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# INEQUALITIES INVOLVING NUMERICAL RADIUS AND OPERATOR NORM FOR A CLASS OF OPERATORS RELATED TO $(\alpha,\beta)$-CLASS $(\mathcal{Q})$ OPERATORS

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Abstract. In this paper, we study some properties of the class of operators related to the class $(\alpha,\beta)$-Class $(\mathcal{Q})$ operators. Some inequalities linked to concepts of numerical radius and norm of $m$-quasi-$(\alpha,\beta)$-Class $(\mathcal{Q})$ operators..

## 1. Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space. If $\mathbf{R}\in\mathcal{B}\big[\mathcal{H}\big]$ , we denote by $\mathbf{R}^*$ its adjoint, $\mathbf{Ran}(\mathbf{R})$ its range and $\ker(\mathbf{R})$ for its kernel. Any operator $\mathbf{R}\in\mathcal{B}\big[\mathcal{H}\big]$ is said to be

(1) Class $(\mathcal{Q})$ operator if $\mathbf{R}^{*2}\mathbf{R}^2=\big(\mathbf{R}^*\mathbf{R}\big)^2$ or $\|\mathbf{R}^2w\|=\|\mathbf{R}^*\mathbf{R}w\|\quad\forall\, w\in\mathcal{H}$ ([16]),

(2) Almost Class $(\mathcal{Q})$ if $\big(\mathbf{R}^*\mathbf{R}\big)^2\leq\big(\mathbf{R}^*\big)^2\mathbf{R}^2\quad\big(\|\mathbf{R}^*\mathbf{R}w\|\leq\|\mathbf{R}^2w\|\quad\forall\; w\in\mathcal{H}\big)$ ([26]),

(3) $(n,m)$-power class $(\mathcal{Q})$ if $\big(\mathbf{R}^{*m}\mathbf{R}^n\big)^2=\big(\mathbf{R}^{*m}\big)^2\mathbf{R}^{2n}$ ([27]),

(4) $n$-Almost Class $(\mathcal{Q})$ operators if $\big(\mathbf{R}^*\mathbf{R}^n\big)^2\leq\big(\mathbf{R}^*\big)^2\mathbf{R}^{2n}$ ([26]),

(5) $(n,m)$-Almost Class $(\mathcal{Q})$ operators if $\big(\mathbf{R}^{*m}\mathbf{R}^n\big)^2\leq\big(\mathbf{R}^{*m}\big)^2\mathbf{R}^{2n}$ ([26]),

(6) $(\alpha,\beta)$-Class $(\mathcal{Q})$ operators $(0\leq\alpha\leq1\leq\beta)$ ([25]) if

$$\alpha^2\mathbf{R}^{*2}\mathbf{R}^2\leq\big(\mathbf{R}^*\mathbf{R}\big)^2\leq\beta^2\mathbf{R}^{*2}\mathbf{R}^2,$$

or

$$\alpha\|\mathbf{R}^2w\|\leq\|\mathbf{R}^*\mathbf{R}w\|\leq\beta\|\mathbf{R}^2w\|\quad\forall\, w\in\mathcal{H},$$

(7) $(\alpha,\beta)$-normal operator $(0\leq\alpha\leq1\leq\beta)$ ( [11], [8], [21]) if

$$\alpha^2\mathbf{R}^*\mathbf{R}\leq\mathbf{R}\mathbf{R}^*\leq\beta^2\mathbf{R}^*\mathbf{R},$$

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(8) $m$-quasi-$(\alpha, \beta)$-normal operator $(0 \leq \alpha \leq 1 \leq \beta)$ ([22]) if

$$\alpha^2\big(\mathbf{R}^*\big)^{m+1}\mathbf{R}^{m+1} \leq \big(\mathbf{R}^*\big)^m\mathbf{R}\mathbf{R}^*\big(\mathbf{R}\big)^m \leq \beta^2\big(\mathbf{R}^*\big)^{m+1}\mathbf{R}^{m+1}.$$

There are many classes of operators that have been studied in [14, 19, 15, 24].

The authors in [2] has presented a new class of operators termed as $m$-quasi-$(\alpha, \beta)$-Class $(\mathcal{Q})$ operators parallel to $(\alpha, \beta)$-normal operators ([3, 8, 21]) and $m$-quasi-$(\alpha, \beta)$-normal operators ([20, 22]).

The paper will be organized two section. In section 2 we study some additional properties of the class of $m$-quasi-$(\alpha, \beta)$-Class $(\mathcal{Q})$ operators for $0 \leq \alpha \leq 1 \leq \beta$. In Section 3, our attention focuses on studying some inequalities linked to concepts of numerical radius and norm of $m$-quasi-$(\alpha, \beta)$-Class $(\mathcal{Q})$ operators. We will extended some of inequalities to operator norm and numerical radius for $m$-quasi-$(\alpha, \beta)$- Class $(\mathcal{Q})$- operators in Hilbert spaces by using some known results for vectors in inner product spaces.

## 2. Main Results

In this section, we present more properties for the class of $m$-quasi-$(\alpha, \beta)$-Class $(\mathcal{Q})$ operators, under study which is recently introduced in [2].

**Deninition 2.1.** *([2]) An operator $\mathbf{R} \in \mathcal{B}\big[\mathcal{H}\big]$ is said to be an $m$-quasi-$(\alpha, \beta)$-Class $(\mathcal{Q})$ operators for $0 \leq \alpha \leq 1$ and $1 \leq \beta$ if $\mathbf{R}$ satisfies*

$$\alpha^2(\mathbf{R}^*)^{m+2}\mathbf{R}^{m+2} \leq (\mathbf{R}^*)^m(\mathbf{R}^*\mathbf{R})^2\mathbf{R}^m \leq \beta^2(\mathbf{R}^*)^{m+2}\mathbf{R}^{m+2},$$

*or equivalently*

$$\begin{cases} \mathbf{R}^{*m}\Big(\big(\mathbf{R}^*\mathbf{R}\big)^2 - \alpha^2\mathbf{R}^{*2}\mathbf{R}^2\Big)\mathbf{R}^m \geq 0 \\ \\ \mathbf{R}^{*m}\Big(\beta^2\mathbf{R}^{*2}\mathbf{R}^2 - \big(\mathbf{R}^*\mathbf{R}\big)^2\Big)\mathbf{R}^m \geq 0 \end{cases}$$

*for some nonnegative integer $m$.*

The following theorem gives a characterization of $m$-quasi-$(\alpha, \beta)$-Class $(\mathcal{Q})$ operators.

**Theorem 2.2.** *Let $\mathbf{R} \in \mathcal{B}\big[\mathcal{H}\big]$, then $\mathbf{R}$ is an $m$-quasi-$(\alpha, \beta)$-Class $(\mathcal{Q})$ operators if and only if*

$$\begin{cases} \lambda^2\big(\mathbf{R}^*\big)^m\big(\mathbf{R}^*\mathbf{R}\big)^2\mathbf{R}^m + 2\lambda\alpha^2\big(\mathbf{R}^*\big)^{m+2}\mathbf{R}^{m+2} + \big(\mathbf{R}^*\big)^m\big(\mathbf{R}^*\mathbf{R}\big)^2\mathbf{R}^m \geq 0, \\ \\ \lambda^2\big(\mathbf{R}^*\big)^{m+2}\mathbf{R}^{m+2} + 2\lambda\big(\mathbf{R}^*\big)^m\big(\mathbf{R}^*\mathbf{R}\big)^2\mathbf{R}^m + \beta^4\big(\mathbf{R}^*\big)^{m+2}\mathbf{R}^{m+2} \geq 0 \end{cases}$$

*for all $\lambda \in \mathbb{R}$.*

*Proof.* By using elementary properties of real quadratic forms, we infer

$$\begin{aligned} & \lambda^2\big(\mathbf{R}^*\big)^m\big(\mathbf{R}^*\mathbf{R}\big)^2\mathbf{R}^m + 2\lambda\alpha^2\big(\mathbf{R}^*\big)^{m+2}\mathbf{R}^{m+2} + \big(\mathbf{R}^*\big)^m\big(\mathbf{R}^*\mathbf{R}\big)^2\mathbf{R}^m \geq 0 \\ \Leftrightarrow\quad & \lambda^2\|\mathbf{R}^*\mathbf{R}^{m+1}w\|^2 + 2\lambda\alpha^2\|\mathbf{R}^{m+2}w\|^2 + \|\mathbf{R}^*\mathbf{R}^{m+1}w\|^2 \geq 0 \quad \forall\, w \in \mathcal{H} \text{ and } \quad \forall\, \lambda \in \mathbb{R} \\ \Leftrightarrow\quad & \alpha\|\mathbf{R}^{m+2}w\| \leq \|\mathbf{R}^*\mathbf{R}^{m+1}w\| \quad \forall\, w \in \mathcal{H}. \end{aligned}$$

Similarly,

$$\begin{aligned} & \lambda^2\big(\mathbf{R}^*\big)^{m+2}\mathbf{R}^{m+2} + 2\lambda\big(\mathbf{R}^*\big)^m\big(\mathbf{R}^*\mathbf{R}\big)^2\mathbf{R}^m + \beta^4\big(\mathbf{R}^*\big)^{m+2}\mathbf{R}^{m+2} \geq 0 \\ \Leftrightarrow\quad & \lambda^2\|\mathbf{R}^{m+2}w\|^2 + 2\lambda\|\mathbf{R}^*\mathbf{R}^{m+1}w\|^2 + \beta^4\|\mathbf{R}^{m+2}w\|^2 \geq 0 \quad \forall\, w \in \mathcal{H} \text{ and } \quad \forall\, \lambda \in \mathbb{R} \\ \Leftrightarrow\quad & \beta\|\mathbf{R}^{m+2}w\| \geq \|\mathbf{R}^*\mathbf{R}^{m+1}w\| \quad \forall\, w \in \mathcal{H}. \end{aligned}$$

Therefore $\mathbf{R}$ is an $m$-quasi-$(\alpha,\beta)$-Class $(\mathcal{Q})$ operator. □

**Theorem 2.3.** *Let $\mathbf{R} \in \mathcal{B}\big[\mathcal{H}\big]$ be an $m$-quasi-$(\alpha,\beta)$-Class $(\mathcal{Q})$ operators such that $\alpha\beta = 1$. Then*

$$\alpha\|\mathbf{R}^*\mathbf{R}^{m+1}w\| \leq \|\mathbf{R}^{m+2}w\| \leq \beta\|\mathbf{R}^*\mathbf{R}^{m+1}w\|, \ \ \forall\, w \in \mathcal{H}.$$

*that is $\mathbf{R}^*$ is $(\alpha,\beta)$-normal operator on $\overline{\mathbf{Ran}(\mathbf{R}^{m+1})}$.*

*Proof.* We observe that $\mathbf{R}$ is an $m$-quasi-$(\alpha,\beta)$-Class $(\mathcal{Q})$ operator if and only if

$$\alpha^2\big(\mathbf{R}^*\big)^{m+2}\mathbf{R}^{m+2} \leq \big(\mathbf{R}^*\big)^m\big(\mathbf{R}^*\mathbf{R}\big)^2\mathbf{R}^m \leq \beta^2\big(\mathbf{R}^*\big)^{m+2}\mathbf{R}^{m+2}.$$

From which we may write

$$\begin{cases} \alpha^4\big(\mathbf{R}^*\big)^{m+2}\mathbf{R}^{m+2} \leq \alpha^2\big(\mathbf{R}^*\big)^m\big(\mathbf{R}^*\mathbf{R}\big)^2\mathbf{R}^m \leq \alpha^2\beta^2\big(\mathbf{R}^*\big)^{m+2}\mathbf{R}^{m+2} \\ \\ \alpha^2\beta^2\big(\mathbf{R}^*\big)^{m+2}\mathbf{R}^{m+2} \leq \beta^2\big(\mathbf{R}^*\big)^m\big(\mathbf{R}^*\mathbf{R}\big)^2\mathbf{R}^m \leq \beta^4\big(\mathbf{R}^*\big)^{m+2}\mathbf{R}^{m+2} \end{cases}$$

Combining these inequalities and taking into account the condition $\alpha\beta = 1$, we get

$$\alpha^2\big(\mathbf{R}^*\big)^m\big(\mathbf{R}^*\mathbf{R}\big)^2\mathbf{R}^m \leq \big(\mathbf{R}^*\big)^{m+2}\mathbf{R}^{m+2} \leq \beta^2\big(\mathbf{R}^*\big)^m\big(\mathbf{R}^*\mathbf{R}\big)^2\mathbf{R}^m$$

and so,

$$\alpha^2\left\langle\big(\mathbf{R}^*\big)^m\big(\mathbf{R}^*\mathbf{R}\big)^2\mathbf{R}^m w \mid w\right\rangle \leq \left\langle\big(\mathbf{R}^*\big)^{m+2}\mathbf{R}^{m+2}w \mid w\right\rangle \leq \beta^2\left\langle\big(\mathbf{R}^*\big)^m\big(\mathbf{R}^*\mathbf{R}\big)^2\mathbf{R}^m w \mid w\right\rangle.$$

This means that

$$\alpha\|\mathbf{R}^*\mathbf{R}^{m+1}w\| \leq \|\mathbf{R}^{m+2}w\| \leq \beta\|\mathbf{R}^*\mathbf{R}^{m+1}w\|, \ \ \forall\, w \in \mathcal{H}.$$

Hence, the result is proved. □

**Theorem 2.4.** *([2]) Let $\mathbf{R}$ be an $m$-quasi-$(\alpha,\beta)$-Class $(\mathcal{Q})$ operators. If $\overline{\mathbf{Ran}(\mathbf{R}^m)} = \mathcal{H}$, then $\mathbf{R}$ is $(\alpha,\beta)$-Class $(\mathcal{Q})$ operators.*

**Corollary 2.5.** *Let $\mathbf{R}$ be an $m$-quasi-$(\alpha,\beta)$-Class $(\mathcal{Q})$ operators. If $\mathbf{R}^m \neq 0$ and $\mathbf{R}$ has no nontrivial $\mathbf{R}^m$-invariant closed subspace, then $\mathbf{R}$ is $(\alpha,\beta)$-Class $(\mathcal{Q})$ operators.*

*Proof.* By taking into account that $\mathbf{R}^m$ has no nontrivial invariant closed subspace, it has no nontrivial hyperinvariant subspace. But $\ker(\mathbf{R}^m)$ and $\overline{\mathbf{Ran}(\mathbf{R}^m)}$ are hyperinvariant subspaces, and $\mathbf{R}^m \neq \{0\}$, hence $\ker(\mathbf{R}^m) = \{0\}$ and $\overline{\mathbf{Ran}(\mathbf{R}^m)} = \mathcal{H}$. Therefore $\mathbf{R}$ is $(\alpha,\beta)$-Class $(\mathcal{Q})$ operators. □

**Corollary 2.6.** *Let $\mathbf{R} \in \mathcal{B}[\mathcal{H}]$. If $\mu + \lambda\mathbf{R}$ is an $m$-quasi-$(\alpha,\beta)$-Class $(\mathcal{Q})$ operators for all $\mu, \lambda \in \mathbb{R}$, then $\mathbf{R}$ is $(\alpha,\beta)$-Class $(\mathcal{Q})$ operators.*

*Proof.* If $\mathbf{R}$ is an $m$-quasi-$(\alpha,\beta)$-Class $(\mathcal{Q})$ operators but not $(\alpha,\beta)$-Class $(\mathcal{Q})$ operators, then $\mathbf{R}^m$ is not invertible. It is possible to find scalars $\mu_0$ and $\lambda \neq 0$ such that $\mathbf{N} = \mu_0 + \lambda_0\mathbf{R}$ is invertible $m$-quasi-$(\alpha,\beta)$-Class $(\mathcal{Q})$ operators. So $\mathbf{N}$ is $(\alpha,\beta)$-Class $(\mathcal{Q})$ operators. Since

$$\mathbf{N} = \mu_0 + +\lambda_0\mathbf{R} \Rightarrow \mathbf{R} = \frac{1}{\lambda_0}\big(\mathbf{N} - \mu_0\big).$$

Consequently, $\mathbf{R}$ is also $(\alpha,\beta)$-Class $(\mathcal{Q})$ operators. □

The following theorem represents a matrix representation.

**Theorem 2.7.** *([2]) Let $\mathbf{R} \in \mathcal{B}\big[\mathcal{H}\big]$ such that $\overline{\mathbf{Ran}(\mathbf{R}^m)} \neq \mathcal{H}$. Then the following are equivalent.*

(1) *$\mathbf{R}$ is an $m$-quasi-$(\alpha,\beta)$-class-$(\mathcal{Q})$ operators.*

(2) $\mathbf{R} = \begin{pmatrix} \mathbf{R}_1 & \mathbf{R}_2 \\ 0 & \mathbf{R}_3 \end{pmatrix}$ *on $\mathcal{H} = \overline{\mathbf{Ran}(\mathbf{R}^m)} \oplus ker(\mathbf{R}^{*m})$, where*

$$\alpha^2\mathbf{R}_1^{*2}\mathbf{R}_1^2 \leq \big(\mathbf{R}_1^*\mathbf{R}_1\big)^2 + \mathbf{R}_1^*\mathbf{R}_2\mathbf{R}_2^*\mathbf{R}_1 \leq \beta^2\mathbf{R}_1^{*2}\mathbf{R}_1^2,$$

*and $\mathbf{R}_3^m = 0$. Furthermore $\sigma(\mathbf{R}) = \sigma(\mathbf{R}_1) \cup \{0\}$.*

**Corollary 2.8.** *Let $\mathbf{R} \in \mathbf{B}\big[\mathcal{H}\big]$ be an $m$-quasi-$(\alpha,\beta)$-Class $(\mathcal{Q})$ operators. If the restriction $\mathbf{R}_1 = \mathbf{R}\big|_{\overline{\mathbf{Ran}(\mathbf{R}^m)}}$ is invertible, then $\mathbf{R}$ is similar to a direct sum of an $(\alpha,\beta)$-Class $(\mathcal{Q})$ operators and a nilpotent operator with index of nilpotence less than or equal $m$.*

*Proof.* Let

$$\mathbf{R} = \begin{pmatrix} \mathbf{R}_1 & \mathbf{R}_2 \\ 0 & \mathbf{R}_3 \end{pmatrix} \quad \text{on } \mathcal{H} = \overline{\mathbf{Ran}(\mathbf{R}^m)} \oplus \ker(\mathbf{R}^{*m}).$$

Since $\mathbf{R}_1$ is invertible in view of theorem 2.2, $\mathbf{R}_1$ is $(\alpha,\beta)$-Class $(\mathcal{Q})$ operators and $R_3$ is nilpotent. Moreover by the fact that $\mathbf{R}_1$ is invertible, it follows that $0 \notin \sigma(R_1)$. and hence, $\sigma(\mathbf{R}_1) \cap \sigma(\mathbf{R}_3) = \emptyset$. By Rosenblum's Corollary [23], there exists $\mathbf{T} \in \mathbf{B}\big[\mathbf{H}\big]$ for which $\mathbf{R}_1\mathbf{T} - \mathbf{T}\mathbf{R}_3 = \mathbf{R}_2$. Thus, we have

$$\begin{aligned} \mathbf{R} &= \begin{pmatrix} \mathbf{I} & -\mathbf{T} \\ 0 & \mathbf{I} \end{pmatrix}\begin{pmatrix} \mathbf{R}_1 & 0 \\ 0 & \mathbf{R}_3 \end{pmatrix}\begin{pmatrix} \mathbf{I} & \mathbf{T} \\ 0 & \mathbf{I} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{I} & \mathbf{T} \\ 0 & I \end{pmatrix}^{-1}\begin{pmatrix} \mathbf{R}_1 & 0 \\ 0 & \mathbf{R}_3 \end{pmatrix}\begin{pmatrix} \mathbf{I} & \mathbf{T} \\ 0 & \mathbf{I} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{I} & \mathbf{T} \\ 0 & \mathbf{I} \end{pmatrix}^{-1}\Big(\mathbf{R}_1 \oplus \mathbf{R}_3\Big)\begin{pmatrix} \mathbf{I} & \mathbf{T} \\ 0 & \mathbf{I} \end{pmatrix}. \end{aligned}$$

□

**Question.** Let $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3 \in \mathcal{B}\big[\mathcal{H}\big]$. If $\mathbf{R}_1$ is $(\alpha,\beta)$-Class $(\mathcal{Q})$ operators and $\mathbf{R}_3^m = 0$. Is the operator matrix $\mathbf{R} = \begin{pmatrix} \mathbf{R}_1 & \mathbf{R}_2 \\ 0 & \mathbf{R}_3 \end{pmatrix} \in \mathcal{B}\big[\mathcal{H} \oplus \mathcal{H}\big]$ $m$-quasi-$(\alpha,\beta)$-Class $(\mathcal{Q})$ operators?

**Theorem 2.9.** *Let $\mathbf{N} \in \mathcal{B}[\mathcal{H}]$ be an invertible operator and $\mathbf{R} \in \mathcal{B}[\mathcal{H}]$ be an operator such that $\mathbf{R}$ commutes with $\mathbf{N}^*\mathbf{N}$. Then $\mathbf{R}$ is an $m$-quasi-$(\alpha,\beta)$-Class $(\mathcal{Q})$ operators if and only if $\mathbf{N}\mathbf{R}\mathbf{N}^{-1}$ is an $m$-quasi-$(\alpha,\beta)$-Class $(\mathcal{Q})$ operators.*

*Proof.* Assume that $\mathbf{R}$ be an $m$-quasi-$(\alpha,\beta)$-Class $(\mathcal{Q})$ operators. Then

$$\begin{cases} \mathbf{R}^{*m}\Big(\big(\mathbf{R}^*\mathbf{R}\big)^2 - \alpha^2\mathbf{R}^{*2}\mathbf{R}^2\Big)\mathbf{R}^m \geq 0 \\ \\ \mathbf{R}^{*m}\Big(\beta^2\mathbf{R}^{*2}\mathbf{R}^2 - \big(\mathbf{R}^*\mathbf{R}\big)^2\Big)\mathbf{R}^m \geq 0. \end{cases}$$

From this we have that

$$\begin{cases} \mathbf{N}\mathbf{R}^{*m}\Big(\big(\mathbf{R}^*\mathbf{R}\big)^2 - \alpha^2\mathbf{R}^{*2}\mathbf{R}^2\Big)\mathbf{R}^m\mathbf{N}^* \geq 0 \\ \\ \mathbf{N}\mathbf{R}^{*m}\Big(\beta^2\mathbf{R}^{*2}\mathbf{R}^2 - \big(\mathbf{R}^*\mathbf{R}\big)^2\Big)\mathbf{R}^m\mathbf{N}^* \geq 0. \end{cases}$$

A computation shows that

$$\begin{aligned} & \mathbf{N}\mathbf{R}^{*m}\Big(\big(\mathbf{R}^*\mathbf{R}\big)^2 - \alpha^2\mathbf{R}^{*2}\mathbf{R}^2\Big)\mathbf{R}^m\mathbf{N}^*(\mathbf{N}\mathbf{N}^*) \\ = \; & \mathbf{N}\mathbf{R}^{*m}\Big(\big(\mathbf{R}^*\mathbf{R}\big)^2 - \alpha^2\mathbf{R}^{*2}\mathbf{R}^2\Big)\mathbf{R}^m(\mathbf{N}^*\mathbf{N})\mathbf{N}^* \\ = \; & \mathbf{N}(\mathbf{N}^*\mathbf{N})\mathbf{R}^{*m}\Big(\big(\mathbf{R}^*\mathbf{R}\big)^2 - \alpha^2\mathbf{R}^{*2}\mathbf{R}^2\Big)\mathbf{R}^m\mathbf{N}^* \\ = \; & (\mathbf{N}\mathbf{N}^*)\mathbf{N}\mathbf{R}^{*m}\Big(\big(\mathbf{R}^*\mathbf{R}\big)^2 - \alpha^2\mathbf{R}^{*2}\mathbf{R}^2\Big)\mathbf{R}^m\mathbf{N}^*. \end{aligned}$$

This shows that the operator $\mathbf{N}\mathbf{N}^*$ commutes with operator

$$\mathbf{N}\mathbf{R}^{*m}\Big(\big(\mathbf{R}^*\mathbf{R}\big)^2 - \alpha^2\mathbf{R}^{*2}\mathbf{R}^2\Big)\mathbf{R}^m\mathbf{N}^*.$$

Hence the operator $(\mathbf{N}\mathbf{N}^*)^{-1}$ also commutes with operator

$$\mathbf{N}\mathbf{R}^{*m}\Big(\big(\mathbf{R}^*\mathbf{R}\big)^2 - \alpha^2\mathbf{R}^{*2}\mathbf{R}^2\Big)\mathbf{R}^m\mathbf{N}^*.$$

Using the fact that the operators $(\mathbf{N}\mathbf{N}^*)^{-1}$ and $\mathbf{N}\mathbf{R}^{*m}\Big(\big(\mathbf{R}^*\mathbf{R}\big)^2 - \alpha^2\mathbf{R}^{*2}\mathbf{R}^2\Big)\mathbf{R}^m\mathbf{N}^*$ are positive and since they commute with each other we get that their product is also positive operator

$$\mathbf{N}\mathbf{R}^{*m}\Big(\big(\mathbf{R}^*\mathbf{R}\big)^2 - \alpha^2\mathbf{R}^{*2}\mathbf{R}^2\Big)\mathbf{R}^m\mathbf{N}^*(\mathbf{N}\mathbf{N}^*)^{-1} \geq 0.$$

This implies that

$$\mathbf{N}\mathbf{R}^{*m}\Big(\big(\mathbf{R}^*\mathbf{R}\big)^2 - \alpha^2\mathbf{R}^{*2}\mathbf{R}^2\Big)\mathbf{R}^m\mathbf{N}^{-1} \geq 0.$$

Similar technics show that

$$\mathbf{N}\mathbf{R}^{*m}\Big(\beta^2\mathbf{R}^{*2}\mathbf{R}^2 - \big(\mathbf{R}^*\mathbf{R}\big)^2\Big)\mathbf{R}^m\mathbf{N}^{-1} \geq 0.$$

From the fact that $\mathbf{RN}^*\mathbf{N} = \mathbf{N}^*\mathbf{NR}$ it follows that

$$\big(\mathbf{NRN}^{-1}\big)^{*m} = \big(\mathbf{NRN}^{-1}\big)^{*}\mathbf{NRN}^{-1}\big)^{*}\cdots\mathbf{NRN}^{-1}\big)^{*} = \big(\mathbf{N}^*\big)^{-1}\mathbf{R}^{*m}\mathbf{N}^*,$$

$$\big(\mathbf{NRN}^{-1}\big)^{m} = \mathbf{NR}^m\mathbf{N}^{-1},$$

$$\big(\mathbf{NRN}^{-1}\big)^{*}\big(\mathbf{NRN}^{-1}\big) = \big(\mathbf{N}^*\big)^{-1}\mathbf{R}^*\mathbf{N}^*\mathbf{NRN}^{-1} = \mathbf{NR}^*\mathbf{RN}^{-1}.$$

$$\big(\mathbf{NRN}^{-1}\big)^{*2}\big(\mathbf{NRN}^{-1}\big)^{2} = \big(\mathbf{N}^*\big)^{-1}\mathbf{R}^{*2}\mathbf{N}^*\mathbf{NR}^2\mathbf{N}^{-1} = \mathbf{NR}^{*2}\mathbf{R}^2\mathbf{N}^{-1}.$$

Now we show that $\mathbf{NRN}^{-1}$ is an $m$-quasi-$(\alpha,\beta)$-Class $(\mathcal{Q})$ operators. In fact

$$\begin{aligned}
&\big(\mathbf{NRN}^{-1}\big)^{*m}\Big(\big(\big(\mathbf{NRN}^{-1}\big)^{*}\big(\mathbf{NRN}^{-1}\big)\big)^{2} - \alpha^2\big(\mathbf{NRN}^{-1}\big)^{*2}\big(\mathbf{NRN}^{-1}\big)^{2}\Big)\big(\mathbf{NRN}^{-1}\big)^{m}\\
= \;& (\mathbf{N}^*)^{-1}\mathbf{R}^{*m}\mathbf{N}^*\Big(\mathbf{N}\big(\mathbf{R}^*\mathbf{R}\big)^2\mathbf{N}^{-1} - \alpha^2\mathbf{NR}^{*2}\mathbf{R}^2\mathbf{N}^{-1}\Big)\mathbf{NR}^m\mathbf{N}^{-1}\\
= \;& (\mathbf{N}^*)^{-1}\mathbf{R}^{*m}\mathbf{N}^*\mathbf{N}\Big(\big(\mathbf{R}^*\mathbf{R}\big)^2 - \alpha^2\mathbf{R}^{*2}\mathbf{R}^2\Big)\mathbf{N}^{-1}\mathbf{NR}^m\mathbf{N}^{-1}\\
= \;& (\mathbf{N}^*)^{-1}\mathbf{N}^*\mathbf{NR}^{*m}\Big(\big(\mathbf{R}^*\mathbf{R}\big)^2 - \alpha^2\mathbf{R}^{*2}\mathbf{R}^2\Big)\mathbf{R}^m\mathbf{N}^{-1}\\
= \;& \mathbf{NR}^{*m}\Big(\big(\mathbf{R}^*\mathbf{R}\big)^2 - \alpha^2\mathbf{R}^{*2}\mathbf{R}^2\Big)\mathbf{R}^m\mathbf{N}^{-1}\\
\geq \;& 0.
\end{aligned}$$

By similar steps we prove that

$$\begin{aligned}
&\big(\mathbf{NRN}^{-1}\big)^{*m}\Big(\beta^2\big(\mathbf{NRN}^{-1}\big)^{*2}\big(\mathbf{NRN}^{-1}\big)^{2} - \big(\big(\mathbf{NRN}^{-1}\big)^{*}\big(\mathbf{NRN}^{-1}\big)\big)^{2}\Big)\big(\mathbf{NRN}^{-1}\big)^{m}\\
= \;& \mathbf{NR}^{*m}\Big(\big(\beta^2\mathbf{R}^{*2}\mathbf{R}^2 - \mathbf{R}^*\mathbf{R}\big)^2\Big)\mathbf{R}^m\mathbf{N}^{-1}\\
\geq \;& 0.
\end{aligned}$$

Based on these calculations we conclude that $\mathbf{NRN}^{-1}$ is $m$-quasi-$(\alpha,\beta)$-Class $(\mathcal{Q})$ operators.

Conversely, assume that $\mathbf{NRN}^{-1}$ is an $m$-quasi-$(\alpha,\beta)$-Class $(\mathcal{Q})$ operators. Then

$$\big(\mathbf{NRN}^{-1}\big)^{*m}\Big(\beta^2\big(\mathbf{NRN}^{-1}\big)^{*2}\big(\mathbf{NRN}^{-1}\big)^{2} - \big(\big(\mathbf{NRN}^{-1}\big)^{*}\big(\mathbf{NRN}^{-1}\big)\big)^{2}\Big)\big(\mathbf{NRN}^{-1}\big)^{m} \geq 0$$

and

$$\big(\mathbf{NRN}^{-1}\big)^{*m}\Big(\big(\big(\mathbf{NRN}^{-1}\big)^{*}\big(\mathbf{NRN}^{-1}\big)\big)^{2} - \alpha^2\big(\mathbf{NRN}^{-1}\big)^{*2}\big(\mathbf{NRN}^{-1}\big)^{2}\Big)\big(\mathbf{NRN}^{-1}\big)^{m} \geq 0.$$

Similar as before we get

$$\mathbf{NR}^{*m}\Big(\big(\mathbf{R}^*\mathbf{R}\big)^2 - \alpha^2\mathbf{R}^{*2}\mathbf{R}^2\Big)\mathbf{R}^m\mathbf{N}^{-1} \geq 0.$$

and

$$\mathbf{NR}^{*m}\Big(\beta^2\mathbf{R}^{*2}\mathbf{R}^2 - \big(\mathbf{R}^*\mathbf{R}\big)^2\Big)\mathbf{R}^m\mathbf{N}^{-1} \geq 0.$$

Hence,

$$\mathbf{N}^*\mathbf{NR}^{*m}\Big(\big(\mathbf{R}^*\mathbf{R}\big)^2 - \alpha^2\mathbf{R}^{*2}\mathbf{R}^2\Big)\mathbf{R}^m\mathbf{N}^{-1}\mathbf{N} \geq 0.$$

and

$$\mathbf{N}^*\mathbf{N}\mathbf{R}^{*m}\Big(\beta^2\mathbf{R}^{*2}\mathbf{R}^2-\big(\mathbf{R}^*\mathbf{R}\big)^2\Big)\mathbf{R}^m\mathbf{N}^{-1}\mathbf{N}\geq 0.$$

or equivalently

$$\mathbf{N}^*\mathbf{N}\mathbf{R}^{*m}\Big(\big(\mathbf{R}^*\mathbf{R}\big)^2-\alpha^2\mathbf{R}^{*2}\mathbf{R}^2\Big)\mathbf{R}^m\geq 0.$$

and

$$\mathbf{N}^*\mathbf{N}\mathbf{R}^{*m}\Big(\beta^2\mathbf{R}^{*2}\mathbf{R}^2-\big(\mathbf{R}^*\mathbf{R}\big)^2\Big)\mathbf{R}^m\geq 0.$$

By using that $\mathbf{N}^*\mathbf{N}$ commutes with operator $\mathbf{R}$ and hence commute with operators

$$\mathbf{N}^*\mathbf{N}\mathbf{R}^{*m}\Big(\big(\mathbf{R}^*\mathbf{R}\big)^2-\alpha^2\mathbf{R}^{*2}\mathbf{R}^2\Big)\mathbf{R}^m$$

and

$$\mathbf{N}^*\mathbf{N}\mathbf{R}^{*m}\Big(\beta^2\mathbf{R}^{*2}\mathbf{R}^2-\big(\mathbf{R}^*\mathbf{R}\big)^2\Big)\mathbf{R}^m.$$

It follows that $\big(\mathbf{N}^*\mathbf{N}\big)^{-1}$ commute with

$$\mathbf{N}^*\mathbf{N}\mathbf{R}^{*m}\Big(\big(\mathbf{R}^*\mathbf{R}\big)^2-\alpha^2\mathbf{R}^{*2}\mathbf{R}^2\Big)\mathbf{R}^m$$

and with

$$\mathbf{N}^*\mathbf{N}\mathbf{R}^{*m}\Big(\beta^2\mathbf{R}^{*2}\mathbf{R}^2-\big(\mathbf{R}^*\mathbf{R}\big)^2\Big)\mathbf{R}^m.$$

By observing that $\big(\mathbf{N}^*\mathbf{N}\big)^{-1}$ and $\mathbf{N}^*\mathbf{N}\mathbf{R}^{*m}\Big(\big(\mathbf{R}^*\mathbf{R}\big)^2-\alpha^2\mathbf{R}^{*2}\mathbf{R}^2\Big)\mathbf{R}^m$ are positive and since they commutes with each other we have

$$\big(\mathbf{N}^*\mathbf{N}\big)^{-1}\mathbf{N}^*\mathbf{N}\mathbf{R}^{*m}\Big(\big(\mathbf{R}^*\mathbf{R}\big)^2-\alpha^2\mathbf{R}^{*2}\mathbf{R}^2\Big)\mathbf{R}^m\geq 0.$$

For similar reason

$$\big(\mathbf{N}^*\mathbf{N}\big)^{-1}\mathbf{N}^*\mathbf{N}\mathbf{R}^{*m}\Big(\beta^2\mathbf{R}^{*2}\mathbf{R}^2-\big(\mathbf{R}^*\mathbf{R}\big)^2\Big)\mathbf{R}^m\geq 0.$$

Therefore,

$$\mathbf{R}^{*m}\Big(\big(\mathbf{R}^*\mathbf{R}\big)^2-\alpha^2\mathbf{R}^{*2}\mathbf{R}^2\Big)\mathbf{R}^m\geq 0$$

and

$$\mathbf{R}^{*m}\Big(\beta^2\mathbf{R}^{*2}\mathbf{R}^2-\big(\mathbf{R}^*\mathbf{R}\big)^2\Big)\mathbf{R}^m\geq 0.$$

Whit does it mean that $\mathbf{R}$ is an $m$-quasi-$(\alpha,\beta)$-Class $(\mathcal{Q})$ operators. □

**Corollary 2.10.** *Let* $\mathbf{N}\in\mathcal{B}\big[\mathcal{H}\big]$ *be an* $m$*-quasi-*$(\alpha,\beta)$*-Class* $(\mathcal{Q})$*-operators and* $\mathbf{T}$ *any invertible operator such that* $\mathbf{T}^{-1}=\mathbf{T}^*$*. Then* $\mathbf{R}=\mathbf{T}^{-1}\mathbf{N}\mathbf{T}$ *is an* $m$*-quasi-*$(\alpha,\beta)$*-Class* $(\mathcal{Q})$ *operators.*

*Proof.* Assume that $\mathbf{N}$ be an $m$-quasi-$(\alpha,\beta)$-Class $(\mathbf{Q})$ operator. Then

$$\begin{cases} \left\langle \mathbf{N}^{*m}\Big(\big(\mathbf{N}^*\mathbf{N}\big)^2-\alpha^2\mathbf{N}^{*2}\mathbf{N}^2\Big)\mathbf{N}^m w \mid w\right\rangle \geq 0 \ \ \forall\, w\in\mathcal{H} \\ \\ \left\langle \mathbf{N}^{*m}\Big(\beta^2\mathbf{N}^{*2}\mathbf{N}^2-\big(\mathbf{N}^*\mathbf{N}\big)^2\Big)\mathbf{N}^m w \mid w\right\rangle \geq 0 \ \ \forall\, w\in\mathcal{H} \end{cases}$$

By expanding this and by simple calculations we have

$$\begin{aligned} & \mathbf{R}^{*m}\Big(\big(\mathbf{R}^*\mathbf{R}\big)^2-\alpha^2\mathbf{R}^{*2}\mathbf{R}^2\Big)\mathbf{R}^m \\ = \;& \big(\mathbf{T}^{-1}\mathbf{N}\mathbf{T}\big)^{*m}\Big(\big(\mathbf{T}^{-1}\mathbf{N}\mathbf{T}\big)^*\big(\mathbf{T}^{-1}\mathbf{N}\mathbf{T}\big)\big)^2-\alpha^2\big(\mathbf{T}^{-1}\mathbf{N}\mathbf{T}\big)^{*2}\big(\mathbf{T}^{-1}\mathbf{N}\mathbf{T}\big)^2\Big)\big(\mathbf{T}^{-1}\mathbf{N}\mathbf{T}\big)^m \\ = \;& \mathbf{T}^{-1}\mathbf{N}^{*m}\Big(\big(\mathbf{N}^*\mathbf{N}\big)^2-\alpha^2\mathbf{N}^{*2}\mathbf{N}^2\Big)\mathbf{N}^m\mathbf{T} \\ = \;& \mathbf{T}^*\mathbf{N}^m\Big(\big(\mathbf{N}^*\mathbf{N}\big)^2-\alpha^2\mathbf{N}^{*2}\mathbf{N}^2\Big)\mathbf{N}^m\mathbf{T} \\ \geq \;& 0. \end{aligned}$$

Similarly,

$$\begin{aligned} & \mathbf{R}^{*m}\Big(\beta^2\mathbf{R}^{*2}\mathbf{R}^2-\big(\mathbf{R}^*\mathbf{R}\big)^2\Big)\mathbf{R}^m \\ = \;& \big(\mathbf{T}^{-1}\mathbf{N}\mathbf{T}\big)^{*m}\Big(\beta^2\big(\mathbf{T}^{-1}\mathbf{N}\mathbf{T}\big)^{*2}\big(\mathbf{T}^{-1}\mathbf{N}\mathbf{T}\big)^2-\big(\mathbf{T}^{-1}\mathbf{N}\mathbf{T}\big)^*\big(\mathbf{T}^{-1}\mathbf{N}\mathbf{T}\big)\big)^2-\Big)\big(\mathbf{T}^{-1}\mathbf{N}\mathbf{T}\big)^m \\ = \;& \mathbf{T}^{-1}\mathbf{N}^{*m}\Big(\beta^2\mathbf{N}^{*2}\mathbf{N}^2-\big(\mathbf{N}^*\mathbf{N}\big)^2\Big)\mathbf{N}^m\mathbf{T} \\ = \;& \mathbf{T}^*\mathbf{N}^m\Big(\beta^2\mathbf{N}^{*2}\mathbf{N}^2-\big(\mathbf{N}^*\mathbf{N}\big)^2\Big)\mathbf{N}^m\mathbf{T} \\ \geq \;& 0. \end{aligned}$$

Hence, $\mathbf{R}=\mathbf{T}^{-1}\mathbf{N}\mathbf{T}$ is $m$-quasi-$(\alpha,\beta)$-Class $(\mathcal{Q})$ operators as required. □

## 3. Inequalities Involving Numerical Radius and Operator Norm

In this section, our attention focuses on studying some inequalities linked to concepts of numerical radius and norm of $m$-quasi-$(\alpha,\beta)$-Class $(\mathcal{Q})$- operators. It represents an extension of many relationships that were previously proven through [4, 5, 6, 7, 8, 9, 10, 12, 21]. The authors in [22] have given various inequalities between the operator norm and the numerical radius of $m$-quasi-$(\alpha,\beta)$-normal operators in Hilbert spaces. Motivated by this work, we will extended some of these inequalities to operator norm and numerical radius for $m$-quasi-$(\alpha,\beta)$- Class $(\mathcal{Q})$-operators in Hilbertian spaces by using some known results for vectors in inner product spaces.

Recall that the numerical radius of an operator $\mathbf{R}\in\mathcal{B}\big[\mathcal{H}\big]$ is defined by

$$\omega(R)=\sup\{|\langle\mathbf{R}\psi\mid\psi\rangle|\,,\;\;\psi\in\mathcal{H}:\;\;\|\psi\|=1\}.$$

**Theorem 3.1.** *Let $\mathbf{R} \in \mathcal{B}[\mathcal{H}]$ be an $m$-quasi-$(\alpha, \beta)$-Class $(\mathcal{Q})$-operators. Then the following inequality hold.*

$$\left(1+\left(\frac{\alpha}{\beta}\right)^{2q}\right)\|\mathbf{R}^{m+2}\|^2 \leq \frac{2}{\beta}\omega\left(\mathbf{R}^{*(m+1)}\mathbf{R}^{m+3}\right)+\frac{q^2}{\beta^2}\|\beta\mathbf{R}^{m+2}-\mathbf{R}^*\mathbf{R}^{m+1}\|^2, \textit{ if } q \geq 1. \tag{3.1}$$

*and*

$$\left(1+\left(\frac{\alpha}{\beta}\right)^{2q}\right)\|\mathbf{R}^{m+2}\|^2 \leq \frac{2}{\beta}\omega\left(\mathbf{R}^{*(m+1)}\mathbf{R}^{m+3}\right)+\frac{1}{\beta^2}\|\beta\mathbf{R}^{m+2}-\mathbf{R}^*R^{m+1}\|^2, \textit{ if } q < 1. \tag{3.2}$$

*Proof.* Referring to[12] the following inequalities can be used for $q \in \mathbb{R}$ and $\varphi_1, \varphi_2 \in \mathcal{H}$ with $\|\varphi_1\| \geq \|\varphi_2\|$,

$$\|\varphi_1\|^{2q}+\|\varphi_2\|^{2q}-2\|\varphi_1\|^{q-1}\|\varphi_2\|^{q-1}Re\left\langle\varphi_1 \mid \varphi_2\right\rangle \leq q^2\|\varphi_1\|^{2q-2}\|\varphi_1-\varphi_2\|^2, \text{for } \ q \geq 1,$$

and

$$\|\varphi_1\|^{2q}+\|\varphi_2\|^{2q}-2\|\varphi_1\|^{q-1}\|\varphi_2\|^{q-1}Re\left\langle\varphi_1 \mid \varphi_2\right\rangle \leq \|\varphi_2\|^{2q-2}\|\varphi_1-\varphi_2\|^2, \text{for } \ q < 1.$$

Since $\mathbf{R}$ is an $m$-quasi-$(\alpha, \beta)$-Class $(\mathcal{Q})$-operators, it follows that

$$\alpha\|\mathbf{R}^{m+2}\psi\| \leq \|\mathbf{R}^*\mathbf{R}^{m+1}\psi\| \leq \beta\|R^{m+2}\psi\|, \quad \forall\, \psi \in \mathcal{H}.$$

By making the right choice with $\varphi_1 = \beta\mathbf{R}^{m+2}\psi$ and $\varphi_2 = \mathbf{R}^*\mathbf{R}^{m+1}\psi$ we obtain

$$\begin{aligned}\|\beta\mathbf{R}^{m+2}\psi\|^{2q}+\|\mathbf{R}^*\mathbf{R}^{m+1}\psi\|^{2q} \quad &\leq \quad q^2\|\beta\mathbf{R}^{m+2}\psi\|^{2q-2}\|\beta\mathbf{R}^{m+2}\psi-\mathbf{R}^*\mathbf{R}^{m+1}\psi\|^2\\ &\quad +2\|\beta\mathbf{R}^{m+2}x\|^{q-1}\|\mathbf{R}^*\mathbf{R}^{m+1}\psi\|^{q-1}Re\left\langle\beta\mathbf{R}^{m+2}\psi \mid \mathbf{R}^*\mathbf{R}^{m+1}\psi\right\rangle\end{aligned}$$

for any $\psi \in \mathcal{H}$ with $\|\psi\| = 1$ and $q \geq 1$.

Hence,

$$\begin{aligned}\left(\alpha^{2q}+\beta^{2q}\right)\|\mathbf{R}^{m+2}\psi\|^{2q} \quad &= \quad \alpha^{2q}\|\mathbf{R}^{m+2}\psi\|^{2q}+\beta^{2q}\|\mathbf{R}^{m+2}\psi\|^{2q}\\ &\leq \quad \|\mathbf{R}^*\mathbf{R}^{m+1}x\|^{2q}+\beta^{2q}\|\mathbf{R}^{m+2}x\|^{2q}\\ &\leq \quad q^2\beta^{2q-2}\|\mathbf{R}^{m+2}\psi\|^{2q-2}\|\beta\mathbf{R}^{m+2}\psi-\mathbf{R}^*\mathbf{R}^{m+}\psi\|^2\\ &\quad +2\beta^{2q-1}\|\mathbf{R}^{m+2}\psi\|^{2q-2}|\left\langle\mathbf{R}^{*m}\mathbf{R}^{m+3}\psi \mid \psi\right\rangle|.\end{aligned}$$

Applying sup over $\psi \in \mathcal{H}$, $\|\psi\| = 1$ of both sides of the above inequality, we infer

$$\begin{aligned}\left(\alpha^{2q}+\beta^{2q}\right)\|\mathbf{R}^{m+2}\|^{2q} \quad &\leq \quad q^2\beta^{2q-2}\|\mathbf{R}^{m+2}\|^{2q-2}\|\beta\mathbf{R}^{m+2}-\mathbf{R}^*\mathbf{R}^{m+1}\|^2\\ &\quad +2\beta^{2q-1}\|\mathbf{R}^{m+2}\|^{2q-2}\omega\left(\left(\mathbf{R}^{*(m+1)}\mathbf{R}^{m+3}\right).\right.\end{aligned}$$

This proves the first inequality.

Analogously, with similar steps we prove the second inequality for $q < 1$. □

**Theorem 3.2.** *Let $\mathbf{R} \in \mathcal{B}[\mathcal{H}]$ be an $m$-quasi-$(\alpha, \beta)$-Class $(\mathcal{Q})$ operators. If $\mu \in \mathbb{C}$, then*

$$\alpha\|\mathbf{R}^{m+2}\|^2 \leq \omega\left(\left(\mathbf{R}^{*m}\mathbf{R}^{m+3}\right)+\frac{2\beta}{\left(1+|\mu|\alpha\right)^2}\|\mathbf{R}^{m+2}-\mu\mathbf{R}^*\mathbf{R}^{m+1}\|^2.\right.$$

*Proof.* We give our proof depending on the following inequality inspired from [10],

$$\|\varphi_1\|\|\varphi_2\| \leq |\langle\varphi_1 \mid \varphi_2\rangle| + \frac{2\|\varphi_1\|\|\varphi_2\|\|\varphi_1 - \varphi_2\|^2}{(\|\varphi_1\| + \|\varphi_2\|)^2},$$

for $\varphi_1, \varphi_2 \in \mathcal{H}\setminus\{0\}$. Since $\mathbf{R}$ is an $m$-quasi-$(\alpha,\beta)$-Class $(\mathcal{Q})$ operators, it follows that

$$\alpha\|\mathbf{R}^{m+2}\psi\| \leq \|\mathbf{R}^*\mathbf{R}^{m+1}\psi\| \leq \beta\|R^{m+2}\psi\|, \quad \forall\, \psi \in \mathcal{H}.$$

By making the right choice with $\varphi_1 = \mathbf{R}^{m+2}\psi$ and $\varphi_2 = \mu\mathbf{R}^*\mathbf{R}^{m+1}\psi$ we infer

$$\begin{aligned}\|\mathbf{R}^{m+2}\psi\|\|\mu\mathbf{R}^*\mathbf{R}^{m+1}\psi\| \quad &\leq \quad |\left\langle\mathbf{R}^{m+2}\psi \mid \mu\mathbf{R}^*\mathbf{R}^{m+1}\psi\right\rangle| \\ &\quad + \frac{2\|\mathbf{R}^{m+2}\psi\|\|\mu\mathbf{R}^*\mathbf{R}^{m+1}\psi\|\|\mathbf{R}^{m+2}\psi - \mu\mathbf{R}^*\mathbf{R}^{m+1}\psi\|^2}{(\|\mathbf{R}^{m+2}\psi\| + \|\mu\mathbf{R}^*\mathbf{R}^{m+1}\psi\|)^2}.\end{aligned}$$

This gives

$$\alpha\|\mathbf{R}^{m+2}\psi\|^2 \leq |\left\langle\mathbf{R}^{*(m+1)}\mathbf{R}^{m+3}\psi \mid \psi\right\rangle| + \frac{2\beta}{(1+|\lambda|\alpha)^2}\|\mathbf{R}^{m+2}\psi - \mu\mathbf{R}^*\mathbf{R}^{m+1}\psi\|^2.$$

Applying the $\sup\limits_{\|\psi\|=1}$ of both sides of the last inequality,the requirement is proven. □

**Theorem 3.3.** *Let $\mathbf{R} \in \mathcal{B}[\mathcal{H}]$ be an $m$-quasi-$(\alpha,\beta)$-Class $(\mathcal{Q})$ operator. If $\mu \in \mathbb{C}\setminus\{0\}$, then*

$$\left[\alpha^2 - \left(\tfrac{1}{|\mu|} + \beta\right)^2\right]\|\mathbf{R}^{m+2}\|^4 \leq \omega\left(\mathbf{R}^{*(m+1)}\mathbf{R}^{m+3}\right)^2.$$

*Proof.* We give our proof depending on the following inequality inspired from [7],

$$\|\varphi_1\|^2\|\varphi_2\|^2 \leq |\langle\varphi_1 \mid \varphi_2\rangle|^2 + \frac{1}{|\mu|^2}\|\varphi_1\|^2\|\varphi_1 - \mu\varphi_2\|^2,$$

provided $\varphi_1, \varphi_2 \in \mathcal{H}$ and $\mu \in \mathbb{C}\setminus\{0\}$. By choosing $\varphi_1 = \mathbf{R}^{m+2}\psi, \varphi_2 = \mathbf{R}^*\mathbf{R}^{m+1}\psi$, we infer

$$\|\mathbf{R}^{m+2}\psi\|^2\|\mathbf{R}^*\mathbf{R}^{m+1}\psi\|^2 \leq |\left\langle\mathbf{R}^{m+2}\psi \mid \mathbf{R}^*\mathbf{R}^{m+1}\psi\right\rangle|^2 + \frac{1}{|\mu|^2}\|\mathbf{R}^{m+2}\psi\|^2\|\mathbf{R}^{m+2}\psi - \mu\mathbf{R}^*\mathbf{R}^{m+1}\psi\|^2.$$

This gives

$$\alpha^2\|\mathbf{R}^{m+2}\psi\|^4 - \frac{1}{|\mu|^2}\|\mathbf{R}^{m+2}\psi\|^4(1+|\mu|\beta)^2 \leq |\left\langle\mathbf{R}^{*(m+1)}\mathbf{R}^{m+3}\psi \mid \psi\right\rangle|^2.$$

Applying the $\sup\limits_{\|\psi\|=1}$ of both sides of the last inequality,the requirement is proven. □

**Theorem 3.4.** *Let $\mathbf{R} \in \mathcal{B}[\mathcal{H}]$ be an $m$-quasi-$(\alpha,\beta)$-Class $(\mathcal{Q})$ operators. Then*

$$2\omega\left(\mathbf{R}^{m+2}\right)\omega\left(\mathbf{R}^*\mathbf{R}^{m+1}\right) \leq \beta\|\mathbf{R}^{m+2}\|^2 + \omega\left(R^{*(m+1)}R^{m+3}\right).$$

*Proof.* According to the following inequality mentioned in [4],we may write

$$2|\langle\varphi \mid \varphi_1\rangle\langle\langle\varphi_1 \mid \varphi_2\rangle| \leq \|\varphi\|\|\varphi_2\| + |\langle\varphi \mid \varphi_2\rangle|$$

where $\psi, \varphi_1, \varphi_2 \in \mathcal{H}$ and $\|\psi\| = 1$,

Choosing $\varphi_1 = \mathbf{R}^{m+2}\psi$ and $\varphi_2 = \mathbf{R}^*\mathbf{R}^{m+1}\psi$ with $\|\psi\| = 1$, it follows

$$2\left|\left\langle \mathbf{R}^{m+2}\psi_1 \mid \psi\right\rangle \left\langle \psi \mid \mathbf{R}^*\mathbf{R}^{m+1}\psi\right\rangle\right| \leq \|\mathbf{R}^{m+2}\psi\|\|\mathbf{R}^*\mathbf{R}^{m+1}\psi\| + \left|\left\langle \mathbf{R}^{m+2}\psi \mid \mathbf{R}^*\mathbf{R}^{m+1}\psi\right\rangle\right|.$$

Hence

$$2\left|\left\langle \mathbf{R}^{m+2}\psi \mid \psi\right\rangle \left\langle \mathbf{R}^{*(m+1)}\mathbf{R}\psi \mid \psi\right\rangle\right| \leq \beta\|\mathbf{R}^{m+2}\|^2 + \left|\left\langle \mathbf{R}^{*(m+1)}\mathbf{R}^{m+3}\psi \mid \psi\right\rangle\right|.$$

Applying the $\sup\limits_{\|\psi\|=1}$ of both sides of the last inequality,the requirement is proven. □

**Theorem 3.5.** *Let $\mathbf{R} \in \mathcal{B}\big[\mathcal{H}\big]$ be an $m$-quasi-$(\alpha, \beta)$-Class $(\mathcal{Q})$ operators. If $p \geq 2$, then*

$$(1+\alpha^p)\|\mathbf{R}^{m+2}\|^p \leq \frac{1}{2}\Big(\|\mathbf{R}^{m+2} + \mathbf{R}^*\mathbf{R}^{m+1}\|^p + \|\mathbf{R}^{m+2} - \mathbf{R}^*\mathbf{R}^{m+1}\|^p\Big).$$

*Proof.* We use the following inequality[9],

$$\|\varphi_1\|^p + \|\varphi_2\|^p \leq \tfrac{1}{2}\left(\|\varphi_1 + \varphi_2\|^p + \|\varphi_1 - \varphi_2\|^p\right)$$

for any $\varphi_1, \psi\varphi_2 \in \mathcal{H}$ and $p \geq 2$.

Now, if we choose $\varphi_1 = \mathbf{R}^{m+1}\psi, \varphi_2 = \mathbf{R}^*\mathbf{R}^{m+1}\psi$, we may write

$$\|\mathbf{R}^{m+2}\psi\|^p + \|\mathbf{R}^*\mathbf{R}^{m+1}\psi\|^p \leq \frac{1}{2}\Big(\mathbf{R}^{m+2}\psi + \mathbf{R}^*\mathbf{R}^{m+1}\psi\|^p + \|\mathbf{R}^{m+2}\psi - \mathbf{R}^*\mathbf{R}^{m+1}\psi\|^p\Big).$$

According to the fact that $\mathbf{R}$ is an $m$-quasi-$(\alpha, \beta)$- class $(Q)$ operators, we may write

$$(1+\alpha^p)\|\mathbf{R}^{m+2}\psi\|^p \leq \|\mathbf{R}^{m+2}\psi\|^p + \|\mathbf{R}^*\mathbf{R}^{m+1}\psi\|^p \leq \frac{1}{2}\Big(\|\mathbf{R}^{m+2}\psi + \mathbf{R}^*\mathbf{R}^{m+1}\psi$$

$$|^p + \|\mathbf{R}^{m+2}\psi - \mathbf{R}^*\mathbf{R}^{m+1}\psi\|^p\Big).$$

Taking the sup over all $\psi \in \mathcal{H}$ with $\|\psi\| = 1$ in the above inequality, we get

$$(1+\alpha^p)\|\mathbf{R}^{m+2}\|^p \leq \frac{1}{2}\Big(\|\mathbf{R}^{m+2} + \mathbf{R}^*\mathbf{R}^{m+1}\|^p + \|\mathbf{R}^{m+2} - \mathbf{R}^*\mathbf{R}^{m+1}\|^p\Big).$$

□

**Theorem 3.6.** *Let $R \in \mathcal{B}_b\big[\mathcal{H}\big]$ be an $m$-quasi-$(\alpha, \beta)$-Class $(\mathcal{Q})$ operators and $p \geq 2$. Then the following identity holds.*

$$\omega\left(\frac{\left(\mathbf{R}^*\right)^{m+2}\mathbf{R}^{m+2} + \left(R^*\right)^{m+1}\mathbf{R}\mathbf{R}^*R^{m+1}}{2}\right)^{\frac{p}{2}} \leq \frac{1+\beta^p}{4(1+\alpha^p)}\Big(\|\mathbf{R}^{m+2} + \mathbf{R}^*\mathbf{R}^{m+1}\|^p + \|\mathbf{R}^{m+2} - \mathbf{R}^*\mathbf{R}^{m+1}\|^p\Big).$$

*Proof.* Using the following elementary inequality,

$$2^{1-q}\left(a+b\right)^q \leq a^q + b^q$$

for $a, b \geq 0$ and $q \geq 1$.

For any $\psi \in \mathcal{H}$ with $\|\psi\| = 1$, take $a = \|\mathbf{R}^{m+2}\psi\|^2, b = \|\mathbf{R}^*\mathbf{R}^{m+1}\psi\|^2$ and $q = \frac{p}{2}$ in the above inequality we get

$$2^{1-\frac{p}{2}}\left(\|\mathbf{R}^{m+2}\psi\|^2 + \|\mathbf{R}^*\mathbf{R}^{m+1}\psi\|^2\right)^{\frac{p}{2}} \leq \left(\mathbf{R}^{m+2}\psi\|^2\right)^{\frac{p}{2}} + \left(\|\mathbf{R}^*\mathbf{R}^{m+1}\psi\|^2\right)^{\frac{p}{2}}.$$

Therefore,

$$\left(\frac{\|\mathbf{R}^{m+2}\psi\|^2 + \|\mathbf{R}^*\mathbf{R}^{m+1}\psi\|^2}{2}\right)^{\frac{p}{2}} \leq \frac{1}{2}\|\mathbf{R}^{m+2}\psi\|^p + \|\mathbf{R}^*\mathbf{R}^{m+1}\psi\|^p.$$

According to $\mathbf{R}$ is an $m$-quasi-$(\alpha,\beta)$- class $(\mathcal{Q})$-operators, it follows that

$$\|\mathbf{R}^{m+2}\psi\|^p + \|\mathbf{R}^*\mathbf{R}^{m+1}\psi\|^p \leq (1+\beta^p)\|\mathbf{R}^{m+2}\psi\|^p.$$

By taking into account Theorem 3.5, we obtain

$$\left(\frac{\|\mathbf{R}^{m+2}\psi\|^2 + \|\mathbf{R}^*\mathbf{R}^{m+1}\psi\|^2}{2}\right)^{\frac{p}{2}} \leq \frac{1+\beta^p}{4(1+\alpha^p)}\left(\|\mathbf{R}^{m+2} + \mathbf{R}^*\mathbf{R}^{m+1}\|^p + \|\mathbf{R}^{m+2} - \mathbf{R}^*\mathbf{R}^{m+1}\|^p\right)\|\psi\|.$$

On the other hand,

$$|\left\langle (\frac{\left(\mathbf{R}^*\right)^{m+2}\mathbf{R}^{m+2} + \left(\mathbf{R}^*\right)^{m+1}\mathbf{R}\mathbf{R}^*\mathbf{R}^{m+1})}{2}\psi \mid \psi\right\rangle|^{\frac{p}{2}} = \left(\frac{\|\mathbf{R}^{m+2}\psi\|^2 + \|\mathbf{R}^*\mathbf{R}^{m+1}\psi\|^2}{2}\right)^{\frac{p}{2}}.$$

Therefore

$$\begin{aligned}&\left|\left\langle\left(\frac{\left(\mathbf{R}^*\right)^{m+2}\mathbf{R}^{m+2} + \left(\mathbf{R}^*\right)^{m+1}\mathbf{R}\mathbf{R}^*\mathbf{R}^{m+1}}{2}\right)\psi \mid \psi\right\rangle\right|^{\frac{p}{2}}\\ \leq\ & \frac{\left(1+\beta^p\right)\|\psi\|}{4(1+\alpha^p)}\left(\|\mathbf{R}^{m+2} + \mathbf{R}^*\mathbf{R}^{m+1}\|^p + \|\mathbf{R}^{m+2} - \mathbf{R}^*\mathbf{R}^{m+1}\|^p\right).\end{aligned}$$

Applying the $\sup\limits_{\|\psi\|=1}$ of both sides of the last inequality,the requirement is proven.

□

**Theorem 3.7.** *Let* $\mathbf{R} \in \mathcal{B}[\mathcal{H}]$ *be an* $m$*-quasi-*$(\alpha,\beta)$*-Class* $(\mathcal{Q})$ *operators. If* $p \in \mathbb{R}$ *with* $1 < p < 2$ *and* $\lambda, \mu \in \mathbb{C}$ *such that* $|\lambda| - \beta|\mu| \geq 0$ *or* $\alpha|\mu| - |\lambda| \geq 0$*, then the following inequality holds.*

$$\begin{aligned}&\left(|\left(\lambda|+\alpha|\mu|\right)\right)^p\|\mathbf{R}^{m+2}\|^p + \left(\max\left(\left\{|\lambda|-|\mu|\beta, \alpha|\mu|-|\lambda|\right\}\right)\right)^p\|\mathbf{R}^{m+2}\|^p\\ \leq\ & \|\lambda\mathbf{R}^{m+2}+\mu\mathbf{R}^*\mathbf{R}^{m+1}\|^p + \|\lambda\mathbf{R}^{m+2}-\mu\mathbf{R}^*\mathbf{R}^{m+1}\|^p.\end{aligned}$$

*Proof.* According to the following inequality mentioned in [9],we have

$$\left(\|\varphi_1\| + \|\varphi_2\|\right)^p + \left|\|\varphi_1\| - \|\varphi_2\|\right|^p \leq \|\varphi_1 + \varphi_2\|^p + \|\varphi_1 - \varphi_2\|^p,$$

for any $\varphi_1, \varphi_2 \in \mathcal{H}$ and $p \in \mathbb{R}:\ 1 < p < 2$.

Put $\varphi_1 = \lambda\mathbf{R}^{m+2}\psi$ and $\varphi_2 = \mu\mathbf{R}^*\mathbf{R}^{m+1}\psi$ for $\psi \in \mathcal{H}$, to get

$$\begin{aligned}&\left(\|\lambda\mathbf{R}^{m+2}\psi\| + \|\mu\mathbf{R}^*\mathbf{R}^{m+1}\psi\|\right)^p + \left|\|\lambda\mathbf{R}^{m+2}\psi\| - \|\mu\mathbf{R}^*\mathbf{R}^{m+2}\psi\|\right|^p\\ \leq\ & \|\lambda\mathbf{R}^{m+2}\psi + \mathbf{R}^*\mathbf{R}^{m+1}\psi\|^p + \|\lambda\mathbf{R}^{m+2}\psi - \mu\mathbf{R}^*\mathbf{R}^{m+1}\psi\|^p.\end{aligned}$$

As $\mathbf{R}$ is an $m$-quasi-$(\alpha,\beta)$-Class $(\mathcal{Q})$ operators, it follows that

$$(|\lambda|+\alpha|\mu|)^p\,\|\mathbf{R}^{m+2}\psi\|^p \leq \left(\|\lambda\mathbf{R}^{m+2}\psi\|+\|\mu\mathbf{R}^*\mathbf{R}^{m+1}\psi\|\right)^p$$

and

$$(|\lambda|-|\mu|\beta)\,\|\mathbf{R}^{m+2}\psi\| \leq \|\lambda\mathbf{R}^{m+2}\psi\|-\|\mu\mathbf{R}^*\mathbf{R}^{m+1}\psi\|^p \leq (|\lambda|-\alpha|\mu|)\,\|\mathbf{R}^{m+2}\psi\|.$$

Therefore

$$\begin{aligned} &\left((|\lambda|+\beta|\mu|)^p\,\|\mathbf{R}^{m+2}\psi\|^p+max\,\{|\lambda|-|\mu|\beta,\alpha|\mu|-|\lambda|\}\right)\|\mathbf{R}^{m+2}\psi\|^p \\ \leq\ &\left(\|\lambda\mathbf{R}^{m+2}\psi\|+\|\mu\mathbf{R}^*\mathbf{R}^{m+1}\psi\|\right)^p+\left|\|\lambda\mathbf{R}^{m+2}\psi\|-\|\mu R^{m+1}u\|_A\right|^p \\ \leq\ &\|\mu\mathbf{R}^{m+1}u+\mu\mathbf{R}^\sharp\mathbf{R}^m u\|_A^p+\|\lambda\mathbf{R}^{m+1}u-\mu\mathbf{R}^\sharp\mathbf{R}^m u\|_A^p. \end{aligned}$$

Applying the $\sup\limits_{\|\psi\|=1}$ of both sides of the last inequality,the requirement is proven. □

**Theorem 3.8.** *Let $\mathbf{R}\in\mathcal{B}[\mathcal{H}]$ be an $m$-quasi-$(\alpha,\beta)$ Class-$(\mathcal{Q})$ operators and $s\geq 0$. If*

$$\|\lambda\mathbf{R}^*\mathbf{R}^{m+1}-\mathbf{R}^{m+2}\|\leq s\leq \inf\left\{|\lambda|\|\mathbf{R}^*\mathbf{R}^{m+1}\psi\|,\ \ \psi\in\mathcal{H},\ \ \|\psi\|=1\right\},$$

*then the following inequality holds.*

$$\alpha^2\|\mathbf{R}^{m+2}\|^4\leq\omega\left(\mathbf{R}^{*(m+1)}\mathbf{R}^{m+3}\right)^2+\frac{s^2}{|\lambda|^2}\|\mathbf{R}^{m+2}\|^2.$$

*Proof.* We use the following inequality[5],

$$\|\varphi_1\|^2\|\varphi_2\|^2\leq\left[Re\left\langle\varphi_1\mid\varphi_2\right\rangle\right]^2+s^2\|\varphi_2\|^2$$

for which $\|\varphi_1-\varphi_2\|\leq s\leq\|\varphi_2\|$.

Put $\varphi_2=\lambda\mathbf{R}^*\mathbf{R}^{m+1}\psi$ and $\varphi_1=\mathbf{R}^{m+2}\psi$ for $\psi\in\mathcal{H}$ to get,

$$\|\mathbf{R}^{m+2}\psi\|^2\|\lambda\mathbf{R}^*\mathbf{R}^m\psi\|^2\leq\left[Re\left\langle\mathbf{R}^{m+2}\psi\mid\lambda\mathbf{R}^*\mathbf{R}^{m+1}\psi\right\rangle\right]^2+s^2\|\mathbf{R}^{m+2}\psi\|^2.$$

Therefore

$$|\lambda|^2\alpha^2\|\mathbf{R}^{m+2}\psi\|^4\leq|\lambda|^2\left[Re\left\langle\mathbf{R}^{*m+1}\mathbf{R}^{m+3}\psi\mid\psi\right\rangle\right]^2+s^2\|\mathbf{R}^{m+2}\|^2.$$

Applying the $\sup\limits_{\|\psi\|=1}$ of both sides of the last inequality, the requirement is proven. □

**Theorem 3.9.** *Let $\mathbf{R}\in\mathcal{B}[\mathcal{H}]$ be an $m$-quasi-$(\alpha,\beta)$-Class $(\mathcal{Q})$- operator, $s\geq 0$. If $\|\lambda\mathbf{R}^*\mathbf{R}^{m+1}-\mathbf{R}^{m+2}\|\leq s$ for $\lambda\in\mathbb{C},\lambda\neq 0$, then the following holds.*

$$\alpha\|\mathbf{R}^{m+2}\|^2\leq\omega\left[\left(\mathbf{R}^*\right)^{m+1}\mathbf{R}^{m+3}\right]+\frac{s^2}{2|\lambda|}.$$

*Proof.* Using the following inequality mentioned in [6],

$$\|\varphi_1\|\|\ \varphi_2\|\leq\left[Re\left\langle\varphi_1\mid\varphi_2\right\rangle\right]+\frac{s^2}{2},$$

for which $\|\varphi_1-\varphi_2\|\leq s$.

By considering $\varphi_2=\lambda\mathbf{R}^*\mathbf{R}^{m+1}\psi$ and $\varphi_1=\mathbf{R}^{m+2}\psi$ for $\psi\in\mathcal{H}$ to get

$$\|\mathbf{R}^{m+2}\psi\|\|\lambda\mathbf{R}^*\mathbf{R}^m\psi\|\leq\left[Re\left\langle\mathbf{R}^{m+2}\psi\mid\lambda\mathbf{R}^*\mathbf{R}^{m+1}\psi\right\rangle\right]+\tfrac{s^2}{2}.$$

From this we obtain

$$|\lambda|\alpha\|\mathbf{R}^{m+2}\psi\|^2 \leq |\lambda||Re\left\langle \mathbf{R}^{*(m+1)}\mathbf{R}^{m+3}\psi \mid \psi\right\rangle| + \tfrac{s^2}{2}.$$

Applying the $\sup\limits_{\|\psi\|=1}$ of both sides of the last inequality,the requirement is proven. □

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**Remark 1.** *Let $\mathbf{R} \in \mathcal{B}[\mathcal{H}]$,then we have*

*(1) if $\mathbf{R}$ is normal, then $r(\mathbf{R}) = \|\mathbf{R}\|$..*

*(2) if $\mathbf{R}$ is hyponormal, then $r(\mathbf{R}) = \|\mathbf{R}\|$.*

The following theorem presents a generalization of these results to $(\alpha, \beta)$- Class $(\mathcal{Q})$-operator. Our inspiration cames from [13, Theorem 2.5].

**Theorem 3.10.** *Let $\mathbf{R} \in \mathcal{B}[\mathcal{H}]$ be an $(\alpha, \beta)$- Class $(\mathcal{Q})$ operators such that $\mathbf{R}^{2^n}$ is $(\alpha, \beta)$-Class $(\mathcal{Q})$ operators for every $n \in \mathbb{N}$,too. Then,*

$$\frac{1}{\beta}\|\mathbf{R}\| \leq r(\mathbf{R}) \leq \|\mathbf{R}\|.$$

*Proof.* It is well known that if $\mathbf{R} \in \mathcal{B}[\mathcal{H}]$ then

$$\|\mathbf{R}^*\mathbf{R}\| = \|\mathbf{R}\mathbf{R}^*\| = \|\mathbf{R}\|^2$$

and moreover if $\mathbf{R}$ is selfadjoint then

$$\left\|\mathbf{R}^2\right\| = \|\mathbf{R}\|^2.$$

Since $\mathbf{R}$ is $(\alpha, \beta)$-Class $(\mathcal{Q})$ operators, it follows that

$$\alpha^2\left(\mathbf{R}^*\right)^2\mathbf{R}^2 \leq \left(\mathbf{R}^*\mathbf{R}\right)^2 \leq \beta^2\left(\mathbf{R}^*\right)^2\mathbf{R}^2$$

and so

$$\frac{1}{\beta^2}\sup_{\|\psi\|=1}\left\langle\left(\mathbf{R}^*\mathbf{R}\right)^2\psi \mid \psi\right\rangle \leq \sup_{\|\psi\|=1}\left\langle\left(\mathbf{R}^*\right)^2\mathbf{R}^2\psi \mid \psi\right\rangle.$$

Therefore

$$\frac{1}{\beta^2}\|\mathbf{R}\|^4 = \frac{1}{\beta^2}\left\|\left(\mathbf{R}^*\mathbf{R}\right)^2\right\|^2 \leq \left\|\left(\mathbf{R}^*\right)^{\mathbf{2}}\mathbf{R}^2\right\|^2.$$

According to a mathematical induction principal, we can prove that for every positive integer number $n$,

$$\frac{1}{\beta^{2^{n+1}-2}}\|\mathbf{R}\|^{2^{n+1}} \leq \left\|\left(\mathbf{R}^*\right)^{2^n}\mathbf{R}^{2^n}\right\|.$$

We have

$$\begin{aligned} r(\mathbf{R})^2 = r(\mathbf{R}^*)r(\mathbf{R}) &= \lim\sup_{n\longrightarrow\infty}\left\|\left(\mathbf{R}^*\right)^{2^n}\right\|^{\frac{1}{2^n}}\lim\sup_{n\longrightarrow\infty}\left\|\mathbf{R}^{2^n}\right\|^{\frac{1}{2^n}} \\ &\geq \lim_{n\longrightarrow\infty}\left(\left\|\left(\mathbf{R}^*\right)^{2^n}\right\|\left\|\mathbf{R}^{2^n}\right\|\right)^{\frac{1}{2^n}} \\ &\geq \lim_{n\longrightarrow\infty}\left(\left\|\left(\mathbf{R}^\sharp\right)^{2^n}\mathbf{R}^{2^n}\right\|\right)^{\frac{1}{2^n}} \\ &\geq \frac{1}{\beta^2}\|\mathbf{R}\|^2. \end{aligned}$$

Therefore, we get

$$\frac{1}{\beta}\|\mathbf{R}\| \leq r(\mathbf{R}) \leq \|\mathbf{R}\|.$$

This completes the proof. □

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## References

[1] P. Aiena, *Fredholm and local spectral theory with applications to multipliers*, Kluwer Academic Publishers (2004), Dordrecht, Boston, London.

[2] A. M. Ayed AL-Ahmadi, N. M. Saqer Alruwaili and O.A. Mahmoud Sid Ahmed, *Class of operators related to* $(\alpha, \beta)$*-Class* $(\mathcal{Q})$*-operators.* Bulletin of Mathematical Analysis and Applications Volume 15 Issue 4(2023), Pages 12-20. https://doi.org/10.54671/BMAA-2023-4-2

[3] A. Benali1 and O. A. Mahmoud Sid Ahmed, $(\alpha, \beta)$*-A-Normal operators in semi-Hilbertian spaces*,Afrika Matematika (2019) pp. 1012-9405, (2019)

[4] M. L. Buzano, *Generalizzatione della disiguaglianza di Cauchy-Schwaz,*, Rend. Sem. Mat. Univ. E Politech. Torino **31** (1974), 405–409.

[5] S. S. Dragomir, *Reverse of Schwarz, triangle and Bessel inequalities in inner product spaces*, J. Inequal. Pure and Appl. Math. **5** Article 76(2004).

[6] S. S. Dragomir, *Advances in inequalities of the Schwarz, Gruss and Bessel Type in Inner Product Spaces*, Nova Science Publishers (2005), Newyork.

[7] S. S. Dragomir, *A potpourri of Schwarz related inequalities in inner product spaces (II)*, J. Ineq. Pure Appl. Math. **7** (2006), Art 14.

[8] S. S. Dragomir and M.S. Moslehian, *Some Inequalities for* $(\alpha, \beta)$*-normal Operators in Hilbert Spaces*, Ser. Math. Inform. **23** (2008),39–47 .

[9] S. S. Dragomir and J. Sandor, *Some inequalities in pre-Hilertian spaces*, Studia Univ. Babes-Bolyai-Mathematic, **32** (1987), 71–78.

[10] C. F. Dunkl and K.S. Williams, *A simple norm inequality*,Amer. Math. Monthly **71** (1964), 43–44 .

[11] R. Eskandari, F. Mirzapour, A. Morassaei, *More on* $(\alpha, \beta)$*-normal operators in Hilbert spaces.* Abstr. Appl. Anal. 2012, Article ID 204031 (2012).

[12] A. Goldstein, J.V. Ryff and L.E. Clarke, *Problem 5473*, Amer. Math. Monthly **75** (1968), 309.

[13] A. Gupta and P. Sharma, $(\alpha, \beta)$*-Normal Composition Operators*, Thai Journal of Mathematics Volume 14 (2016) Number 1 : 83–92.

[14] V.R. Hamiti, *On k-quasi class* $\mathcal{Q}$ *operators.* Bulletin of Mathematical Analysis and Applications, 6:31-37, (2014)

[15] Sh. Lohaj and V. R. Hamiti. *A note on class* $\mathcal{Q}(N)$ *operators*, Missouri Journal of Math. Sci., 2:185196, 2018.

[16] A. A. Jibril, *On Operators for which* $T^{*2}T^2 = \left(T^*T\right)^2$, International Mathematical Forum. 5 (46): 2255–2262.

[17] J.K. Han, H.Y. Lee and W.Y. Lee, *Invertible completions of* $2 \times 2$ *upper triangular operator matrices*, Proc. Amer. Math. Soc. **128** N0.1,(1999), 119-123.

[18] K.B. Laursen and M. M. Neumann, *An introduction to local spectral theory*, Oxford University Press, 2000

[19] S. Lohaj, *Structural and Spectral Properties of k-Quasi Class* $\mathcal{Q}(N)$ *and k-Quasi Class* $\mathcal{Q}^*(N)$ *Operators*,European Journal of pure and applied mathematics.

[20] O. A. Mahmoud Sid Ahmed and S. Hamidou Jah, *Inequalities Involving A-Numerical Radius and Operator A-Norm for a Class of Operators Related to* $(\alpha, \beta)$*-A-Normal Operators*, Volume 2022, Article ID 1506330, 15 pages.

[21] M. S. Moslehian, *On* $(\alpha, \beta)$*-normal operators in Hilbert spaces*, Image, vol. 39, 2007.

[22] R. Pradeep , P. Maheswari and O. A. Mahmoud Sid Ahmed, *On $m$-Quasi-totaly-$(\alpha, \beta)$-Normal operators*, Operators and Matrices Volume 15, Number 3 (2021), 1055–1072.
[23] M. A. Rosenblum, *On the operator equation* $BX - XA = Q$, Duke Math. J. 23(1956) 263–269.
[24] J. L. Shen, F. Zuo and C. S. Yang, *On Operators Satisfying* $T^*|T^2|T \geq T^*|T^*|^2T$, Acta Mathematica Sinica, English Series Nov., 2010, Vol. 26, No. 11, pp. 2109-2116.
[25] W.Victor1, and A. M. Nyongesa1, *On $(\alpha, \beta)$-Class $(\mathcal{Q})$ Operators*, Int. J. Math. And Appl., 9(2)(2021), 111–113.
[26] W. Victor1, and B. A. Obiero, *On Almost Class $(Q)$ and Class $(M, n)$ Operators*,Int. J. Math. And Appl., 9(2)(2021), 115–118.
[27] W. Victor and B. A. Obiero, *On class $(n, mB\mathcal{Q})$ Operators*, World Journal of Advanced Research and Reviews, 2021, 11(02), 053-057.

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