-point proximal method for solving the problem (2.8).
For $A = I$, Algorithm 1 reduces to

Algorithm 5. For given $u_0 \in C_{hg}$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho T(u_{n+1} + \eta(u_n - u_{n+1}), \frac{vg(u_{n+1})}{v - g(u_{n+1})}\rangle + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0, \quad \forall v \in C_{hg}.$$

for solving general harmonic variational inequality.
For the convergence analysis of Algorithm 2, we recall the following concepts and results.

Definition 10. For $u, v, z \in H$, an operator T is said to be:
(i) general harmonic monotone with respect to an operator g, if and only if,

$$\langle Tu - Tv, \frac{ug(v)}{u - g(v)}\rangle \geq 0;$$

(ii) general harmonic pseudomonotone with respect to an operator g, if and only if,

$$\langle Tu, \frac{ug(v)}{u - g(v)}\rangle \geq 0 \Longrightarrow -\langle Tv, \frac{ug(v)}{u - g(v)}\rangle \geq 0;$$

(iii) partially relaxed strongly general harmonic monotone with respect to an operator g, if there exists a constant $\zeta > 0$ such that

$$\langle Tu - Tv, \frac{zg(v)}{z - g(v)}\rangle \geq \zeta\|z - u\|^2.$$

Note that, for $z = u$, partially relaxed strongly general harmonic monotonicity re- duces to monotonicity. It is known that partially relaxed strongly harmonic monotonicity is general monotone, but the converse is not true. It is known that general harmonic monotonicity implies general harmonic pseudomonotonicity, but the converse is not true. Consequently, the class of general harmonic pseudomonotone operators is bigger than the one of general harmonic monotone operators.
We now consider the convergence criteria of Algorithm 2.

Theorem 3. *Let $u \in C_{hg}$ be a solution of (2.8) and let η_{n+1} be the approximated solution obtained from Algorithm 2. Let T be a general harmonic speudomonotone operator. If the operator A is a strongly monotone operator with constant $\eta > 0$ and Lipschitz continuous with constant $\zeta > 0$, then*

$$\eta\|u_{n+1} - u\| \leq \zeta\|u_n - u\|. \tag{3.3}$$

Proof. Let $\eta \in C_{hg}$ be a solution of (2.8). Then

$$-\langle Tv, \frac{ug(v)}{u - g(v)}\rangle \geq 0, \quad \forall v \in C_{hg}, \tag{3.4}$$

since T is a general harmonic pseudomonotone operator.
Taking $v = u_{n+1}$ in (3.4), we have

$$\langle Tu_{n+1}, \frac{ug(u_{n+1})}{g(u_{n+1}) - u}\rangle \geq 0. \tag{3.5}$$

Taking $v = u$ in (3.2), we get

$$\langle \rho Tu_{n+1}, \frac{ug(u_{n+1})}{u - g(u_{n+1})}\rangle + \langle A(u_{n+1}) - A(u_n), u - u_{n+1}\rangle \geq 0, \quad \forall v \in C_{hg}. \tag{3.6}$$

From (3.5) and (3.8), we have

$$\langle Au_{n+1} - A_n, u - u_{n+1}\rangle \geq \langle \rho Tu_{n+1}, \frac{ug(u_{n+1})}{u - g(u_{n+1})}\rangle \geq 0. \tag{3.7}$$

from which, using (3.5), we have

$$\begin{aligned} 0 &\leq \langle Au_{n+1} - A_n, u - u_{n+1}\rangle = \langle Au_{n+1} - A_n, u - u_n + u_n - u_{n+1}\rangle \\ &= \langle Au_{n+1} - Au_n, u_n - u_{n+1}\rangle + \langle Au_{n+1} - Au_n, u - u_n\rangle \end{aligned}$$

Consequently

$$\langle Au_{n+1} - Au_n, u_{n+1}| - u_n\rangle \leq \langle Au_{n+1} - Au_n, u - u_n\rangle.$$

Since the operator A is strongly monotone with constant η and Lipschitz continuous with constant ζ, applying Cauchy-Scharzt inequality, we obtain

$$\eta\|u_{n+1} - u_n\|^2 \leq \|Tu_{n+1} - Tu_n\|\|u_{n+1} - u_n\| \leq \zeta\|u_{n+1} - u_n\|\|u - u_n\|.$$

This implies that

$$\eta\|u_{n+1} - u_n\| \leq \zeta\|u - u_n\|,$$

which is the required result (3.3). □

Theorem 4. *Let H be a finite dimensional space and all the assumptions of Theorem 3 hold. Then the sequence $\{u_n\}_1^\infty$ given by Algorithm 2 converges to a solution $u \in C_{hg}$ of the problem (2.8).*

Proof. Let $u \in C_{hg}$ be a solution of (2.8). From (3.3), we see that the sequence $\{\|\eta - \eta_n\|\}$ is nondecreasing and consequently, the sequence $\{u_n\}$ is bounded. Also from (3.3), we obtain

$$\frac{\eta}{\zeta}\sum_{n=0}^{\infty}\|u_n - u_{n+1}\| \leq \|u - u_0\|,$$

which implies that

$$\|u_n - u_{n+1}\| = 0. \tag{3.8}$$

Let $\hat{u}$ be the limit point of $\{u_n\}_0^\infty$, whose subsequent of $\{u_n\}_1^\infty$ of $\{u_n\}_0^\infty$ Replacing u_n by η_{n_j} in (3.2), taking the limit as $n_j \longrightarrow \infty$ and using (3.8), we obtain

$$\langle T\hat{u}, \frac{g(v)\hat{u}}{\hat{u} - g(v)}\rangle \geq 0, \quad \forall v \in C_{hg},$$

which shows that $\hat{u}$ solves the general harmonic variational inequality (2.8) and

$$\eta\|\hat{u} - u_{n+1}\| \leq \zeta\|\hat{u} - u_n\|.$$

Thus, it follows from the above inequality that the $\{u_n\}_1^\infty$ has exactly one limit point $\hat{u}$ and

$$\lim_{n\longrightarrow\infty} u_n = \hat{u}.$$

This is the required result. □

We again consider the auxiliary principle technique to suggest some hybrid inertial proximal point methods for solving the problem (2.8). Polyak [32] considered the inertial methods to speed up convergence of the iterative methods. For the applications of the inertial type methods and its variant forms, see [17, 20, 28, 29]. We again apply the auxiliary principle idea to suggest some inertial iterative methods for solving the harmonic inequalities (2.8).

For a given $u \in C_{hg}$ satisfying (2.8), consider the problem of finding $w \in C_{hg}$ such that

$$\langle T(w + \eta(u - w)), \frac{vg(w)}{v - g(w)}\rangle + \langle A(w) - A(u) + \alpha(u - u), v - w\rangle \geq 0, \quad \forall v \in C_{hg}. \tag{3.9}$$

where $\rho, \alpha, \eta \in [0, 1]$ are constants and $A : H \rightarrow H$ is an arbitrary operator.
Clearly, for $w = u$, w is a solution of (2.8). This fact motivated us to suggest the following inertial iterative method for solving (2.8).

Algorithm 6. For given $u_0, u_1 \in C_{hg}$, compute the approximated solution u_{n+1} by the iterative scheme

$$\langle \rho T(u_{n+1} + \eta(u_n - u_{n+1})), \frac{vg(u_{n+1})}{v - g(u_{n+1})}\rangle + \langle A(u_{n+1}) - A(u_n) + \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle \geq 0, \quad \forall v \in C_{hg}.$$

For $\alpha = 0$, Algorithm 6 is exactly Algorithm 1. Using essentially the technique of Theorem 3 and Noor [17], one can study the convergence of Algorithm 6.
If $\eta = \frac{1}{2}, \eta = 1, \eta = 0$, then Algorithm 6 reduces to the following inertial iterative schemes for solving the problem (2.8).

Algorithm 7. For given $u_0, u_1 \in C_{hg}$, compute the approximated solution u_{n+1} by the iterative scheme

$$\langle \rho T(\frac{u_{n+1} + u_n}{2}), \frac{vg(u_{n+1})}{v - g(u_{n+1})}\rangle$$
$$+\langle A(u_{n+1}) - A(u_n) + \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle \geq 0, \quad \forall v \in C_{hg},$$

which is known as the hybrid inertial mid point proximal method.

Algorithm 8. For given $u_0, u_1 \in C_{hg}$, compute the approximated solution u_{n+1} by the iterative scheme

$$\langle \rho T u_n, \frac{vg(u_{n+1})}{v - g(u_{n+1})}\rangle$$
$$+\langle A(u_{n+1}) - A(u_n) + \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle \geq 0, \quad \forall v \in C_{hg},$$

which is known as the hybrid inertial explicit iterative method.

Algorithm 9. For given $u_0, u_1 \in C_{hg}$, compute the approximated solution u_{n+1} by the iterative scheme

$$\langle \rho T u_{n+1}, \frac{vg(u_{n+1})}{v - g(u_{n+1})}\rangle$$
$$+\langle A(u_{n+1}) - A(u_n) + \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle \geq 0, \quad \forall v \in C_{hg},$$

which is known as the hybrid inertial implicit iterative method.

For different and appropriate values of the parameters η, α, the operators T, A and spaces, one can obtain a wide class of inertial type iterative methods for solving the general harmonic variational inequalities and related optimization problems.

Conclusion. Some new classes of general harmonic variational inequalities are introduced in this paper. It is shown that several important problems such as harmonic complementarity problems, system of harmonic absolute value problems and related problems can be obtained as special cases. The auxiliary principle technique involving an arbitrary operator is applied to suggest several inertial type methods for solving general harmonic variational inequalities with suitable modifications. We note that this technique is independent of the projection and the resolvent of the operator. Moreover, we have studied the convergence analysis of these new methods under weaker conditions. Applications of the fuzzy set theory, stochastic, quantum calculus, fractal, fractional and random can be found in many branches of mathematical and engineering sciences including artificial intelligence, computer science, control engineering, management science, operations research and variational inequalities. One may explore these aspects for the harmonic variational inequality and its variant forms. We have only considered the theoretical aspects of the hybrid inertial proximal methods. It is an interesting problem to implement these methods numerically and compare with other iterative schemes.

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