Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 15 Issue 2 (2024), Pages 34-47
https://doi.org/10.54379/jma-2024-2-3.

COMMON FIXED POINT RESULTS IN COMPLEX VALUED b-METRIC SPACES WITH APPLICATIONS

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Abstract. The purpose of this research article is to point out some fallacies in the statements of the main results of Rao et al. [*Bull. Maths. Stat. Res*] and Berrah et al. [*AIMS Mathematics*, 4(3)(2023), 1019–1033] and give a genuine contractive condition in the framework of complex valued b-metric spaces. We also introduce interpolative rational contractions in complex valued b-metric spaces and prove some fixed point results for such contractions. A non-trivial example is also provided to demonstrate the validity of obtained theorems. As an application, we investigate the existence of solutions for integral equations.

1. Introduction

The concept of classical metric spaces is a basic and elementary tool in topology, functional analysis and nonlinear analysis. This structure has attracted a substantial concentration from researchers because of the evolution of the fixed point theory in classical metric spaces. In the recent past, many generalizations of classical metric spaces have appeared. Czerwik [1] introduced the notion of b-metric space by replacing a number $s \geq 1$ outside the triangle inequality. In 2006, Long-Guang et al. [2] gave the concept of cone metric space (CMS) by replacing the real numbers with ordering Banach space. As a particular case of CMS, Azam et al. [3] initiated the theory of a complex valued metric space (CVMS). Since the idea to introduce CVMS is designed to define rational inequalities which cannot be defined in the framework of CMS. Thus many results of fixed point theory cannot be established to CMS. Hence, we can find many generalizations of fixed point results in the literature regarding divisions in CVMS. Furthermore, this conception is also used to give the idea of complex valued Banach spaces [4]. Rao et al. [5] combined the concepts of b-metric spaces and complex valued metric spaces to introduce the notion of complex valued b-metric space (CVbMS). They proved common fixed point results in this generalized metric space. Subsequently, Berrah et al. [6] used the concept of CVbMS to prove common fixed points of four self mappings. Mukheimer [7] used the contraction of Azam et al. [3] in the context of CVbMS and establish

2000 *Mathematics Subject Classification.* 47H10; 54H25, 46S40.
Key words and phrases. Fixed point, complex valued b-metric space, interpolative rational contraction, integral equations.
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Submitted December 4, 2023. Published February 29, 2024.
Communicated by H. Aydi.

common fixed points of single valued mappings. Dubey et al. [10] proved the existence and uniqueness of a common fixed point for a pair of mappings satisfying generalized contraction under rational expressions having point-dependent control functions as coefficients in complex valued b-metric spaces. In this context, Marzouki et al. [8] and Dubey et al. [9] obtained common fixed point theorems for different generalized contractions. For further features, we refer the researchers to [11-19].

In this research work, we point out some fallacies in the statements of the main results of Rao et al. [5] and Berrah et al. [6] and give a genuine contractive condition to prove common fixed point results in the framework of CVbMS. We also introduce interpolative rational contractions in CVbMS and established some fixed point results for such contractions. Some non-trivial examples are also provided to manifest the validity of obtained results. We solve integral equations by applying our result.

2. Preliminaries

The notion of CVMS is introduced by Azam et al. [3] in this way.

Definition 2.1. *([3]) Let $\omega_1, \omega_2 \in \mathbb{C}$ and $\precsim$ be a partial order on $\mathbb{C}$ which is defined by*

$$\omega_1 \precsim \omega_2 \;\Leftrightarrow\; Re\,(\omega_1) \leqslant Re\,(\omega_2)\,,\; Im\,(\omega_1) \leqslant Im\,(\omega_2)\,.$$

It follows that

$$\omega_1 \precsim \omega_2$$

if atleast one of following axioms is satisfied:

$$\begin{aligned} \text{(a) } \mathrm{Re}\,(\omega_1) &= \mathrm{Re}\,(\omega_2)\,,\; \mathrm{Im}\,(\omega_1) < \mathrm{Im}\,(\omega_2)\,,\\ \text{(b) } \mathrm{Re}\,(\omega_1) &< \mathrm{Re}\,(\omega_2)\,,\; \mathrm{Im}\,(\omega_1) = \mathrm{Im}\,(\omega_2)\,,\\ \text{(c) } \mathrm{Re}\,(\omega_1) &< \mathrm{Re}\,(\omega_2)\,,\; \mathrm{Im}\,(\omega_1) < \mathrm{Im}\,(\omega_2)\,,\\ \text{(d) } \mathrm{Re}\,(\omega_1) &= \mathrm{Re}\,(\omega_2)\,,\; \mathrm{Im}\,(\omega_1) = \mathrm{Im}\,(\omega_2)\,. \end{aligned}$$

Definition 2.2. *([3]) Let $\Omega \neq \emptyset$ and $\sigma : \Omega \times \Omega \to \mathbb{C}$ be a function satisfies*

(i) $0 \precsim \sigma(o, \tau)$, for all $o, \tau \in \Omega$ and $\sigma(o, \tau) = 0 \Longleftrightarrow o = \tau$;
(ii) $\sigma(o, \tau) = \sigma(\tau, o)$ for all $o, \tau \in \Omega$;
(iii) $\sigma(o, \tau) \precsim \sigma(o, \nu) + \sigma(\nu, \tau)$, for all $o, \tau, \nu \in \Omega$,

then (Ω, σ) is called a CVMS.

Example 2.3. *([3]) Let $\Omega = [0, 1]$ and $o, \tau \in \Omega$. Define $\sigma : \Omega \times \Omega \to \mathbb{C}$ by*

$$\sigma(o, \tau) = \begin{cases} 0, & \text{if } o = \tau, \\ \frac{i}{2}, & \text{if } o \neq \tau. \end{cases}$$

Then (Ω, σ) is a CVMS.

Rao et al. [5] furnished with the concept of CVbMS in such a way.

Definition 2.4. *([5]) Let $\Omega \neq \emptyset$, $s \geq 1$ and $\sigma : \Omega \times \Omega \to \mathbb{C}$ be a function fulfilling*

(i) $0 \precsim \sigma(o, \tau)$, and $\sigma(o, \tau) = 0 \Leftrightarrow o = \tau$;
(ii) $\sigma(o, \tau) = \sigma(\tau, o)$,
(iii) $\sigma(o, \tau) \precsim s\,[\sigma(o, \nu) + \sigma(\nu, \tau)]\,,$

for all $o, \tau, \nu \in \Omega$, then (Ω, σ, s) is claimed as a CVbMS.

Example 2.5. *([5])Let $\Omega = [0, 1]$. Define $\sigma : \Omega \times \Omega \to \mathbb{C}$ by*

$$\sigma(o, \tau) = |o - \tau|^2 + i|o - \tau|^2,$$

for all $o, \tau \in \Omega$. Then $(\Omega, \sigma, s = 2)$ is a CVbMS.

Rao et al. [5] proved the following fixed point theorem in CVbMS.

Theorem 2.6. *([6]) Let (Ω, σ) be a complete CVbMS and let $L, M, P, Q : \Omega \to \Omega$. Assume that these conditions are satisfied*

(i) $L(\Omega) \subset Q(\Omega)$ and $M(\Omega) \subset P(\Omega)$,
(ii)

$$\sigma(Lo, M\tau) \preceq \lambda \max\left\{\sigma(Po, Q\tau), \sigma(Po, Lo), \sigma(Q\tau, M\tau), \sigma(Po, M\tau), \sigma(Q\tau, Lo)\right\},$$

for all $o, \tau \in \Omega$, and λ a non negative real number with $0 \leq \lambda < \frac{1}{s^2+s}$,
(iii) the pairs (L, P) and (M, Q) are weakly compatible and $M(\Omega)$ is closed.
Then there exists a unique point $o^* \in \Omega$ such that $Lo^* = Mo^* = Po^* = Qo^* = o^*$.

Recently, Berrah et al. [6] obtained the following fixed point result in the context of CVbMS.

Theorem 2.7. *([6]) Let (Ω, σ) be a complete CVbMS and let $L, M, P, Q : \Omega \to \Omega$. Assume that these conditions are satisfied*

(i) $L(\Omega) \subset Q(\Omega)$ and $M(\Omega) \subset P(\Omega)$,
(ii)

$$\sigma(Lo, M\tau) \preceq \frac{\lambda}{s^2} R(o, \tau),$$

for all $o, \tau \in \Omega$, and λ a non negative reals with $\lambda \in (0, 1)$, $s \geq 1$ where

$$R(o, \tau) = \max\left\{\sigma(Po, Q\tau), \sigma(Po, Lo), \sigma(Q\tau, M\tau), \frac{\sigma(Po, Lo)\,\sigma(Q\tau, M\tau)}{1 + \sigma(Po, Q\tau)}\right\},$$

(iii) (M, Q) is weakly compatible and (L, P) is compatible,
(iv) either P or L is continuous.
Then there exists a unique point $o^* \in \Omega$ such that $Lo^* = Mo^* = Po^* = Qo^* = o^*$.

On the other hand, Mukheimer et al. [7] presented the following theorem.

Theorem 2.8. [7]*Let (Ω, σ) be a complete CVbMS and $L : \Omega \to \Omega$. If there exist some $\lambda_1, \lambda_2 \in [0, 1)$ with $\lambda_1 + \lambda_2 < 1$ such that*

$$\sigma(Lo, L\tau) \precsim \lambda\sigma(o, \tau) + \lambda_2 \left(\frac{\sigma(o, Lo)\sigma(\tau, L\tau)}{1 + \sigma(o, \tau)}\right),$$

for all $o, \tau \in \Omega$, then there exists a unique point $o^* \in \Omega$ such that $Lo^* = o^*$.

Before establishing our main theorem, let us point out a fallacy in the statement of Theorem 3.1 of Rao et al. [5] and Theorem 2.1 of Berrah et al. [6] wherein authors have taken the maximum of complex numbers because elements of complex valued b-metric space are complex numbers, so their statements of the theorems are not well defined since maximum of two complex numbers does not exist.

In this this research article, we give a genuine contractive condition in the setting of complex valued b-metric space and prove common fixed point theorems.

Lemma 2.9. *([5]) Let (Ω, σ) be a CVbMS and let $\{o_n\} \subseteq \Omega$. Then $\{o_n\}$ converges to o if and only if $|\sigma(o_n, o)| \to 0$ when $n \to \infty$.*

Lemma 2.10. *([5]) Let (Ω, σ) be a CVbMS and let $\{o_n\} \subseteq \Omega$. Then $\{o_n\}$ is Cauchy if and only if $|\sigma(o_n, o_{n+m})| \to 0$ when $n \to \infty$, for each $m \in \mathbb{N}$.*

3. Main Results

In this section, we prove some common fixed point theorems for rational type contraction conditions. Our main result runs as follows.

Theorem 3.1. *Let (Ω, σ) be a complete CVbMS with the coefficient $s \geq 1$ and let σ be a continuous functional. Assume that $L, M : \Omega \to \Omega$ be self mappings satisfying the following condition:*

$$\sigma\left(Lo, M\tau\right) \preceq \frac{\lambda}{s^2} R\left(o, \tau\right), \tag{3.1}$$

for all $o, \tau \in \Omega$, and λ a non negative reals with $\lambda \in (0,1)$, $s \geq 1$ where

$$R\left(o, \tau\right) \in \left\{\sigma(o,\tau), \sigma\left(o, Lo\right), \sigma\left(\tau, M\tau\right), \frac{\sigma\left(o, Lo\right)\sigma\left(\tau, M\tau\right)}{1+\sigma\left(o,\tau\right)}\right\}.$$

Then L and M have a unique common fixed point in Ω .

Proof. Let $o_0 \in \Omega$ be an arbitrary point. We generate a sequence $\{o_n\}$ in Ω in this way

$$o_{2n+1} = Lo_{2n} \text{ and } \quad o_{2n+2} = Mo_{2n+1} \tag{3.2}$$

for all $n \in \mathbb{N}$. By (3.1), we get

$$\sigma\left(o_{2n+1}, o_{2n+2}\right) = \sigma\left(Lo_{2n}, Mo_{2n+1}\right) \preceq \frac{\lambda}{s^2} R\left(o_{2n}, o_{2n+1}\right), \tag{3.3}$$

where

$$\begin{aligned} R\left(o_{2n}, o_{2n+1}\right) &\in \left\{ \begin{array}{c} \sigma\left(o_{2n}, o_{2n+1}\right), \sigma\left(o_{2n}, Lo_{2n}\right), \sigma\left(o_{2n+1}, Mo_{2n+1}\right), \\ \frac{\sigma(o_{2n}, Lo_{2n})\sigma(o_{2n+1}, Mo_{2n+1})}{1+\sigma(o_{2n}, o_{2n+1})} \end{array} \right\} \\ &= \left\{ \begin{array}{c} \sigma\left(o_{2n}, o_{2n+1}\right), \sigma\left(o_{2n}, o_{2n+1}\right), \sigma\left(o_{2n+1}, o_{2n+2}\right), \\ \frac{\sigma(o_{2n}, o_{2n+1})\sigma(o_{2n+1}, o_{2n+2})}{1+\sigma(o_{2n}, o_{2n+1})} \end{array} \right\} \\ &= \left\{ \begin{array}{c} \sigma\left(o_{2n}, o_{2n+1}\right), \sigma\left(o_{2n+1}, o_{2n+2}\right), \\ \frac{\sigma(o_{2n}, o_{2n+1})\sigma(o_{2n+1}, o_{2n+2})}{1+\sigma(o_{2n}, o_{2n+1})} \end{array} \right\}. \end{aligned} \tag{3.4}$$

Now we have possible three cases. If

$$R\left(o_{2n}, o_{2n+1}\right) = \sigma\left(o_{2n}, o_{2n+1}\right).$$

Then by (3.3), we have

$$\sigma\left(o_{2n+1}, o_{2n+2}\right) \preceq \frac{\lambda}{s^2}\sigma\left(o_{2n}, o_{2n+1}\right)$$

which implies that

$$\left|\sigma\left(o_{2n+1}, o_{2n+2}\right)\right| \leq \frac{\lambda}{s^2}\left|\sigma\left(o_{2n}, o_{2n+1}\right)\right|. \tag{3.5}$$

If

$$R\left(o_{2n}, o_{2n+1}\right) = \sigma\left(o_{2n+1}, o_{2n+2}\right).$$

Then by (3.3), we have

$$\sigma\left(o_{2n+1}, o_{2n+2}\right) \preceq \frac{\lambda}{s^2}\sigma\left(o_{2n+1}, o_{2n+2}\right)$$

which implies that

$$\left|\sigma\left(o_{2n+1}, o_{2n+2}\right)\right| \leq \frac{\lambda}{s^2}\left|\sigma\left(o_{2n+1}, o_{2n+2}\right)\right| \tag{3.6}$$

which is a contradiction, because $\lambda \in (0,1)$ and $s \geq 1$. □

If $R\left(o_{2n}, o_{2n+1}\right) = \frac{\sigma(o_{2n}, o_{2n+1})\sigma(o_{2n+1}, o_{2n+2})}{1+\sigma(o_{2n}, o_{2n+1})}$. Then by (3.3), we have

$$\sigma\left(o_{2n+1}, o_{2n+2}\right) \preceq \frac{\lambda}{s^2}\frac{\sigma\left(o_{2n}, o_{2n+1}\right)\sigma\left(o_{2n+1}, o_{2n+2}\right)}{1+\sigma\left(o_{2n}, o_{2n+1}\right)}$$

which impies that

$$\left|\sigma\left(o_{2n+1}, o_{2n+2}\right)\right| \leq \frac{\lambda}{s^2}\frac{\left|\sigma\left(o_{2n}, o_{2n+1}\right)\right|\left|\sigma\left(o_{2n+1}, o_{2n+2}\right)\right|}{\left|1+\sigma\left(o_{2n}, o_{2n+1}\right)\right|}.$$

As $\left|1+\sigma\left(o_{2n}, o_{2n+1}\right)\right| > \left|\sigma\left(o_{2n}, o_{2n+1}\right)\right|$, so from above inequality, we have

$$\left|\sigma\left(o_{2n+1}, o_{2n+2}\right)\right| \leq \frac{\lambda}{s^2}\left|\sigma\left(o_{2n+1}, o_{2n+2}\right)\right| \tag{3.7}$$

which is a contradiction, because $\lambda \in (0,1)$ and $s \geq 1$. Thus we conclude that

$$\left|\sigma\left(o_{2n+1}, o_{2n+2}\right)\right| \leq \frac{\lambda}{s^2}\left|\sigma\left(o_{2n}, o_{2n+1}\right)\right|. \tag{3.8}$$

for all $n \in \mathbb{N}$. Similarly from (3.1), we get

$$\sigma\left(o_{2n+2}, o_{2n+3}\right) = \sigma(Mo_{2n+1}, Lo_{2n+2}) = \sigma(Lo_{2n+2}, Mo_{2n+1}) \preceq \frac{\lambda}{s^2}R(o_{2n+2}, o_{2n+1}), \tag{3.9}$$

where

$$\begin{aligned} R\left(o_{2n+2}, o_{2n+1}\right) &\in \left\{\begin{array}{c} \sigma\left(o_{2n+2}, o_{2n+1}\right), \sigma\left(o_{2n+2}, Lo_{2n+2}\right), \sigma\left(o_{2n+1}, Mo_{2n+1}\right), \\ \frac{\sigma(o_{2n+2}, Lo_{2n+2})\sigma(o_{2n+1}, Mo_{2n+1})}{1+\sigma(o_{2n+2}, o_{2n+1})} \end{array}\right\} \\ &= \left\{\begin{array}{c} \sigma\left(o_{2n+2}, o_{2n+1}\right), \sigma\left(o_{2n+2}, o_{2n+3}\right), \sigma\left(o_{2n+1}, o_{2n+2}\right), \\ \frac{\sigma(o_{2n+2}, o_{2n+3})\sigma(o_{2n+1}, o_{2n+2})}{1+\sigma(o_{2n+2}, o_{2n+1})} \end{array}\right\} \\ &= \left\{\begin{array}{c} \sigma\left(o_{2n+2}, o_{2n+1}\right), \sigma\left(o_{2n+2}, o_{2n+3}\right), \\ \frac{\sigma(o_{2n+2}, o_{2n+3})\sigma(o_{2n+1}, o_{2n+2})}{1+\sigma(o_{2n+2}, o_{2n+1})} \end{array}\right\} \\ &= \left\{\begin{array}{c} \sigma\left(o_{2n+1}, o_{2n+2}\right), \sigma\left(o_{2n+2}, o_{2n+3}\right), \\ \frac{\sigma(o_{2n+2}, o_{2n+3})\sigma(o_{2n+1}, o_{2n+2})}{1+\sigma(o_{2n+1}, o_{2n+2})} \end{array}\right\}. \end{aligned} \tag{3.10}$$

Now we have possible three cases. If $R\left(o_{2n+2}, o_{2n+1}\right) = \sigma\left(o_{2n+1}, o_{2n+2}\right)$, then by (3.9), we have

$$\sigma\left(o_{2n+2}, o_{2n+3}\right) \preceq \frac{\lambda}{s^2}\sigma\left(o_{2n+1}, o_{2n+2}\right)$$

which implies

$$\left|\sigma\left(o_{2n+2}, o_{2n+3}\right)\right| \leq \frac{\lambda}{s^2}\left|\sigma\left(o_{2n+1}, o_{2n+2}\right)\right|. \tag{3.11}$$

If $R\left(o_{2n+2}, o_{2n+1}\right)=\sigma\left(o_{2n+2}, o_{2n+3}\right)$, then by (3.9), we have

$$\sigma\left(o_{2n+2}, o_{2n+3}\right) \preceq \frac{\lambda}{s^{2}} \sigma\left(o_{2n+2}, o_{2n+3}\right)$$

which implies that

$$\left|\sigma\left(o_{2n+2}, o_{2n+3}\right)\right| \leq \frac{\lambda}{s^{2}}\left|\sigma\left(o_{2n+2}, o_{2n+3}\right)\right| .$$

which is a contradiction, because $\lambda \in(0,1)$ and $s \geq 1$. Now if $R\left(o_{2n+2}, o_{2n+1}\right)=\frac{\sigma\left(o_{2n+2}, o_{2n+3}\right) \sigma\left(o_{2n+1}, o_{2n+2}\right)}{1+\sigma\left(o_{2n+1}, o_{2n+2}\right)}$, then by (3.9), we have

$$\sigma\left(o_{2n+2}, o_{2n+3}\right) \preceq \frac{\lambda}{s^{2}} \frac{\sigma\left(o_{2n+2}, o_{2n+3}\right) \sigma\left(o_{2n+1}, o_{2n+2}\right)}{1+\sigma\left(o_{2n+1}, o_{2n+2}\right)}$$

which implies

$$\left|\sigma\left(o_{2n+2}, o_{2n+3}\right)\right| \preceq \frac{\lambda}{s^{2}} \frac{\left|\sigma\left(o_{2n+2}, o_{2n+3}\right)\right|\left|\sigma\left(o_{2n+1}, o_{2n+2}\right)\right|}{\left|1+\sigma\left(o_{2n+1}, o_{2n+2}\right)\right|} .$$

As $\left|1+\sigma\left(o_{2n+1}, o_{2n+2}\right)\right|>\left|\sigma\left(o_{2n+1}, o_{2n+2}\right)\right|$, so from above inequality we have

$$\left|\sigma\left(o_{2n+2}, o_{2n+3}\right)\right| \preceq \frac{\lambda}{s^{2}}\left|\sigma\left(o_{2n+2}, o_{2n+3}\right)\right|$$

which is a contradiction, because $\lambda \in(0,1)$ and $s \geq 1$. Thus we conclude that

$$\left|\sigma\left(o_{2n+2}, o_{2n+3}\right)\right| \leq \frac{\lambda}{s^{2}}\left|\sigma\left(o_{2n+1}, o_{2n+2}\right)\right| \tag{3.12}$$

for all $n \in \mathbb{N}$. Hence by (3.8) and (3.12), we have

$$\left|\sigma(o_{n}, o_{n+1})\right| \leq \frac{\lambda}{s^{2}}\left|\sigma(o_{n-1}, o_{n})\right| \leq \cdots \leq\left(\frac{\lambda}{s^{2}}\right)^{n}\left|\sigma(o_{0}, o_{1})\right| \tag{3.13}$$

for all $n \in \mathbb{N}$. Now by (3.13) and triangle inequality for $m < n$, we have

$$\begin{aligned}
|\sigma(o_n, o_m)| &\leq s\left(\frac{\lambda}{s^2}\right)|\sigma(o_n, o_{n+1})| + s^2\left(\frac{\lambda}{s^2}\right)|\sigma(o_{n+1}, o_{n+2})| \\
&\quad +.... + s^{m-1}\left(\frac{\lambda}{s^2}\right)|\sigma(o_{m-1}, o_m)| \\
&\leq s\left(\frac{\lambda}{s^2}\right)^n |\sigma(o_0, o_1)| + s^2\left(\frac{\lambda}{s^2}\right)^{n+1} |\sigma(o_0, o_1)| \\
&\quad +... + s^{m-n}\left(\frac{\lambda}{s^2}\right)^{m-1} |\sigma(o_0, o_1)| \\
&\leq \sum_{i=1}^{m-n} s^i \left(\frac{\lambda}{s^2}\right)^{i+n-1} |\sigma(o_0, o_1)| \\
&\leq \sum_{i=1}^{m-n} s^{i+n-1} \left(\frac{\lambda}{s^2}\right)^{i+n-1} |\sigma(o_0, o_1)| \\
&= \sum_{i=1}^{m-n} \left(\frac{\lambda}{s}\right)^{i+n-1} |\sigma(o_0, o_1)| \\
&= \sum_{t=n}^{m-1} \left(\frac{\lambda}{s}\right)^{t} |\sigma(o_0, o_1)| \\
&\leq \sum_{t=n}^{\infty} \left(\frac{\lambda}{s}\right)^{t} |\sigma(o_0, o_1)| \leq \frac{\left(\frac{\lambda}{s}\right)^n}{\left(1-\frac{\lambda}{s}\right)} |\sigma(o_0, o_1)| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Hence (o_n) is a Cauchy sequence in complete CVbMS (Ω, σ). Thus there exists $o^* \in \Omega$ such that $o_n \to o^*$ as $n \to \infty$. Now we prove that o^* is a fixed point of L. We suppose on the contrary that $o^* \neq Lo^*$. Then $\sigma(o^*, Lo^*) \succeq 0$. Now we have

$$\begin{aligned}
\sigma(o^*, Lo^*) &\preceq s\left(\sigma(o^*, o_{2n+2}) + \sigma(o_{2n+2}, Lo^*)\right) \\
&= s\left(\sigma(o^*, o_{2n+2}) + \sigma(Mo_{2n+1}, Lo^*)\right) \\
&= s\left(\sigma(o^*, o_{2n+2}) + \sigma(Lo^*, Mo_{2n+1})\right) \\
&\preceq s\left(\sigma(o^*, o_{2n+2}) + \frac{\lambda}{s^2} R(o^*, o_{2n+1})\right) \qquad (3.14)
\end{aligned}$$

where

$$\begin{aligned}
R(o^*, o_{2n+1}) &\in \left\{ \begin{array}{c} \sigma(o^*, o_{2n+1}), \sigma(o^*, Lo^*), \sigma(o_{2n+1}, Mo_{2n+1}), \\ \frac{\sigma(o^*, Lo^*)\sigma(o_{2n+1}, Mo_{2n+1})}{1+\sigma(o^*, o_{2n+1})} \end{array} \right\} \\
&= \left\{ \begin{array}{c} \sigma(o^*, o_{2n+1}), \sigma(o^*, Lo^*), \sigma(o_{2n+1}, o_{2n+2}), \\ \frac{\sigma(o^*, Lo^*)\sigma(o_{2n+1}, o_{2n+2})}{1+\sigma(o^*, o_{2n+1})} \end{array} \right\}.
\end{aligned}$$

If $R(o^*, o_{2n+1}) = \sigma(o^*, o_{2n+1})$. Then by (3.14), we have

$$\sigma(o^*, Lo^*) \preceq s\left(\sigma(o^*, o_{2n+2}) + \frac{\lambda}{s^2}\sigma(o^*, o_{2n+1})\right).$$

This implies that

$$|\sigma(o^*, Lo^*)| \leq s\left(|\sigma(o^*, o_{2n+2})| + \frac{\lambda}{s^2}|\sigma(o^*, o_{2n+1})|\right).$$

Taking the limit as $n\to\infty$, we get $|\sigma(o^*,Lo^*)|=0$, a contradiction and, hence $o^*=Lo^*$.

If $R(o^*,o_{2n+1})=\sigma(o^*,Lo^*)$. Then by (3.14), we have

$$\sigma(o^*,Lo^*)\preceq s\left(\sigma(o^*,o_{2n+2})+\frac{\lambda}{s^2}\sigma(o^*,Lo^*)\right).$$

This implies that

$$|\sigma(o^*,Lo^*)|\leq s\left(|\sigma(o^*,o_{2n+2})|+\frac{\lambda}{s^2}|\sigma(o^*,Lo^*)|\right).$$

Taking the limit as $n\to\infty$, we get $|\sigma(o^*,Lo^*)|\leq\frac{\lambda}{s}|\sigma(o^*,Lo^*)|$, a contradiction because $\lambda\in(0,1)$ and $s\geq 1$ and, hence $o^*=Lo^*$.

If $R(o^*,o_{2n+1})=\sigma(o_{2n+1},o_{2n+2})$. Then by (3.14), we have

$$\sigma(o^*,Lo^*)\preceq s\left(\sigma(o^*,o_{2n+2})+\frac{\lambda}{s^2}\sigma(o_{2n+1},o_{2n+2})\right).$$

This implies that

$$|\sigma(o^*,Lo^*)|\leq s\left(|\sigma(o^*,o_{2n+2})|+\frac{\lambda}{s^2}|\sigma(o_{2n+1},o_{2n+2})|\right).$$

Taking the limit as $n\to\infty$, we get $|\sigma(o^*,Lo^*)|=0$, a contradiction and, hence $o^*=Lo^*$.

If $R(o^*,o_{2n+1})=\frac{\sigma(o^*,Lo^*)\sigma(o_{2n+1},o_{2n+2})}{1+\sigma(o^*,o_{2n+1})}$. Then by (3.14), we have

$$\sigma(o^*,Lo^*)\preceq s\left(\sigma(o^*,o_{2n+2})+\frac{\lambda}{s^2}\frac{\sigma(o^*,Lo^*)\,\sigma(o_{2n+1},o_{2n+2})}{1+\sigma(o^*,o_{2n+1})}\right).$$

This implies that

$$|\sigma(o^*,Lo^*)|\leq s\left(|\sigma(o^*,o_{2n+2})|+\frac{\lambda}{s^2}\left|\frac{\sigma(o^*,Lo^*)\,\sigma(o_{2n+1},o_{2n+2})}{1+\sigma(o^*,o_{2n+1})}\right|\right).$$

Taking the limit as $n\to\infty$, we get $|\sigma(o^*,Lo^*)|=0$, a contradiction and, hence $o^*=Lo^*$. Similarly, we can prove that $o^*=Mo^*$

Thus o^* is a common fixed point of the mappings L and M. Let $o^/$ be another common fixed point of L and M, then $Lo^/=Mo^/=o^/$ and $o^/\neq o^*$.

Now by (3.1) with $o=o^/,\tau=o^*$, we have

$$\sigma(o^/,o^*)=\sigma(Lo^/,Mo^*)\preceq\frac{\lambda}{s^2}R(o^/,o^*) \tag{3.15}$$

where

$$\begin{aligned} R\left(o^/,o^*\right) &\in \left\{\begin{array}{c}\sigma\left(o^/,o^*\right),\sigma\left(o^/,Lo^/\right),\sigma(o^*,Mo^*),\\ \frac{\sigma\left(o^/,Lo^/\right)\sigma(o^*,Mo^*)}{1+\sigma\left(o^/,o^*\right)}\end{array}\right\}\\ &= \left\{\begin{array}{c}\sigma\left(o^/,o^*\right),\sigma\left(o^/,o^/\right),\sigma(o^*,o^*),\\ \frac{\sigma\left(o^/,o^/\right)\sigma(o^*,o^*)}{1+\sigma\left(o^/,o^*\right)}\end{array}\right\}\\ &= \left\{\sigma\left(o^/,o^*\right),0\right\}.\end{aligned}$$

If $R\left(o^{/},o^{*}\right)=\sigma\left(o^{/},o^{*}\right)$, then from (3.15), we have

$$\sigma\left(o^{/},o^{*}\right)\preceq\frac{\lambda}{s^{2}}\sigma\left(o^{/},o^{*}\right),$$

which implies that

$$\left|\sigma\left(o^{/},o^{*}\right)\right|\leq\frac{\lambda}{s^{2}}\left|\sigma\left(o^{/},o^{*}\right)\right|,$$

a contradiction because because $\lambda\in(0,1)$ and $s\geq 1$. Hence $o^{/}=o^{*}$. Thus o^{*} is the unique common fixed point of L and M.

Corollary 3.2. *Let (Ω,σ) be a complete CVbMS with the coefficient $s\geq 1$ and let σ be a continuous functional. Assume that $L:\Omega\to\Omega$ be a self mapping satisfying the following condition:*

$$\sigma\left(Lo,L\tau\right)\preceq\frac{\lambda}{s^{2}}R\left(o,\tau\right), \tag{3.16}$$

for all $o,\tau\in\Omega$, and λ a non negative reals with $\lambda\in(0,1)$, $s\geq 1$ where

$$R\left(o,\tau\right)\in\left\{\sigma(o,\tau),\sigma\left(o,Lo\right),\sigma\left(\tau,L\tau\right),\frac{\sigma\left(o,Lo\right)\sigma\left(\tau,L\tau\right)}{1+\sigma\left(o,\tau\right)}\right\}.$$

Then L has a unique fixed point in Ω .

Proof. Taking $L=M$ in Theorem 3.1. □

Corollary 3.3. *Let (Ω,σ) be a complete CVbMS with the coefficient $s\geq 1$ and let σ be a continuous functional. Assume that $L:\Omega\to\Omega$ be a self mapping satisfying the following condition:*

$$\sigma\left(L^{n}o,L^{n}\tau\right)\preceq\frac{\lambda}{s^{2}}R\left(o,\tau\right),$$

for all $o,\tau\in\Omega$, and λ a non negative reals with $\lambda\in(0,1)$, $s\geq 1$ where

$$R\left(o,\tau\right)\in\left\{\sigma(o,\tau),\sigma\left(o,L^{n}o\right),\sigma\left(\tau,L^{n}\tau\right),\frac{\sigma\left(o,L^{n}o\right)\sigma\left(\tau,L^{n}\tau\right)}{1+\sigma\left(o,\tau\right)}\right\}.$$

Then L has a unique fixed point in Ω .

Proof. By Corollary 3.2 we obtain that o^{*} is a unique fixed point of L^{n}, that is, $L^{n}o^{*}=o^{*}$. Now using this fact, we have

$$\sigma\left(o^{*},Lo^{*}\right)=\sigma\left(L^{n}o^{*},LL^{n}o^{*}\right)=\sigma\left(L^{n}o^{*},L^{n}Lo^{*}\right)\preceq\frac{\lambda}{s^{2}}R\left(o^{*},Lo^{*}\right) \tag{3.17}$$

where

$$\begin{aligned}R\left(o^{*},Lo^{*}\right) &\in \left\{\sigma\left(o^{*},Lo^{*}\right),\sigma\left(o^{*},L^{n}o^{*}\right),\sigma\left(Lo^{*},L^{n}Lo^{*}\right),\frac{\sigma\left(o^{*},L^{n}o^{*}\right)\sigma\left(Lo^{*},L^{n}Lo^{*}\right)}{1+\sigma\left(o^{*},Lo^{*}\right)}\right\}\\ &= \left\{\sigma\left(o^{*},Lo^{*}\right),\sigma\left(o^{*},o^{*}\right),\sigma\left(Lo^{*},Lo^{*}\right),\frac{\sigma\left(o^{*},o^{*}\right)\sigma\left(Lo^{*},Lo^{*}\right)}{1+\sigma\left(o^{*},Lo^{*}\right)}\right\}\\ &= \sigma\left(o^{*},Lo^{*}\right).\end{aligned}$$

Thus by (3.17), we have

$$\sigma\left(o^{*},Lo^{*}\right)\preceq\frac{\lambda}{s^{2}}\sigma\left(o^{*},Lo^{*}\right)$$

which implies that

$$\left(1-\frac{\lambda}{s^2}\right)\sigma\left(o^*,Lo^*\right)\preceq 0,$$

which is possible only if $o^*=Lo^*$. Thus we proved that L has a unique fixed point in Ω. □

Remark. *If we take $s=1$ in the Definition (2.4), then the notion of CVbMS is reduced to CVMS and we derive some results of Hussain et al.*[20].

Example 3.4. *Let $\Omega=\{(1,2),(2,3),(3,4),(4,5),(5,6)\}$ and define a mapping $\sigma:\Omega\times\Omega\to\mathbb{C}$ as*

$$\sigma(z_1,z_2)=|o_1-o_2|^2+i|\tau_1-\tau_2|^2$$

where $z_1=o_1+i\tau_1, z_2=o_2+i\tau_2$, then (Ω,σ) is a complete CVbMS. Define a self mappings L on Ω (with $z=o+i\tau$) as

$$Lz=|o-\tau|+2i|o-\tau|$$

for all $z\in\Omega$. By a routine calculation, one can easily verify that L satisfies the contractive condition (3.16) and has a unique fixed point $(1,2)$ in Ω.

Definition 3.5. *Let (Ω,σ) be a CVbMS. A mapping $L:\Omega\to\Omega$ is said to be interpolative rational contraction if there exist $\lambda\in[0,1)$ and $\alpha\in(0,1)$ such that*

$$\sigma(Lo,L\tau)\precsim\lambda\left(\frac{\sigma(o,Lo)\sigma(\tau,L\tau)}{1+\sigma(o,\tau)}\right)^{\alpha}(\sigma(o,\tau))^{1-\alpha} \tag{3.18}$$

for all $o,\tau\in\Omega\backslash Fix(L)$, where $Fix(L)=\{o\in\Omega:o=Lo\}$.

Theorem 3.6. *Let (Ω,σ) be a complete CVbMS and the mapping $L:\Omega\to\Omega$ is an interpolative rational contraction, then L has a fixed point.*

Proof. Let $o_0\in\Omega$ be an arbitrary point. We generate a sequence $\{o_n\}$ as $o_{n+1}=To_n$ for each positive integer n. If there exists some n_0 such that $o_{n_0+1}=To_{n_0}$, then o_{n_0} is a fixed point of mapping L. The proof is completed. So, assume that $o_{n+1}=To_n$ for all $n\geq 0$. Now by (3.18), we have

$$\begin{aligned}\sigma(o_n,o_{n+1}) &= \sigma(Lo_{n-1},Lo_n)\precsim\lambda\left(\frac{\sigma(o_{n-1},Lo_{n-1})\sigma(o_n,Lo_n)}{1+\sigma(o_{n-1},o_n)}\right)^{\alpha}(\sigma(o_{n-1},o_n))^{1-\alpha}\\ &= \lambda\left(\frac{\sigma(o_{n-1},o_n)\sigma(o_n,o_{n+1})}{1+\sigma(o_{n-1},o_n)}\right)^{\alpha}(\sigma(o_{n-1},o_n))^{1-\alpha}\end{aligned}$$

which implies that

$$\begin{aligned}|\sigma(o_n,o_{n+1})| &\leq \lambda\left(\left|\frac{\sigma(o_{n-1},o_n)}{1+\sigma(o_{n-1},o_n)}\right||\sigma(o_n,o_{n+1})|\right)^{\alpha}(|\sigma(o_{n-1},o_n)|)^{1-\alpha}\\ &\leq \lambda\left(|\sigma(o_n,o_{n+1})|\right)^{\alpha}(|\sigma(o_{n-1},o_n)|)^{1-\alpha}\end{aligned}$$

which implies that

$$|\sigma(o_n,o_{n+1})|^{1-\alpha}\leq\lambda\,|\sigma(o_{n-1},o_n)|^{1-\alpha}. \tag{3.19}$$

Suppose that $|\sigma(o_n,o_{n+1})|\geq|\sigma(o_{n-1},o_n)|$, then from (3.19), we have

$$|\sigma(o_n,o_{n+1})|^{1-\alpha}\leq\lambda\,|\sigma(o_{n-1},o_n)|^{1-\alpha}\leq\lambda\,|\sigma(o_n,o_{n+1})|^{1-\alpha}$$

which is a contradiction because $\lambda<1$. Thus we we conclude that $|\sigma(o_n,o_{n+1})|\leq|\sigma(o_{n-1},o_n)|$ for all $n\geq 1$. Thus $\{|\sigma(o_{n-1},o_n)|\}$ is a non-increasing sequence and by the inequality (3.19) that there exists a nonnegative constant $l \geq 0$ such that $\lim_{n\to\infty} |\sigma(o_{n-1}, o_n)| = l$. Then the inequality (3.19) yields that

$$|\sigma(o_n, o_{n+1})| \leq \lambda |\sigma(o_{n-1}, o_n)| \leq ... \leq \lambda^n |\sigma(o_0, o_1)| . \tag{3.20}$$

Now since $\lambda < 1$, so if we take the limit as $n \to \infty$ in the inequality (3.20), we get $l = 0$. Now we prove that $\{o_n\}$ is a Cauchy sequence. By triangle inequality for $m, n \in \mathbb{N}$ with $m < n$, we have

$$\begin{aligned} |\sigma(o_m, o_n)| &\leq s|\sigma(o_m, o_{m+1})| + s^2|\sigma(o_{m+1}, o_{m+2})| \\ &\quad +.... + s^{n-m-1}|\sigma(o_{n-1}, o_n)| \\ &\leq s(\lambda)^m |\sigma(o_0, o_1)| + s^2 (\lambda)^{m+1} |\sigma(o_0, o_1)| \\ &\quad +... + s^{n-m-1} (\lambda)^{n-1} |\sigma(o_0, o_1)| \\ &\leq s\lambda^m \left(1 + s\lambda + s^2\lambda^2 + ... + s^{n-m-1}\lambda^{n-m-1}\right) |\sigma(o_0, o_1)| \\ &\leq s\lambda^m \left[\sum_{i=0}^{\infty} (s\lambda)^i\right] |\sigma(o_0, o_1)| \\ &\leq \frac{s\lambda^m}{(1 - s\lambda)} |\sigma(o_0, o_1)| \to 0 \text{ as } m \to \infty \end{aligned}$$

which yields that $\{o_n\}$ is a Cauchy sequence in the complete CVbMS (Ω, σ). Hence, there exists $o^* \in \Omega$ such that

$$\lim_{n\to\infty} o_n = o^*.$$

Now we prove that o^* is a fixed point of mapping L. We assume on the contrary that $o^* \neq Lo^*$. By (3.18), we have

$$\begin{aligned} \sigma(o_{n+1}, Lo^*) &= \sigma(Lo_n, Lo^*) \precsim \lambda \left(\frac{\sigma(o_n, Lo_n)\sigma(o^*, Lo^*)}{1 + \sigma(o_n, o^*)}\right)^\alpha (\sigma(o_n, o^*))^{1-\alpha} \\ &= \lambda \left(\frac{\sigma(o_n, o_{n+1})\sigma(o^*, Lo^*)}{1 + \sigma(o_n, o^*)}\right)^\alpha (\sigma(o_n, o^*))^{1-\alpha} . \end{aligned} \tag{3.21}$$

Letting $n \to \infty$, in the inequality (3.21), we find that $\sigma(o^*, Lo^*) = 0$, which is a contradiction. Thus, $o^* = Lo^*$. □

Example 3.7. *Let* $\Omega = \{0, 1, 2, 3\}$ *and* $\sigma : \Omega \times \Omega \to \mathbb{C}$ *is given as*

$$\sigma(o, \tau) = |o - \tau|^2 + i|o - \tau|^2,$$

then (Ω, σ) *is a complete CVbMS. Define* $L : \Omega \to \Omega$ *by*

$$Lo = \begin{cases} 1, & \text{if } o \in \{0, 1, 3\} \\ 2, & \text{if } o = 2. \end{cases}$$

for all $o \in \Omega$*. First we prove that Theorem 2.8 of Mukheimer* [7] *is not satisfied for* $\lambda_1 = \frac{1}{10}$, $\lambda_2 = \frac{5}{10}$, $o = 1$ *and* $\tau = 2$.

$$\sigma(Lo, L\tau) = 1 + i \succ \frac{1}{10}(1 + i) = \lambda_1\sigma(o, \tau) + \lambda_2 \left(\frac{\sigma(o, Lo)\sigma(\tau, L\tau)}{1 + \sigma(o, \tau)}\right).$$

On the other side, we take $\lambda = \frac{1}{2}$ *and* $\alpha = \frac{1}{2}$*. Now let* $o, \tau \in \Omega \backslash Fix(L)$*, then* $o, \tau \in \{(0, 3), (3, 0)\}$ *and we have two cases.*

Case 1. If $o = 0$ and $\tau = 3$, then

$$\sigma(L0, L3) = 0 + 0i \prec \frac{1}{2}\left(\frac{(1+i)(4+4i)}{(10+9i)}\right)^{\frac{1}{2}} (9+9i)^{\frac{1}{2}}.$$

Case 2. If $o = 3$ and $\tau = 0$, then

$$\sigma(L3, L0) = 0 + 0i \prec \frac{1}{2}\left(\frac{(4+4i)(1+i)}{(10+9i)}\right)^{\frac{1}{2}} (9+9i)^{\frac{1}{2}}.$$

Hence all the conditions of Theorem 3.6 are satisfied and L has two fixed points that are 1 and 2.

4. Applications

Fixed point theorems for operators in ordered Banach spaces are widely investigated and have found various applications in differential and integral equations (see [13, 19, 20] and references therein).

Theorem 4.1. *Let $\Omega = C([a,b], \mathbb{R}^n)$, $a > 0$ and $\sigma : \Omega \times \Omega \to \mathbb{C}$ be defined as follows:*

$$\sigma(o, \tau) = \max_{t\in[a,b]} \|o(t) - \tau(t)\|_\infty e^{i\frac{\pi}{2}}.$$

Then (Ω, σ) is complete CVbMS (see. [6]). Consider

$$o(t) = \int_a^b K_1(t, s, o(s))\sigma s + \varphi(t), \tag{4.1}$$

$$o(t) = \int_a^b K_2(t, s, o(s))\sigma s + \psi(t), \tag{4.2}$$

for $t \in [a,b] \subset \mathbb{R}$, $o, \varphi, \psi \in \Omega$.

Assume that $K_1, K_2 : [a,b] \times [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ are such that $F_o, G_o \in \Omega$ for each $o \in \Omega$, where,

$$F_o(t) = \int_a^b K_1(t, s, o(s))\sigma s,$$

and

$$G_o(t) = \int_a^b K_2(t, s, o(s))\sigma s$$

for all $t \in [a,b]$. If there exist $s \geq 1$ and $0 < \lambda < 1$ such that

$$\|F_o(t) - G_\tau(t) + \varphi(t) - \psi(t)\|_\infty e^{i\frac{\pi}{2}} \precsim \frac{\lambda}{s^2} R(o,\tau)(t), \tag{4.3}$$

where

$$R(o,\tau)(t) \in \{A(o,\tau)(t), B(o,\tau)(t), C(o,\tau)(t), E(o,\tau)(t)\}, \tag{4.4}$$

$$\begin{aligned}
A(o,\tau)(t) &= \|o(t) - \tau(t)\|_\infty e^{i\frac{\pi}{2}}, \\
B(o,\tau)(t) &= \|F_o(t) + \varphi(t) - o(t)\|_\infty e^{i\frac{\pi}{2}}, \\
C(o,\tau)(t) &= \|G_\tau(t) + \psi(t) - \tau(t)\|_\infty e^{i\frac{\pi}{2}} \\
E(o,\tau)(t) &= \frac{\|F_o(t) + \varphi(t) - o(t)\|_\infty \|G_\tau(t) + \psi(t) - \tau(t)\|_\infty}{1 + A(o,\tau)(t)} e^{i\frac{\pi}{2}}
\end{aligned}$$

holds for all $o, \tau \in \Omega$, then the system of integral equations (4.1) and (4.2) have a unique common solution.

Proof. Define $L, M : \Omega \to \Omega$ by

$$Lo = F_o + \varphi, \quad Mo = G_o + \psi.$$

Then

$$\sigma\left(Lo, M\tau\right) = \max_{t\in[a,b]} \left\|F_o\left(t\right) - G_\tau\left(t\right) + \varphi(t) - \psi(t)\right\|_\infty e^{i\frac{\pi}{2}},$$

$$\sigma\left(o, \tau\right) = \max_{t\in[a,b]} A\left(o, \tau\right)\left(t\right),$$

$$\sigma\left(o, Lo\right) = \max_{t\in[a,b]} B\left(o, \tau\right)\left(t\right),$$

$$\sigma\left(\tau, M\tau\right) = \max_{t\in[a,b]} C\left(o, \tau\right)\left(t\right)$$

$$\frac{\sigma(o, Lo)\sigma(\tau, M\tau)}{1 + \sigma(o, \tau)} = \max_{t\in[a,b]} E\left(o, \tau\right)\left(t\right).$$

Thus from assumptions (4.3) and (4.4), for each $t \in [a, b]$, we have

$$\sigma(Lo, M\tau) \precsim \frac{\lambda}{s^2} R(o, \tau)$$

where

$$R(o, \tau) \in \left\{\sigma(o, \tau), \sigma(o, Lo), \sigma(\tau, M\tau), \frac{\sigma(o, Lo)\sigma(\tau, M\tau)}{1 + \sigma(o, \tau)}\right\}$$

for every $o, \tau \in \Omega$. Thus by Theorem 3.1, the system of integral equations (4.1) and (4.2) have a unique common solution. □

5. Conclusion

In this research article, we have pointed out some fallacies in the statments of the main results of Rao et al. [*Bull. Maths. Stat. Res*] and Berrah et al. [*AIMS Mathematics*, 4(3)(2023), 1019–1033] and gave a genuine contractive condition to prove common fixed point results in the background of CVbMS. We also introduced interpolative rational contractions in CVbMS and established some fixed point results for such contractions. Some non-trivial examples are also provided to manifest the validity of obtained results. By applying our prime result, we have investigated the existence of solutions for integral equations.

The given results in this research work can be augmented to some multivalued mappings and fuzzy mappings in the framework of CVbMS. Additionally, common fixed point results for self and non self mappings can be proved in this context. As utilizations of these outcomes in the background of CVbMS, some differential and integral inclusions can be explored.

Conflicts of Interest:

The authors declare that they have no conflicts of interest.

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

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