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ON LACUNARY STATISTICAL BOUNDEDNESS OF ORDER α OF GENERALIZED DIFFERENCE SEQUENCES

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Abstract. In this work, a new generalization of statistical boundedness is provided for difference sequences in regard to lacunary $\alpha-$density and lacunary statistical sense. Apart from examining some inclusion theorems on related sequence spaces, we show that $\Delta_{\theta}^{m}(S_{b}^{\alpha})$ does not form a sequence algebra unlike $S_{\theta}^{\alpha}(b)$.

1. Introduction

In 1935, Zygmund [34] mentioned the idea of statistical convergence in the first edition of his monograph puplished in Warsaw. This concept, as a generalized type of ordinary convergence, appeared in the papers of Steinhaus [31] and Fast [13] independently. Later, Schoenberg [30] studied it as a summability method. Researches on statistical convergence have had an obvious rise after the frequently cited papers of Salat [28] and Fridy [15]. It was further investigated from the sequence space point of view and linked with summability theory by Bhardwaj et al. [4, 5], Braha et al. [6, 7], Colak [8], Connor [9], Et et al. [3, 20, 32, 33], Fridy and Orhan [16], Işık et al. [2, 21, 22], Mohiuddine et al. [25], Mursaleen [26], Rath and Tripathy [27] and many others.

Before advancing more, we pause to compile some notation and definitions. Throughout the sequel, we let
$\omega :=$ the set of all real (or complex) valued sequences
$\ell_{\infty} :=$ the set of all bounded sequences
$c :=$ the set of all convergent sequences
$c_0 :=$ the set of all null sequences
$\chi :=$ the set of all sequences of *zero*s and *one*s.

We also would like to recall some concepts on seqeunce spaces that will take place in following sections. A sequence space X is said to be

Solid (or *normal)*, if $(\alpha_k x_k) \in X$ for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$, whenever $(x_k) \in X$

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Monotone, if $(u_k x_k) \in X$ for all $(u_k) \in \chi$ and $(x_k) \in X$
Symmetric, if $(x_k) \in X$ implies $(x_{\pi(k)}) \in X$, where π is any permutation of $\mathbb{N}$
Sequence algebra, if $(x_k y_k) \in X$ whenever (x_k) and $(y_k) \in X$.

The definition of statistical convergence relies on the density of subsets of the set $\mathbb{N}$ of natural numbers. The density of a subset $\mathbb{E}$ of $\mathbb{N}$ is defined by

$$\delta(\mathbb{E}) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{\mathbb{E}}(k), \quad \text{provided that the limit exists.}$$

A sequence $x = (x_k)$ is said to be statistically convergent to L if $\delta\left(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}\right) = 0$ for every $\varepsilon > 0$.

The concept of statistical boundedness was given by Fridy and Orhan [17] as follows:

The real number sequence $x = (x_k)$ is statistically bounded if there is a number $M \geq 0$ such that $\delta\left(\{k : |x_k| > M\}\right) = 0$.

It is well known that every bounded sequence is statistically bounded, but the converse is not true.

The order of statistical convergence of a sequence of numbers was given by Gadjiev and Orhan [18] and afterwards statistical convergence of order α was studied by Çolak [8].

By a lacunary sequence we mean an increasing sequence $\theta = (k_r)$ of non-negative integers such that $k_0 = 0$ and $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r, and $q_1 = k_1$ for convenience. In recent years, lacunary sequences have been studied in [4, 7, 11, 14, 16, 24].

Study of difference sequence spaces is quite new in summability theory. This concept was introduced by Kızmaz [23] and generalized by Et and Çolak [10]. Afterwards Et and Nuray [12] studied it in order to mainly generalize statistical convergence with respect to Δ^m difference operator as follows

$$\Delta^m(X) = \{x = (x_k) : (\Delta^m x_k) \in X\}$$

where X is any sequence space, $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^m x = (\Delta^m x_k) = \left(\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}\right)$ and so $\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}$.

If $x \in \Delta^m(X)$ then there exists one and only one $y = (y_k) \in X$ such that $y_k = \Delta^m x_k$ and

$$x_k = \sum_{v=1}^{k-m} (-1)^m \binom{k-v-1}{m-1} y_v = \sum_{v=1}^{k} (-1)^m \binom{k+m-v-1}{m-1} y_{v-m}, \quad (1)$$
$$y_{1-m} = y_{2-m} = \cdots = y_0 = 0$$

for sufficiently large k, for instance $k > 2m$. For more information reader is referred to [1, 19, 29, 32].

In this paper we introduce Δ^m-lacunary statistical boundedness of order α and give some inclusion theorems on this concept. We wish to state that main definitions and results generalize that of some former works mentioned above.

2. Main Results

In this section we present the main results of the paper.

Definition 2.1. [11] *Let $\theta = (k_r)$ be a lacunary sequence and $0 < \alpha \leq 1$ be given. The lacunary $\alpha-$density of a subset $\mathbb{E}$ of $\mathbb{N}$ is defined by*

$$\delta_\theta^\alpha(\mathbb{E}) = \lim_{r\to\infty} \frac{1}{h_r^\alpha} \left|\{k_{r-1} < k \leq k_r : k \in \mathbb{E}\}\right| \text{ provided the limit exists.}$$

Lacunary $\alpha-$density $\delta_\theta^\alpha(\mathbb{E})$ reduces to natural density $\delta(\mathbb{E})$ in the special case $\alpha = 1$ and $\theta = (2^r)$.

Throughout this work, $|.|$ will denote the cardinality of the enclosed set.

Proposition 2.2. [11] *Let $\theta = (k_r)$ be a lacunary sequence and $\alpha, \beta \in (0, 1]$ such that $\alpha \leq \beta$, then $\delta_\theta^\alpha(\mathbb{E}) \leq \delta_\theta^\beta(\mathbb{E})$.*

Definition 2.3. [1] *Let $\theta = (k_r)$ be a lacunary sequence and $0 < \alpha \leq 1$ be given. The sequence $x = (x_k) \in \omega$ is said to be Δ^m-lacunary statistically convergent of order α (or $\Delta_\theta^m(S^\alpha)-$convergent to L) if there is a real number L such that*

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \left|\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}\right| = 0 \text{ for each } \varepsilon > 0 \tag{2}$$

where h_r^α denotes the α^{th} power $(h_r)^\alpha$ of h_r, that is $h^\alpha = (h_r^\alpha) = (h_1^\alpha, h_2^\alpha, ..., h_r^\alpha, ...)$.

In this case we write $\Delta_\theta^m(S^\alpha) - \lim x_k = L$. By $\Delta_\theta^m(S^\alpha)$, we will denote the set of such sequences.

In the sequel, $\Delta_\theta^m(S^\alpha)$ shall be replaced by
$\Delta^m(S^\alpha)$ for the special case $\theta = (2^r)$,
$\Delta_\theta^m(S)$ for the special case $\alpha = 1$,
$\Delta^m(S)$ for the special cases $\theta = (2^r)$ and $\alpha = 1$.

Now we provide the main definition of this work.

Definition 2.4. *Let $\theta = (k_r)$ be a lacunary sequence and $0 < \alpha \leq 1$ be given. The sequence $x = (x_k) \in \omega$ is said to be Δ^m-lacunary statistically bounded of order α (or $\Delta_\theta^m(S_b^\alpha)-$bounded) if there is an $M \geq 0$ such that*

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \left|\{k \in I_r : |\Delta^m x_k| > M\}\right| = 0. \tag{3}$$

By $\Delta_\theta^m(S_b^\alpha)$, we will denote the set of such sequences.

In the following, $\Delta_\theta^m(S_b^\alpha)$ shall be replaced by

$\Delta^m(S_b^\alpha)$, which was studied in [32], for the special case $\theta = (2^r)$,
$\Delta_\theta^m(S_b)$ for the special case $\alpha = 1$,
$\Delta^m(S_b)$ for the special cases $\theta = (2^r)$ and $\alpha = 1$.

We begin with a basic theorem between the spaces $\Delta^m(\ell_\infty)$ and $\Delta_\theta^m(S_b^\alpha)$.

Theorem 2.5. *Every Δ^m-bounded sequence is Δ^m-lacunary statistically bounded of order α, but the converse is not true.*

Proof. Let $\theta = (k_r)$, $0 < \alpha \leq 1$ be given and $x \in \Delta^m(\ell_\infty)$. Then there exists such a number $M \geq 0$ that $|\Delta^m x_k| \leq M$ for every $k \in \mathbb{N}$. So $\{k \in I_r : |\Delta^m x_k| > M\} = \emptyset$ which yields

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r : |\Delta^m x_k| > M\}| = 0.$$

Thus, x is $\Delta_\theta^m(S_b^\alpha)-$bounded. To show the strictness, take $\theta = (2^r)$, $\alpha = 1$, $m = 1$ and define x as follows:

$$x_k = \begin{cases} 0, & k = 1, \\ \frac{n(1-n)}{2}, & (n-1)! + 1 \leq k \leq n!. \end{cases} \tag{4}$$

We obtain $\Delta x_k = \begin{cases} n, & k = n!, \\ 0, & \text{else}, \end{cases}$ which means $x \in \Delta(Sc_o) \subset \Delta(S_b)$ $(= \Delta_\theta(S_b^\alpha))$. But it is obvious that x is not $\Delta-$bounded. □

Theorem 2.5 yields the following result.

Corollary 2.6. *Every Δ^m-convergent sequence is Δ^m-lacunary statistically bounded of order α, not conversely.*

Theorem 2.7. *Δ^m-lacunary statistically convergent sequences of order α are strictly included by Δ^m-lacunary statistically bounded sequences of order α.*

Proof. Let $x \in \Delta_\theta^m(S^\alpha)$ and $\varepsilon > 0$ be given. Then there exists an $L \in \mathbb{R}$ such that

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}| = 0.$$

The result follows from the following inequality

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r : |\Delta^m x_k| > |L| + \varepsilon\}| \leq \lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}|.$$

For the opposite, let $\theta = (2^r)$ be given and consider the sequence $y = (y_k)$ defined by

$$y_k = \begin{cases} 1, & k = 2n \\ -1, & k \neq 2n \end{cases} \quad k, n \in \mathbb{N}. \tag{5}$$

In view of (1) we can determine a sequence x such that $\Delta^m x_k = y_k$. Then we have $x \in \Delta_\theta^m(S_b^\alpha) \setminus \Delta_\theta^m(S^\alpha)$. □

Theorem 2.7 yields the following result.

Corollary 2.8. *Every Δ^m-lacunary statistically convergent sequence is Δ^m-lacunary statistically bounded, but the converse is not true.*

Theorem 2.9.

i) *$\Delta_\theta^m(S_b^\alpha)$ is not symmetric,*
ii) *Although $S_\theta^\alpha(b)$ is normal and monotone, $\Delta_\theta^m(S_b^\alpha)$ is not normal and monotone,*
iii) *Although $S_\theta^\alpha(b)$ is a sequence algebra $\Delta_\theta^m(S_b^\alpha)$ is not a sequence algebra.*

Proof. (i) Let denote the sequence $x=(x_k)$ as follows:

$$(0,-1,-1,-1,-3,-3,-3,-3,-3,-6,-6,-6,-6,-6,-6,-6,\cdots)$$

and take $m=1$. Since $\Delta x=(1,0,0,2,0,0,0,0,3,0,0,..)\in S_\theta^\alpha(b)$ we have $x\in\Delta_\theta(S_b^\alpha)$ where $\theta=(2^r)$. Now define sequence $y=(y_k)$ as a rearrangement of x by $y=(y_k)=(0,-1,-3,-1,-3,-1,-3,-6,-3,-6,-3,-6,..)$ which yields to $\Delta y=(1,2,-2,2,-2,2,3,-3,3,-3,...)$. We observe

$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\left|\{k\in I_r:|\Delta y_k|>M\}\right|\neq 0 \text{ for any } M\geq 0.$$

This means $(x_{\pi(k)})=(y_k)\notin\Delta_\theta(S_b^\alpha)$ and so $\Delta_\theta(S_b^\alpha)$ is not symmetric.

(ii) Take the sequence x defined in part **i**. We showed $x\in\Delta_\theta(S_b^\alpha)$ when $m=1$ and $\theta=(2^r)$. Now picking the sequence $u=(u_k)=(0,1,0,1,...)\in\chi$ we obtain $ux=(0,-1,0,-1,0,-3,0,-3,0,-6,0,-6,0,...)$ which is not in $\Delta_\theta(S_b^\alpha)$ since

$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\left|\{k\in I_r:|\Delta(ux)_k|>M\}\right|\neq 0 \text{ for any } M\geq 0$$

where $\Delta(ux)_k=(1,-1,1,-1,3,-3,3,-3,6,-6,...)$. Thus $\Delta_\theta(S_b^\alpha)$ is not monotone. It follows $\Delta_\theta(S_b^\alpha)$ is not normal from the fact that any normal space is monotone.

(iii) In view of (1) we can construct a sequence $x=(x_k)\in\omega$ such that

$$\Delta^m x_k=\begin{cases} n, & k=n^2 \\ 0, & \text{else}\end{cases}$$

for all $k,\ n\in\mathbb{N}$. Following part (i), we get $x\in\Delta_\theta^m(S_b^\alpha)$ where $\theta=(2^r)$. Now consider the sequence $y=(y_k)=(1,2,3,..)$. Due to y being Δ^m-bounded it is obvious that $y\in\Delta_\theta^m(S_b^\alpha)$. However, we observe $yx=(kx_k)_{k=1}^\infty\notin\Delta_\theta^m(S_b^\alpha)$. Thus $\Delta_\theta^m(S_b^\alpha)$ is not a sequence algebra. □

In the next result we establish an inclusion theorem explaining the relationship between the spaces of Δ^m-lacunary statistically bounded sequences of distinct orders.

Theorem 2.10. *If $0<\alpha\leq\beta\leq 1$ and $\theta=(k_r)$ is a lacunary sequence. Then $\Delta_\theta^m(S_b^\alpha)\subseteq\Delta_\theta^m(S_b^\beta)$ and the inclusion is strict.*

Proof. The first part of proof is straightforward. To show the opposite, observe there exists some sequence $x=(x_k)\in\omega$ such that

$$\Delta^m x_k=\begin{cases} \left[\sqrt{h_r}\right], & k=1,2,3,...,\left[\sqrt{h_r}\right] \\ 0, & \text{else}\end{cases} \quad \text{by (1).} \tag{6}$$

Then $x\in\Delta_\theta^m(S_b^\beta)$ for $\frac{1}{2}<\beta\leq 1$ but $x\notin\Delta_\theta^m(S_b^\alpha)$ for $0<\alpha\leq\frac{1}{2}$. □

Theorem 2.10 yields the following corollary.

Corollary 2.11.

i) *If a sequence is Δ^m-lacunary statistically bounded of order α, then it is Δ^m-lacunary statistically bounded.*
ii) *If a sequence is Δ^m-statistically bounded of order α, then it is Δ^m-statistically bounded of order β.*

iii) *If a sequence is* Δ^m-*statistically bounded of order* α*, then it is* Δ^m-*statistically bounded.*

In the next two theorems, we present some certain conditions on lacunary sequence θ so that the inclusions $\Delta^m(S_b^\alpha) \subset \Delta_\theta^m(S_b^\alpha)$ and $\Delta_\theta^m(S_b^\alpha) \subset \Delta^m(S_b^\alpha)$ occur respectively.

Theorem 2.12. *Let* $0 < \alpha \leq 1$ *and* $\theta = (k_r)$ *be a lacunary sequence. If* $\liminf_r q_r > 1$*, then* $\Delta^m(S_b^\alpha) \subset \Delta_\theta^m(S_b^\alpha)$.

Proof. Suppose that $\liminf_r q_r > 1$, then there exists a $\delta > 0$ such that $q_r \geq 1+\delta$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta} \Longrightarrow \left(\frac{h_r}{k_r}\right)^\alpha \geq \left(\frac{\delta}{1+\delta}\right)^\alpha \Longrightarrow \frac{1}{k_r^\alpha} \geq \frac{\delta^\alpha}{(1+\delta)^\alpha}\frac{1}{h_r^\alpha}.$$

If $x \in \Delta^m(S_b^\alpha)$, then there exists an $M \geq 0$ such that

$$\begin{aligned}&\frac{1}{k_r^\alpha}\left|\{k \leq k_r : |\Delta^m x_k| > M\}\right| \\ &\geq \frac{1}{k_r^\alpha}\left|\{k \in I_r : |\Delta^m x_k| > M\}\right| \geq \frac{\delta^\alpha}{(1+\delta)^\alpha}\frac{1}{h_r^\alpha}\left|\{k \in I_r : |\Delta^m x_k| > M\}\right|\end{aligned}$$

holds for sufficiently large r. Taking limit as $r \to \infty$ we get $x \in \Delta_\theta^m(S_b^\alpha)$. □

Theorem 2.13. *Let* $0 < \alpha \leq 1$ *and* $\theta = (k_r)$ *be a lacunary sequence. If* $\limsup_r q_r < \infty$*, then* $\Delta_\theta^m(S_b^\alpha) \subset \Delta^m(S_b^\alpha)$.

Proof. Omitted. □

Combining Theorem 2.12 and Theorem 2.13 we have what follows.

Corollary 2.14. *Let* $\theta = (k_r)$ *be a lacunary sequence such that* $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$*. Then* $\Delta_\theta^m(S_b^\alpha) = \Delta^m(S_b^\alpha)$.

Theorem 2.15. $\Delta^m(S_b^\alpha) = \bigcap_{\liminf_r q_r > 1} \Delta_\theta^m(S_b^\alpha) = \bigcup_{\limsup_r q_r < \infty} \Delta_\theta^m(S_b^\alpha)$

Proof. $\Delta^m(S_b^\alpha)$ being included by $\bigcap_{\liminf_r q_r > 1} \Delta_\theta^m(S_b^\alpha)$ is a direct result of Theorem 2.12. Now let $\theta = (k_r)$ be Fibonacci sequence with $k_0 = 0$, $k_1 = 1$, $k_2 = 2$ and $k_r = k_{r-2} + k_{r-1}$ for $r \geq 3$. Then $\lim_r q_r \cong 1.618$, the golden ratio. Suppose $x \notin \Delta^m(S_b^\alpha)$ which implies that $x \notin \Delta_\theta^m(S_b^\alpha)$ by Corollary 2.14. This follows that $x \in \bigcup_{\liminf_r q_r > 1} [\Delta_\theta^m(S_b^\alpha)]^c = \left[\bigcap_{\liminf_r q_r > 1} \Delta_\theta^m(S_b^\alpha)\right]^c$ and so $x \notin \bigcap_{\liminf_r q_r > 1} \Delta_\theta^m(S_b^\alpha)$. Thus $\Delta^m(S_b^\alpha) = \bigcap_{\liminf_r q_r > 1} \Delta_\theta^m(S_b^\alpha)$. The remaining equality can be proved analogously hence is omitted. □

Theorem 2.16. *If we have*

$$\lim_{r\to\infty}\inf\frac{h_r^{\alpha}}{k_r}>0, \tag{7}$$

then $\Delta^m(S_b)\subset\Delta_\theta^m(S_b^\alpha)$.

Proof. Let $x\in\Delta^m(S_b)$. Then, knowing $k_r \overset{r\to\infty}{\to} \infty$, there exists some $M\geq 0$ so that

$$\lim_{r\to\infty}\frac{1}{k_r}\left|\{k\leq k_r:|\Delta^m x_k|>M\}\right|=0.$$

Besides, the inclusion

$$\{k\leq k_r:|\Delta^m x_k|>M\}\supset\{k\in I_r:|\Delta^m x_k|>M\}$$

is true. Therefore,

$$\begin{aligned}\frac{1}{k_r}\left|\{k\leq k_r:|\Delta^m x_k|>M\}\right| &\geq \frac{1}{k_r}\{k\in I_r:|x_k|>M\}\\ &=\frac{h_r^\alpha}{k_r}\frac{1}{h_r^\alpha}\{k\in I_r:|\Delta^m x_k|>M\}\ \text{ for all } r\in\mathbb{N}.\end{aligned}$$

Taking limit as $r\to\infty$ and using (7), we get $x\in\Delta_\theta^m(S_b^\alpha)$. □

Theorem 2.17. *Let* $\theta=(k_r)$ *and* $\theta'=(s_r)$ *be two lacunary sequences such that* $I_r\subset J_r$ *for all* $r\in\mathbb{N}$ *where* $I_r=(k_{r-1},k_r]$, $J_r=(s_{r-1},s_r]$, $h_r=k_r-k_{r-1}$ *and* $\ell_r=s_r-s_{r-1}$ *and let* α *and* β *be such that* $0<\alpha\leq\beta\leq 1$.

(*i*) *If*

$$\lim_{r\to\infty}\inf\frac{h_r^\alpha}{\ell_r^\beta}>0 \tag{8}$$

then $\Delta_{\theta'}^m(S_b^\beta)\subset\Delta_\theta^m(S_b^\alpha)$.

(*ii*) *If*

$$\lim_{r\to\infty}\frac{\ell_r}{h_r^\beta}=1 \tag{9}$$

then $\Delta_\theta^m(S_b^\alpha)\subset\Delta_{\theta'}^m(S_b^\beta)$.

Proof. Omitted. □

From Theorem 2.17, we derive the following results.

Corollary 2.18. *If the condition* (8) *is satisfied, then*

(*i*) $\Delta_{\theta'}^m(S_b^\alpha)\subset\Delta_\theta^m(S_b^\alpha)$ *for each* $\alpha\in(0,1]$,

(*ii*) $\Delta_{\theta'}^m(S_b)\subset\Delta_\theta^m(S_b^\alpha)$ *for each* $\alpha\in(0,1]$,

(*iii*) $\Delta_{\theta'}^m(S_b)\subset\Delta_\theta^m(S_b)$.

Furthermore, if the condition (9) *is satisfied, then*

(*i*) $\Delta_\theta^m(S_b^\alpha)\subset\Delta_{\theta'}^m(S_b^\alpha)$ *for each* $\alpha\in(0,1]$,

(*ii*) $\Delta_\theta^m(S_b^\alpha)\subset\Delta_{\theta'}^m(S_b)$ *for each* $\alpha\in(0,1]$,

(*iii*) $\Delta_\theta^m(S_b)\subset\Delta_{\theta'}^m(S_b)$.

In the following, we give inclusion results regarding different lacunary methods in a more generally described way. Before fulfilling that we recall a concept defined in [14]:

A lacunary sequence $\theta' = (s_r)$ is named to be a *lacunary refinement* of another lacunary sequence $\theta = (k_r)$ provided that $(k_r) \subseteq (s_r)$.

Theorem 2.19. *Let θ' be a lacunary refinement of θ and $\alpha \in (0,1]$. If there exists some $\eta > 0$ such that*

$$\frac{|J_j|}{|I_i|} \geq \sqrt[\alpha]{\eta} \text{ for every } J_j \subseteq I_i,$$

Then $\Delta_\theta^m(S_b^\alpha) \subset \Delta_{\theta'}^m(S_b^\alpha)$.

Proof. Let $x = (x_k) \in \Delta_\theta^m(S_b^\alpha)$. This implies that there exists some $M > 0$ such that

$$\lim_{r\to\infty} \frac{1}{|I_r|^\alpha} \left|\{k \in I_r : |\Delta^m x_k| > M\}\right| = 0.$$

Besides, we can find some I_i such that $J_j \subseteq I_i$ for every J_j. Now we have what follows:

$$\begin{aligned}
\frac{1}{|J_j|^\alpha} \left|\{k \in J_j : |\Delta^m x_k| > M\}\right| &= \left(\frac{|I_i|}{|J_j|}\right)^\alpha \frac{1}{|I_i|^\alpha} \left|\{k \in J_j : |\Delta^m x_k| > M\}\right| \\
&\leq \left(\frac{|I_i|}{|J_j|}\right)^\alpha \frac{1}{|I_i|^\alpha} \left|\{k \in I_i : |\Delta^m x_k| > M\}\right| \\
&\leq \left(\frac{1}{\eta}\right) \frac{1}{|I_i|^\alpha} \left|\{k \in I_i : |\Delta^m x_k| > M\}\right|.
\end{aligned}$$

Taking limit as $i \to \infty$ we complete the proof. □

In the remainder we discuss inclusion results of Δ^m-lacunary statistical boundedness of order α via different lacunary methods for $\alpha = 1$ case. We leave the case $\alpha \in (0,1)$ as an open problem.

Proposition 2.20. *If θ' is a lacunary refinement of θ then $\Delta_{\theta'}^m(S_b) \subseteq \Delta_\theta^m(S_b)$.*

Proof. The inclusion follows from Lemma 2.3 of [12] and Theorem 4.1 of [4]. □

Theorem 2.21. *Suppose $\theta' = (s_r)$ and $\theta = (k_r)$ are two arbitrary lacunary sequences with intervals $J_r = (s_{r-1}, s_r]$ and $I_r = (k_{r-1}, k_r]$ respectively. Let $I_{ij} = I_i \cap J_j$, $i, j = 1, 2, 3, \ldots$. If there is some $\eta > 0$ such that*

$$\frac{|I_{ij}|}{|I_i|} \geq \eta \text{ for every } i, j = 1, 2, 3, \ldots$$

provided $I_{ij} \neq \emptyset$, then $\Delta_\theta^m(S_b) \subset \Delta_{\theta'}^m(S_b)$.

Proof. Let $\theta'' = \theta' \cup \theta$. Then θ'' is lacunary refinment of θ' and θ both. Therefore, the set $\{I_{ij} = I_i \cap J_j : I_{ij} \neq \emptyset\}$ forms the sequence of intervals for θ''. In view of Theorem 10, since $\frac{|I_{ij}|}{|I_i|} \geq \eta$ for every $i, j = 1, 2, 3, ...$, provided $I_{ij} \neq \emptyset$ we get the inclusion $\Delta_\theta^m(S_b) \subset \Delta_{\theta''}^m(S_b)$. Moreover, it follows from Proposition 2 that $\Delta_{\theta''}^m(S_b) \subset \Delta_{\theta'}^m(S_b)$ as θ'' is also lacunary refinment of θ'. Hence $\Delta_\theta^m(S_b) \subset \Delta_{\theta'}^m(S_b)$. $\square$

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