Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 15 Issue 2 (2024), Pages 14-33
https://doi.org/10.54379/jma-2024-2-2.

GENERALIZED FOURIER TRANSFORM: ILLUSTRATIVE EXAMPLES AND APPLICATIONS TO DIFFERENTIAL EQUATIONS

ENES ATA AND İ. ONUR KIYMAZ

Abstract. In this paper, we define generalized Fourier and inverse Fourier transforms containing the h-exponential function in their kernels and give the fundamental properties of these transforms. We also compute the transforms of both the classical and the generalized Riemann-Liouville and Caputo fractional operators. In addition, we compute the transforms of some elementary and generalized special functions as well. Finally, as applications, we obtain the solutions of two differential equations with ordinary and fractional derivatives using the transforms we have defined.

1. Introduction

The Fourier transform is named after the famous French mathematician Jean-Baptiste Joseph Fourier. In the 19th century, Jean-Baptiste Joseph Fourier found that any function can be written as the sum of the sine and cosine functions. He later discovered that it was possible to determine the amplitude of sine and cosine waves by means of the integral and thus obtained the Fourier transform, which is of great importance in the world of science. For the famous work of Jean-Baptiste Joseph Fourier, see [10].

The Fourier transform is one of the most important mathematical tools used in a wide variety of fields such as physics, chemistry, biology, medicine, astronomy, engineering and mathematics, and is expressed as the transformation of a signal in a time or space domain into the frequency domain. Also, various integral equations and differential equations, which are difficult to calculate, can be easily calculated via the Fourier transform to obtain simple algebraic structures. Therefore, the Fourier transform is a popular formula of great importance used in the applications of various scientific fields.

The Fourier transform has been of great interest to scientists and has found applications in many fields. For examples; the Fourier transform are used by circuit

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2000 *Mathematics Subject Classification.* 42A38, 26A33, 34B30, 33B15.
Key words and phrases. Fourier transform; fractional derivatives and integrals; ordinary differential equations; fractional differential equations; gamma function; beta function.

Submitted June 20, 2023. Published March 11, 2024.
Communicated by C. Tunc.

designers, in their work on circuits; systems and sound engineers, in their work on audio and video processing and coding; statisticians and probabilists, in their work on characterizing probability distributions; error control code designers, in their work on error correction and detection; radio astronomers and antenna designers, in their work on image formation from antenna data; spectroscopists, in their work on high-resolution spectra; crystallographers, in their work on X-ray; lens designers, in their work on camera performance; psychologists, in their work on studying perception; biomedical engineers, in their work on medical imaging; mathematicians, in their work on various integral and differential equations.

For more detailed information on the Fourier transform, see [9, 11, 12, 14, 22, 29]. Also for various scientific research papers on the Fourier transform, see [13, 18, 20, 21, 26].

Our motivation in this paper is to define a new generalization of the Fourier transform by adding the h-exponential function to the Fourier transform, making it possible and easy to compute the transformation of the generalized special functions and the generalized fractional derivatives, which involving the exponential functions in their kernels. Thus, the application area of the Fourier transform will also expand.

We organize the rest of the paper as follows: In Section 2, we provide the basic materials needed throughout the paper. In Section 3, we describe new generalized Fourier and inverse Fourier transforms and present some fundamental properties. In Section 4, we take the generalized Fourier transform of some elementary functions, the generalized special functions and the generalized fractional derivatives. In Section 5, we obtain the solutions of the ordinary RL electric circuit and the fractional motion differential equations via the generalized Fourier and inverse Fourier transforms. In Section 6, we give conclusion and recommendations.

2. Preliminaries

In this section, we give the basic materials needed throughout the paper, such as the Schwartz and Lizorkin space, the classical Fourier and inverse Fourier transforms, the Fourier convolution, the Dirac delta function, the classical gamma and beta functions, the generalized gamma and beta functions, the Riemann-Liouville fractional integral and derivative, the Caputo fractional derivative, the generalized Riemann-Liouville and Caputo fractional derivatives.

Definition 2.1 ([29])**.** *The Schwartz space is defined by*

$$S(\mathbb{R}) = \left\{ f \in C^{\infty} : \sup \left| \Delta^m f^{(n)}(\Delta) \right| < \infty, \quad \text{for } \forall m, n \in \mathbb{N}_0 \right\}.$$

Definition 2.2 ([27])**.** *The $V(\mathbb{R})$ set be as follows*

$$V(\mathbb{R}) = \left\{ f \in S(\mathbb{R}) : f^{(n)}(0) = 0, \quad \text{for } n \in \mathbb{N}_0 \right\}.$$

Then the Lizorkin space is given by

$$\phi(\mathbb{R}) = \left\{ f \in S(\mathbb{R}) : \mathfrak{F}\left[f \right] \in V(\mathbb{R}) \right\}.$$

Definition 2.3 ([17]). *The Fourier and inverse Fourier transforms respectively are defined by*

$$\mathfrak{F}\Big[f(\Delta)\Big](\omega) = \int_{-\infty}^{+\infty} \exp(i\omega\Delta) f(\Delta) d\Delta, \tag{2.1}$$

$$\mathfrak{F}^{-1}\Big[\mathfrak{F}\Big[f(\Delta)\Big](\omega)\Big](\Delta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(-i\omega\Delta)\mathfrak{F}\Big[f(\Delta)\Big](\omega) d\omega. \tag{2.2}$$

Definition 2.4 ([5]). *The Fourier convolution of functions f and g is defined by*

$$f(\Delta) * g(\Delta) = \int_{-\infty}^{+\infty} f(\Delta - y) g(y) dy, \quad \text{for } \Delta \in \mathbb{R}. \tag{2.3}$$

Definition 2.5 ([8]). *The Dirac delta function is given by*

$$\delta(\Delta) = \begin{cases} 0 & , \textit{ for } \Delta \neq 0, \\ \infty & , \textit{ for } \Delta = 0. \end{cases}$$

Also the following equation in [8] *holds true*

$$\int_{-\infty}^{+\infty} f(\Delta)\delta(\Delta) d\Delta = f(0). \tag{2.4}$$

Theorem 2.1 ([8]). *Let c be an arbitrary constant. Then,*

$$\int_{-\infty}^{+\infty} \exp(i\omega\Delta)\, c \; d\Delta = 2c\pi\delta(\omega). \tag{2.5}$$

Definition 2.6 ([3]). *The gamma and beta functions respectively are defined by*

$$\Gamma(\xi) = \int_{0}^{\infty} u^{\xi-1} \exp(-u)\, du,$$

$$B(\xi, \eta) = \int_{0}^{1} u^{\xi-1} (1-u)^{\eta-1} du,$$

where $\Re(\xi) > 0$, $\Re(\eta) > 0$.

Definition 2.7. *Let* $\Re(\xi) > 0$, $\Re(\eta) > 0$, $\Re(\lambda) > 0$, $\Re(\kappa) > 0$, $\Re(\mu) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\Delta) > 0$. *Then,*

(1) *The generalized gamma function defined by Chaudhry and Zubair* [6] *is as*

$$\Gamma_{\Delta}(\xi) = \int_{0}^{\infty} u^{\xi-1} \exp\left(-u - \frac{\Delta}{u}\right) du. \tag{2.6}$$

(2) *The generalized gamma function defined by Parmar* [25] *is as*

$$\Gamma_{\Delta}^{(\alpha,\beta;\lambda)}(\xi) = \int_{0}^{\infty} u^{\xi-1}{}_1F_1\left(\alpha;\beta;-u-\frac{\Delta}{u^{\lambda}}\right)du. \tag{2.7}$$

(3) *The generalized beta function defined by Chaudhry et al.* [7] *is as*

$$B(\xi,\eta;\Delta) = \int_{0}^{1} u^{\xi-1}(1-u)^{\eta-1}\exp\left(-\frac{\Delta}{u(1-u)}\right)du. \tag{2.8}$$

(4) *The generalized beta function defined by Lee et al.* [19] *is as*

$$B(\xi,\eta;\Delta;\lambda) = \int_{0}^{1} u^{\xi-1}(1-u)^{\eta-1}\exp\left(-\frac{\Delta}{u^{\lambda}(1-u)^{\lambda}}\right)du. \tag{2.9}$$

(5) *The generalized beta function defined by Srivastava et al.* [28] *is as*

$$B_{\Delta}^{(\alpha,\beta;\kappa,\mu)}(\xi,\eta) = \int_{0}^{1} u^{\xi-1}(1-u)^{\eta-1}{}_1F_1\left(\alpha;\beta;-\frac{\Delta}{u^{\kappa}(1-u)^{\mu}}\right)du. \tag{2.10}$$

Definition 2.8 ([17]). *The Riemann-Liouville fractional integral (RLFI), Riemann-Liouville fractional derivative (RLFD) and Caputo fractional derivative (CFD) of order ε respectively are defined by*

$$ {}_{-\infty}I_{\Delta}^{\varepsilon}f(\Delta) = \frac{1}{\Gamma(\varepsilon)}\int_{-\infty}^{\Delta}(\Delta-y)^{\varepsilon-1}f(y)dy,$$

$$ {}_{-\infty}D_{\Delta}^{\varepsilon}f(\Delta) = \frac{1}{\Gamma(r-\varepsilon)}\frac{d^r}{d\Delta^r}\int_{-\infty}^{\Delta}(\Delta-y)^{r-\varepsilon-1}f(y)dy,$$

$$ {}_{-\infty}^{\ c}D_{\Delta}^{\varepsilon}f(\Delta) = \frac{1}{\Gamma(r-\varepsilon)}\int_{-\infty}^{\Delta}(\Delta-y)^{r-\varepsilon-1}f^{(r)}(y)dy,$$

where $\Delta\in\mathbb{R}$, $r\in\mathbb{N}$, $r-1<\Re(\varepsilon)<r$, $\Re(\varepsilon)>0$.

Definition 2.9. *Let $\Re(\Delta)>0$, $\Re(\lambda)>0$, $\Re(\kappa)>0$, $\Re(\mu)>0$, $\Re(\varepsilon)>0$ and $r-1<\Re(\varepsilon)<r$, $r\in\mathbb{N}$. Then,*

(1) *The generalized fractional derivative defined by Agarwal et al.* [1] *is as*

$$D_{v}^{\varepsilon,\Delta;\kappa,\mu}f(v) = \frac{1}{\Gamma(r-\varepsilon)}\frac{d^r}{dv^r}\int_{0}^{v}(v-\rho)^{r-\varepsilon-1}{}_1F_1\left(\alpha;\beta;-\frac{\Delta v^{\kappa+\mu}}{\rho^{\kappa}(v-\rho)^{\mu}}\right)f(\rho)d\rho. \tag{2.11}$$

(2) *The generalized fractional derivative defined by Parmar* [24] *is as*

$$D_{v}^{\varepsilon,\Delta;\lambda}\{f(v)\} = \frac{1}{\Gamma(r-\varepsilon)}\frac{d^r}{dv^r}\int_{0}^{v}(v-\rho)^{r-\varepsilon-1}\exp\left(-\frac{\Delta v^{2\lambda}}{\rho^{\lambda}(v-\rho)^{\lambda}}\right)f(\rho)d\rho. \tag{2.12}$$

(3) *The generalized fractional derivative defined by Agarwal et al.* [2] *is as*

$$D_v^{\varepsilon,\Delta;\lambda} f(v) = \frac{1}{\Gamma(r-\varepsilon)} \int_0^v (v-\rho)^{r-\varepsilon-1} \exp\left(-\frac{\Delta v^{2\lambda}}{\rho^\lambda (v-\rho)^\lambda}\right) f^{(r)}(\rho) d\rho. \tag{2.13}$$

(4) *The generalized fractional derivative defined by Özarslan and Özergin* [23] *is as*

$$D_v^{\varepsilon,\Delta}\left\{f(v)\right\} = \frac{1}{\Gamma(r-\varepsilon)} \frac{d^r}{dv^r} \int_0^v (v-\rho)^{r-\varepsilon-1} \exp\left(-\frac{\Delta v^2}{\rho(v-\rho)}\right) f(\rho) d\rho. \tag{2.14}$$

(5) *The generalized fractional derivative defined by Kıymaz et al.* [16] *is as*

$$D_v^{\varepsilon,\Delta} f(v) = \frac{1}{\Gamma(r-\varepsilon)} \int_0^v (v-\rho)^{r-\varepsilon-1} \exp\left(-\frac{\Delta v^2}{\rho(v-\rho)}\right) f^{(r)}(\rho) d\rho. \tag{2.15}$$

3. New generalized Fourier and inverse Fourier transforms and fundamental properties

In this section, we describe new generalized Fourier and inverse Fourier transforms with h-exponential functions in their kernels. Then, we present basic properties such as linearity, differentiation, convolution theorems. We also apply the generalized Fourier transform to the Riemann-Liouville fractional integral and derivative and the Caputo fractional derivative.

Definition 3.1. *Let $h : [a,b] \to \mathbb{R}$ be a continuous function and $f \in \phi(\mathbb{R})$. Then, the generalized Fourier and inverse Fourier transforms, respectively, are defined as*

$$\widehat{\mathfrak{F}}\Big[f(\Delta)\Big](\omega,\rho) := \int_{-\infty}^{+\infty} \exp\Big(\Delta\big(i\omega + h(\rho)\big)\Big) f(\Delta) d\Delta, \tag{3.1}$$

$$\widehat{\mathfrak{F}}^{-1}\Big[\widehat{\mathfrak{F}}\Big[f(\Delta)\Big](\omega,\rho)\Big](\Delta,\rho) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\Big(-\Delta\big(i\omega + h(\rho)\big)\Big) \widehat{\mathfrak{F}}\Big[f(\Delta)\Big](\omega,\rho) d\omega, \tag{3.2}$$

where $\widehat{\mathfrak{F}}$ is the generalized Fourier transform operator and $\widehat{\mathfrak{F}}^{-1}$ is the generalized inverse Fourier transform operator and the function $\exp\big(\Delta h(\rho)\big)$ is kernel function. For the sake of shortness, we call the generalized Fourier transform and the generalized inverse Fourier transform as the $\widehat{\mathfrak{F}}$ transform and the $\widehat{\mathfrak{F}}^{-1}$ transform, respectively.

Remark. *If we take $h(\rho) = 0$ in Eqs.* (3.1) *and* (3.2)*, we get Eqs.* (2.1) *and* (2.2) *respectively.*

Theorem 3.1. *Let the function f be a function belonging to the Lizorkin space. The product of the function f and the kernel function also belongs to the Lizorkin space.*

Proof. Let $h : [a,b] \to \mathbb{R}$ be a continuous function and $f \in C^\infty$. Then, considering the Maclourin series of the exponential function and for $\forall n \in \mathbb{N}_0$, we have

$$\left|f^{(n)}(\Delta)\exp\big(\Delta h(\rho)\big)\right| = \left|\sum_{k=0}^{\infty} \frac{\big(\Delta h(\rho)\big)^k}{k!} f^{(n)}(\Delta)\right| \le \sum_{k=0}^{\infty} \left|\frac{\big(h(\rho)\big)^k}{k!}\right| \left|\Delta^k f^{(n)}(\Delta)\right|.$$

Hence, using the definition of Lizorkin space and taking the supremum of each side for $\Delta \in \mathbb{R}$, we get

$$\sup_{\Delta\in\mathbb{R}} \left| f^{(n)}(\Delta) \exp\big(\Delta h(\rho)\big) \right| \leq \sum_{k=0}^{\infty} \left| \frac{\big(h(\rho)\big)^k}{k!} \right| \sup_{\Delta\in\mathbb{R}} \left| \Delta^k f^{(n)}(\Delta) \right| < \infty,$$

which completes the proof. $\square$

Theorem 3.2. *Let $h : [a,b] \to \mathbb{R}$ be a continuous function and $f, g \in \phi(\mathbb{R})$. Then,*

$$\widehat{\mathfrak{F}}\Big[\lambda_1\ f(\Delta) + \lambda_2\ g(\Delta)\Big](\omega,\rho) = \lambda_1\ \widehat{\mathfrak{F}}\Big[f(\Delta)\Big](\omega,\rho) + \lambda_2\ \widehat{\mathfrak{F}}\Big[g(\Delta)\Big](\omega,\rho),$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$.

Proof. Using the $\widehat{\mathfrak{F}}$ transform, we have

$$\begin{aligned}
\widehat{\mathfrak{F}}\Big[\lambda_1\ f(\Delta) + \lambda_2\ g(\Delta)\Big](\omega,\rho) &= \int_{-\infty}^{+\infty} \exp\Big(\Delta\big(i\omega + h(\rho)\big)\Big)\big(\lambda_1\ f(\Delta) + \lambda_2\ g(\Delta)\big)d\Delta \\
&= \lambda_1 \int_{-\infty}^{+\infty} \exp\Big(\Delta\big(i\omega + h(\rho)\big)\Big) f(\Delta)d\Delta \\
&+ \lambda_2 \int_{-\infty}^{+\infty} \exp\Big(\Delta\big(i\omega + h(\rho)\big)\Big) g(\Delta)d\Delta \\
&= \lambda_1\ \widehat{\mathfrak{F}}\Big[f(\Delta)\Big](\omega,\rho) + \lambda_2\ \widehat{\mathfrak{F}}\Big[g(\Delta)\Big](\omega,\rho),
\end{aligned}$$

which completes the proof. $\square$

Theorem 3.3. *Let $h : [a,b] \to \mathbb{R}$ be a continuous function and $f \in \phi(\mathbb{R})$. Then,*

$$\widehat{\mathfrak{F}}\Big[f^{(n)}(\Delta)\Big](\omega,\rho) = (-1)^n \big(i\omega + h(\rho)\big)^n\ \widehat{\mathfrak{F}}\Big[f(\Delta)\Big](\omega,\rho), \tag{3.3}$$

where $n \in \mathbb{N}$.

Proof. Let us prove the result by contradiction. For $n = 1$, we have

$$\begin{aligned}
\widehat{\mathfrak{F}}\Big[f'(\Delta)\Big](\omega,\rho) &= \int_{-\infty}^{+\infty} \exp\Big(\Delta\big(i\omega + h(\rho)\big)\Big) f'(\Delta)d\Delta \\
&= \lim_{A\to\infty} \Big[\exp\Big(\Delta\big(i\omega + h(\rho)\big)\Big) f(\Delta)\Big]_{-A}^{A} \\
&- \big(i\omega + h(\rho)\big) \int_{-\infty}^{+\infty} \exp\Big(\Delta\big(i\omega + h(\rho)\big)\Big) f(\Delta)d\Delta \\
&= -\big(i\omega + h(\rho)\big)\widehat{\mathfrak{F}}\Big[f(\Delta)\Big](\omega,\rho),
\end{aligned}$$

where $\lim_{A\to\infty} \Big[\exp\Big(\Delta\big(i\omega + h(\rho)\big)\Big) f(\Delta)\Big]_{-A}^{A} = 0$ since $f \in \phi(\mathbb{R})$.

For $n = k$, let the equation be true:

$$\widehat{\mathfrak{F}}\Big[f^{(k)}(\Delta)\Big](\omega,\rho) = (-1)^k \big(i\omega + h(\rho)\big)^k\ \widehat{\mathfrak{F}}\Big[f(\Delta)\Big](\omega,\rho).$$

For $n = k+1$, considering the above equation, we obtain

$$\begin{aligned}\widehat{\mathfrak{F}}\Big[f^{(k+1)}(\Delta)\Big](\omega,\rho) &= \widehat{\mathfrak{F}}\left[\frac{d}{d\Delta}f^{(k)}(\Delta)\right](\omega,\rho)\\ &= (-1)^k\big(i\omega+h(\rho)\big)^k\,\widehat{\mathfrak{F}}\Big[f'(\Delta)\Big](\omega,\rho)\\ &= (-1)^{k+1}\big(i\omega+h(\rho)\big)^{k+1}\,\widehat{\mathfrak{F}}\Big[f(\Delta)\Big](\omega,\rho).\end{aligned}$$

Noted that since $f, f', f'', \ldots, f^{(n-1)} \in \phi(\mathbb{R})$ for $n \in \mathbb{N}$, the first terms from partial integration become zero when $|\Delta| \to \infty$. □

Theorem 3.4. *Let $h : [a,b] \to \mathbb{R}$ be a continuous function and $f, g \in \phi(\mathbb{R})$. Then,*

$$\widehat{\mathfrak{F}}\Big[f(\Delta)*g(\Delta)\Big](\omega,\rho) = \widehat{\mathfrak{F}}\Big[f(\Delta)\Big](\omega,\rho)\;\widehat{\mathfrak{F}}\Big[g(\Delta)\Big](\omega,\rho).$$

Proof. Using the $\widehat{\mathfrak{F}}$ transform and Eq. (2.3), we have

$$\begin{aligned}\widehat{\mathfrak{F}}\Big[f(\Delta)*g(\Delta)\Big](\omega,\rho) &= \int_{-\infty}^{+\infty}\exp\Big(\Delta\big(i\omega+h(\rho)\big)\Big)\big(f(\Delta)*g(\Delta)\big)d\Delta\\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp\Big(\Delta\big(i\omega+h(\rho)\big)\Big)f(\Delta-y)g(y)dyd\Delta.\end{aligned}$$

Taking $x := \Delta - y$, we get

$$\begin{aligned}\widehat{\mathfrak{F}}\Big[f(\Delta)*g(\Delta)\Big](\omega,\rho) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp\Big((x+y)\big(i\omega+h(\rho)\big)\Big)f(x)g(y)dxdy\\ &= \int_{-\infty}^{+\infty}\exp\Big(x\big(i\omega+h(\rho)\big)\Big)f(x)dx\int_{-\infty}^{+\infty}\exp\Big(y\big(i\omega+h(\rho)\big)\Big)g(y)dy\\ &= \widehat{\mathfrak{F}}\Big[f(\Delta)\Big](\omega,\rho)\;\widehat{\mathfrak{F}}\Big[g(\Delta)\Big](\omega,\rho),\end{aligned}$$

which completes the proof. □

We now apply the $\widehat{\mathfrak{F}}$ transform to the RLFI, RLFD and CFD.

Theorem 3.5. *Let $h : [a,b] \to \mathbb{R}$ be a continuous function and $f \in \phi(\mathbb{R})$. Then,*

$$\widehat{\mathfrak{F}}\Big[\left(I_+^{\varepsilon}f\right)(\Delta)\Big](\omega,\rho) = \big(-i\omega-h(\rho)\big)^{-\varepsilon}\,\widehat{\mathfrak{F}}\Big[f(\Delta)\Big](\omega,\rho),$$

where $\Re(\varepsilon) > 0$.

Proof. The relationship between the RLFI and Eq. (2.3) is as follows

$$\left(I_+^{\varepsilon}f\right)(\Delta) = f(\Delta)*g_+(\Delta), \tag{3.4}$$

where function $g_+(\Delta)$ is

$$g_+(\Delta) = \begin{cases}\frac{\Delta^{\varepsilon-1}}{\Gamma(\varepsilon)} & , \text{ for } \Delta > 0,\\ 0 & , \text{ for } \Delta \le 0.\end{cases}$$

By applying the $\widehat{\mathfrak{F}}$ transform to Eq. (3.4), we have

$$\begin{aligned}\widehat{\mathfrak{F}}\Big[\left(I_{+}^{\varepsilon}f\right)(\Delta)\Big](\omega,\rho) &= \widehat{\mathfrak{F}}\Big[f(\Delta)*g_{+}(\Delta)\Big](\omega,\rho)\\ &= \widehat{\mathfrak{F}}\Big[f(\Delta)\Big](\omega,\rho)\ \widehat{\mathfrak{F}}\Big[g_{+}(\Delta)\Big](\omega,\rho). \qquad (3.5)\end{aligned}$$

Calculating the second $\widehat{\mathfrak{F}}$ transform to the right of Eq. (3.5), we get

$$\begin{aligned}\widehat{\mathfrak{F}}\Big[g_{+}(\Delta)\Big](\omega,\rho) &= \int\limits_{-\infty}^{+\infty}\exp\Big(\Delta\big(i\omega+h(\rho)\big)\Big)g_{+}(\Delta)d\Delta\\ &= \frac{1}{\Gamma(\varepsilon)}\int\limits_{0}^{\infty}\exp\Big(\Delta\big(i\omega+h(\rho)\big)\Big)\Delta^{\varepsilon-1}d\Delta, \quad \text{for } \Delta>0. \qquad (3.6)\end{aligned}$$

Finally, calculating Eq. (3.6) and substituting it into Eq. (3.5), we obtain

$$\widehat{\mathfrak{F}}\Big[\left(I_{+}^{\varepsilon}f\right)(\Delta)\Big](\omega,\rho) = \big(-i\omega-h(\rho)\big)^{-\varepsilon}\ \widehat{\mathfrak{F}}\Big[f(\Delta)\Big](\omega,\rho),$$

which completes the proof. □

Theorem 3.6. *Let $h:[a,b]\to\mathbb{R}$ be a continuous function and $f\in\phi(\mathbb{R})$. Then,*

$$\widehat{\mathfrak{F}}\Big[\left(D_{+}^{\varepsilon}f\right)(\Delta)\Big](\omega,\rho) = \big(-i\omega-h(\rho)\big)^{\varepsilon}\ \widehat{\mathfrak{F}}\Big[f(\Delta)\Big](\omega,\rho),$$

where $r\in\mathbb{N}$, $r-1<\Re(\varepsilon)<r$, $\Re(\varepsilon)>0$.

Proof. The relationship between the RLFI and RLFD is as follows

$$\left(D_{+}^{\varepsilon}f\right)(\Delta) = \frac{d^{r}}{d\Delta^{r}}\ \left(I_{+}^{r-\varepsilon}f\right)(\Delta),$$

where

$$g(\Delta) := \left(I_{+}^{r-\varepsilon}f\right)(\Delta). \qquad (3.7)$$

Then,

$$\left(D_{+}^{\varepsilon}f\right)(\Delta) = g^{(r)}(\Delta). \qquad (3.8)$$

By applying the $\widehat{\mathfrak{F}}$ transform to Eqs. (3.7) and (3.8) respectively, we have

$$\begin{aligned}\widehat{\mathfrak{F}}\Big[g(\Delta)\Big](\omega,\rho) &= \widehat{\mathfrak{F}}\Big[\left(I_{+}^{r-\varepsilon}f\right)(\Delta)\Big](\omega,\rho)\\ &= \big(-i\omega-h(\rho)\big)^{-(r-\varepsilon)}\ \widehat{\mathfrak{F}}\Big[f(\Delta)\Big](\omega,\rho) \qquad (3.9)\end{aligned}$$

and

$$\begin{aligned}\widehat{\mathfrak{F}}\Big[\left(D_{+}^{\varepsilon}f\right)(\Delta)\Big](\omega,\rho) &= \widehat{\mathfrak{F}}\Big[g^{(r)}(\Delta)\Big](\omega,\rho)\\ &= (-1)^{r}\big(i\omega+h(\rho)\big)^{r}\ \widehat{\mathfrak{F}}\Big[g(\Delta)\Big](\omega,\rho). \qquad (3.10)\end{aligned}$$

Finally, using Eq. (3.9) in Eq. (3.10), we obtain

$$\widehat{\mathfrak{F}}\Big[\left(D_{+}^{\varepsilon}f\right)(\Delta)\Big](\omega,\rho) = \big(-i\omega-h(\rho)\big)^{\varepsilon}\ \widehat{\mathfrak{F}}\Big[f(\Delta)\Big](\omega,\rho),$$

which completes the proof. □

Theorem 3.7. *Let $h : [a, b] \to \mathbb{R}$ be a continuous function and $f \in \phi(\mathbb{R})$. Then,*

$$\widehat{\mathfrak{F}}\Big[\left({}^{c}D_{+}^{\varepsilon} f\right)(\Delta)\Big](\omega,\rho) = \big(-i\omega - h(\rho)\big)^{\varepsilon}\,\widehat{\mathfrak{F}}\Big[f(\Delta)\Big](\omega,\rho), \tag{3.11}$$

where $r \in \mathbb{N}$, $r-1 < \Re(\varepsilon) < r$, $\Re(\varepsilon) > 0$.

Proof. The relationship between the RLFI and CFD is as follows

$$\left({}^{c}D_{+}^{\varepsilon} f\right)(\Delta) = \frac{1}{\Gamma(r-\varepsilon)} \int_{-\infty}^{\Delta} (\Delta - y)^{r-\varepsilon-1} g(y)dy = \left(I_{+}^{r-\varepsilon} g\right)(\Delta), \tag{3.12}$$

where

$$g(\Delta) := f^{(r)}(\Delta). \tag{3.13}$$

By applying the $\widehat{\mathfrak{F}}$ transform to Eqs. (3.12) and (3.13) respectively, we have

$$\begin{aligned}\widehat{\mathfrak{F}}\Big[\left({}^{c}D_{+}^{\varepsilon} f\right)(\Delta)\Big](\omega,\rho) &= \widehat{\mathfrak{F}}\Big[\left(I_{+}^{r-\varepsilon} g\right)(\Delta)\Big](\omega,\rho)\\ &= \big(-i\omega - h(\rho)\big)^{-(r-\varepsilon)}\,\widehat{\mathfrak{F}}\Big[g(\Delta)\Big](\omega,\rho)\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\widehat{\mathfrak{F}}\Big[g(\Delta)\Big](\omega,\rho) &= \widehat{\mathfrak{F}}\Big[f^{(r)}(\Delta)\Big](\omega,\rho)\\ &= (-1)^{r}\big(i\omega + h(\rho)\big)^{r}\,\widehat{\mathfrak{F}}\Big[f(\Delta)\Big](\omega,\rho).\end{aligned} \tag{3.15}$$

Finally, using Eq. (3.15) in Eq. (3.14), we have

$$\widehat{\mathfrak{F}}\Big[\left({}^{c}D_{+}^{\varepsilon} f\right)(\Delta)\Big](\omega,\rho) = \big(-i\omega - h(\rho)\big)^{\varepsilon}\,\widehat{\mathfrak{F}}\Big[f(\Delta)\Big](\omega,\rho),$$

which completes the proof. □

Corollary 3.8. *The $\widehat{\mathfrak{F}}$ transform of the RLFD and CFD overlap.*

4. Illustrative Examples

In this section, we give the $\widehat{\mathfrak{F}}$ transform of various elementary functions. We also present the $\widehat{\mathfrak{F}}$ transform of the generalized special functions and the generalized fractional derivatives by means of suitably chosen kernel functions.

Example 4.1. *The $\widehat{\mathfrak{F}}$ transform of the Dirac delta function is obtained as*

$$\widehat{\mathfrak{F}}\Big[\delta(\Delta)\Big](\omega,\rho) = 1. \tag{4.1}$$

Proof. Using the $\widehat{\mathfrak{F}}$ transform and considering Eq. (2.4), we have

$$\widehat{\mathfrak{F}}\Big[\delta(\Delta)\Big](\omega,\rho) = \int_{-\infty}^{+\infty} \exp\Big(\Delta\big(i\omega + h(\rho)\big)\Big)\delta(\Delta)d\Delta = 1.$$ □

Example 4.2. *The $\widehat{\mathfrak{F}}$ transform of function* $\exp(i\lambda\Delta)$ *is obtained as*

$$\widehat{\mathfrak{F}}\Big[\exp(i\lambda\Delta)\Big](\omega,\rho) = 2\pi \exp\left(-h(\rho)\left(\frac{h(\rho)}{i} + \omega + \lambda\right)\right)\delta\left(\frac{h(\rho)}{i} + \omega + \lambda\right). \tag{4.2}$$

Proof. By applying the $\widehat{\mathfrak{F}}^{-1}$ transform to Eq. (4.1), we get

$$\delta(\Delta) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \exp\Big(- \Delta\big(i\omega + h(\rho)\big)\Big) d\omega.$$

Taking $\Delta := \omega$, we have

$$\delta(\omega) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \exp\Big(- \omega\big(i\Delta + h(\rho)\big)\Big) d\Delta.$$

By replacing ω with $\left(-\frac{h(\rho)}{i} - \omega - \lambda\right)$, we get

$$\delta\left(-\frac{h(\rho)}{i} - \omega - \lambda\right) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \exp\left(- \left(-\frac{h(\rho)}{i} - \omega - \lambda\right)\left(i\Delta + h(\rho)\right)\right) d\Delta.$$

Taking into account that the Dirac delta function is an even function and making the necessary arrangements, we have

$$\widehat{\mathfrak{F}}\Big[\exp(i\lambda\Delta)\Big](\omega,\rho) = 2\pi \exp\left(-h(\rho)\left(\frac{h(\rho)}{i} + \omega + \lambda\right)\right)\delta\left(\frac{h(\rho)}{i} + \omega + \lambda\right). \quad \square$$

Corollary 4.3. *In Eq.* (4.2)*, substituting* $-\lambda$ *for* λ*, we get*

$$\widehat{\mathfrak{F}}\Big[\exp(-i\lambda\Delta)\Big](\omega,\rho) = 2\pi \exp\left(-h(\rho)\left(\frac{h(\rho)}{i} + \omega - \lambda\right)\right)\delta\left(\frac{h(\rho)}{i} + \omega - \lambda\right). \quad (4.3)$$

Example 4.4. *The* $\widehat{\mathfrak{F}}$ *transform of function* $\exp\left(-\lambda\Delta^2\right)$ *is obtained as*

$$\widehat{\mathfrak{F}}\Big[\exp\left(-\lambda\Delta^2\right)\Big](\omega,\rho) = \sqrt{\frac{\pi}{\lambda}}\exp\left(\frac{\big(i\omega + h(\rho)\big)^2}{4\lambda}\right), \quad \text{for } \lambda > 0.$$

Proof. Using the $\widehat{\mathfrak{F}}$ transform, we have

$$\begin{aligned}\widehat{\mathfrak{F}}\Big[\exp\left(-\lambda\Delta^2\right)\Big](\omega,\rho) &= \int\limits_{-\infty}^{+\infty} \exp\Big(\Delta\big(i\omega + h(\rho)\big)\Big)\exp\left(-\lambda\Delta^2\right) d\Delta \\ &= \exp\left(\frac{\big(i\omega + h(\rho)\big)^2}{4\lambda}\right)\int\limits_{-\infty}^{+\infty} \exp\left(-\lambda\left(\Delta - \frac{\big(i\omega + h(\rho)\big)}{2\lambda}\right)^2\right) d\Delta.\end{aligned}$$

Taking $x := \Delta - \frac{\left(i\omega + h(\rho)\right)}{2\lambda}$, we get

$$\widehat{\mathfrak{F}}\Big[\exp\left(-\lambda\Delta^2\right)\Big](\omega,\rho) = \exp\left(\frac{\big(i\omega + h(\rho)\big)^2}{4\lambda}\right)\int\limits_{-\infty}^{+\infty} \exp\left(-\lambda x^2\right) dx.$$

Using the formula $\int\limits_{-\infty}^{+\infty} \exp\left(-\lambda x^2\right) dx = \sqrt{\frac{\pi}{\lambda}}$, we have

$$\widehat{\mathfrak{F}}\Big[\exp\left(-\lambda\Delta^2\right)\Big](\omega,\rho) = \sqrt{\frac{\pi}{\lambda}}\exp\left(\frac{\big(i\omega + h(\rho)\big)^2}{4\lambda}\right). \quad \square$$

Example 4.5. *The* $\widehat{\mathfrak{F}}$ *transform of function* $\exp(-\lambda|\Delta|)$ *is obtained as*

$$\widehat{\mathfrak{F}}\Big[\exp(-\lambda|\Delta|)\Big](\omega,\rho)=\frac{-2\lambda}{\big(i\omega+h(\rho)\big)^2-\lambda^2},\quad \text{for } \lambda>0.$$

Proof. Using the $\widehat{\mathfrak{F}}$ transform, we get

$$\begin{aligned}\widehat{\mathfrak{F}}\Big[\exp(-\lambda|\Delta|)\Big](\omega,\rho)&=\int_{-\infty}^{+\infty}\exp\Big(\Delta\big(i\omega+h(\rho)\big)\Big)\exp(-\lambda|\Delta|)\,d\Delta\\&=\int_{-\infty}^{0}\exp\Big(\Delta\big(i\omega+h(\rho)+\lambda\big)\Big)d\Delta\\&+\int_{0}^{+\infty}\exp\Big(\Delta\big(i\omega+h(\rho)-\lambda\big)\Big)d\Delta.\end{aligned}$$

Then,

$$\begin{aligned}\widehat{\mathfrak{F}}\Big[\exp(-\lambda|\Delta|)\Big](\omega,\rho)&=\lim_{A\to-\infty}\left[\frac{\exp\Big(\Delta\big(i\omega+h(\rho)+\lambda\big)\Big)}{i\omega+h(\rho)+\lambda}\right]_A^0\\&+\lim_{A\to+\infty}\left[\frac{\exp\Big(\Delta\big(i\omega+h(\rho)-\lambda\big)\Big)}{i\omega+h(\rho)-\lambda}\right]_0^A\\&=\frac{1}{i\omega+h(\rho)+\lambda}-\frac{1}{i\omega+h(\rho)-\lambda}.\end{aligned}$$

Finally, by making the necessary calculations, we get

$$\widehat{\mathfrak{F}}\Big[\exp(-\lambda|\Delta|)\Big](\omega,\rho)=\frac{-2\lambda}{\big(i\omega+h(\rho)\big)^2-\lambda^2}.\qquad\square$$

Example 4.6. *The* $\widehat{\mathfrak{F}}$ *transform of function* $\sin(\lambda\Delta)$ *is obtained as*

$$\begin{aligned}\widehat{\mathfrak{F}}\Big[\sin(\lambda\Delta)\Big](\omega,\rho)&=i\pi\exp\left(-h(\rho)\left(\frac{h(\rho)}{i}+\omega\right)\right)\\&\cdot\left[\exp\big(\lambda h(\rho)\big)\delta\left(\frac{h(\rho)}{i}+\omega-\lambda\right)-\exp\big(-\lambda h(\rho)\big)\delta\left(\frac{h(\rho)}{i}+\omega+\lambda\right)\right].\end{aligned}$$

Proof. Using the $\widehat{\mathfrak{F}}$ transform, we have

$$\widehat{\mathfrak{F}}\Big[\sin(\lambda\Delta)\Big](\omega,\rho)=\int_{-\infty}^{+\infty}\exp\Big(\Delta\big(i\omega+h(\rho)\big)\Big)\sin(\lambda\Delta)d\Delta.$$

Using the formula $\sin(\lambda\Delta) = \frac{\exp(i\lambda\Delta)-\exp(-i\lambda\Delta)}{2i}$, we get

$$\begin{aligned}\widehat{\mathfrak{F}}\Big[\sin(\lambda\Delta)\Big](\omega,\rho) &= \int_{-\infty}^{+\infty} \exp\Big(\Delta\big(i\omega+h(\rho)\big)\Big)\left(\frac{\exp(i\lambda\Delta)-\exp(-i\lambda\Delta)}{2i}\right)d\Delta \\ &= \frac{1}{2i}\Bigg[\int_{-\infty}^{+\infty} \exp\Big(\Delta\big(i\omega+h(\rho)+i\lambda\big)\Big)d\Delta \\ &- \int_{-\infty}^{+\infty} \exp\Big(\Delta\big(i\omega+h(\rho)-i\lambda\big)\Big)d\Delta\Bigg] \\ &= \frac{1}{2i}\left(\widehat{\mathfrak{F}}\Big[\exp(i\lambda\Delta)\Big](\omega,\rho) - \widehat{\mathfrak{F}}\Big[\exp(-i\lambda\Delta)\Big](\omega,\rho)\right).\end{aligned}$$

Finally, using Eqs. (4.2) and (4.3), we have

$$\begin{aligned}\widehat{\mathfrak{F}}\Big[\sin(\lambda\Delta)\Big](\omega,\rho) &= i\pi\exp\left(-h(\rho)\left(\frac{h(\rho)}{i}+\omega\right)\right) \\ \cdot \Bigg[\exp\big(\lambda h(\rho)\big)\delta\left(\frac{h(\rho)}{i}+\omega-\lambda\right) &- \exp\big(-\lambda h(\rho)\big)\delta\left(\frac{h(\rho)}{i}+\omega+\lambda\right)\Bigg].\end{aligned}$$ □

Example 4.7. *The* $\widehat{\mathfrak{F}}$ *transform of function* $\cos(\lambda\Delta)$ *is obtained as*

$$\begin{aligned}\widehat{\mathfrak{F}}\Big[\cos(\lambda\Delta)\Big](\omega,\rho) &= \pi\exp\left(-h(\rho)\left(\frac{h(\rho)}{i}+\omega\right)\right) \\ \cdot \Bigg[\exp\big(-\lambda h(\rho)\big)\delta\left(\frac{h(\rho)}{i}+\omega+\lambda\right) &+ \exp\big(\lambda h(\rho)\big)\delta\left(\frac{h(\rho)}{i}+\omega-\lambda\right)\Bigg].\end{aligned}$$

Proof. Using the $\widehat{\mathfrak{F}}$ transform, we get

$$\widehat{\mathfrak{F}}\Big[\cos(\lambda\Delta)\Big](\omega,\rho) = \int_{-\infty}^{+\infty} \exp\Big(\Delta\big(i\omega+h(\rho)\big)\Big)\cos(\lambda\Delta)d\Delta.$$

Using the formula $\cos(\lambda\Delta) = \frac{\exp(i\lambda\Delta)+\exp(-i\lambda\Delta)}{2}$, we have

$$\begin{aligned}\widehat{\mathfrak{F}}\Big[\cos(\lambda\Delta)\Big](\omega,\rho) &= \int_{-\infty}^{+\infty} \exp\Big(\Delta\big(i\omega+h(\rho)\big)\Big)\left(\frac{\exp(i\lambda\Delta)+\exp(-i\lambda\Delta)}{2}\right)d\Delta \\ &= \frac{1}{2}\Bigg[\int_{-\infty}^{+\infty} \exp\Big(\Delta\big(i\omega+h(\rho)+i\lambda\big)\Big)d\Delta \\ &+ \int_{-\infty}^{+\infty} \exp\Big(\Delta\big(i\omega+h(\rho)-i\lambda\big)\Big)d\Delta\Bigg] \\ &= \frac{1}{2}\left(\widehat{\mathfrak{F}}\Big[\exp(i\lambda\Delta)\Big](\omega,\rho) + \widehat{\mathfrak{F}}\Big[\exp(-i\lambda\Delta)\Big](\omega,\rho)\right).\end{aligned}$$

Finally, using Eqs. (4.2) and (4.3), we get

$$\widehat{\mathfrak{F}}\Big[\cos(\lambda\Delta)\Big](\omega,\rho) = \pi \exp\left(-h(\rho)\left(\frac{h(\rho)}{i}+\omega\right)\right) \cdot \left[\exp\big(-\lambda h(\rho)\big)\delta\left(\frac{h(\rho)}{i}+\omega+\lambda\right) + \exp\big(\lambda h(\rho)\big)\delta\left(\frac{h(\rho)}{i}+\omega-\lambda\right)\right]. \square$$

We give the $\widehat{\mathfrak{F}}$ transform of the generalized special functions and the generalized fractional derivatives by means of suitably chosen kernel functions.

Example 4.8. *The $\widehat{\mathfrak{F}}$ transform of the generalized gamma function is obtained as*

$$\widehat{\mathfrak{F}}\Big[\Gamma_{\Delta}^{(\alpha,\alpha;\lambda)}(\xi)\Big](\omega,\rho) = 2\pi\delta(\omega)\Gamma(\xi), \quad \text{for } \Re(\xi) > 0.$$

Proof. Let the kernel function of the $\widehat{\mathfrak{F}}$ transform be chosen as follows

$$\exp\big(\Delta h(\rho)\big) := \exp\left(\frac{\Delta}{\rho^{\lambda}}\right). \tag{4.4}$$

By applying the $\widehat{\mathfrak{F}}$ transform with kernel function (4.4) to Eq. (2.7) for $\alpha = \beta$, we have

$$\begin{aligned}\widehat{\mathfrak{F}}\Big[\Gamma_{\Delta}^{(\alpha,\alpha;\lambda)}(\xi)\Big](\omega,\rho) &= \int_{-\infty}^{+\infty}\int_{0}^{\infty} \exp(i\omega\Delta)\exp\left(\frac{\Delta}{\rho^{\lambda}}\right)\rho^{\xi-1}\exp\left(-\rho-\frac{\Delta}{\rho^{\lambda}}\right)d\rho d\Delta \\ &= \int_{-\infty}^{+\infty}\exp(i\omega\Delta)\,d\Delta\int_{0}^{\infty}\rho^{\xi-1}\exp(-\rho)\,d\rho. \end{aligned} \tag{4.5}$$

Using Eq. (2.5) for $c = 1$ in Eq. (4.5), we get

$$\widehat{\mathfrak{F}}\Big[\Gamma_{\Delta}^{(\alpha,\alpha;\lambda)}(\xi)\Big](\omega,\rho) = 2\pi\delta(\omega)\Gamma(\xi). \qquad \square$$

Example 4.9. *The $\widehat{\mathfrak{F}}$ transform of the generalized gamma function is obtained as*

$$\widehat{\mathfrak{F}}\Big[\Gamma_{\Delta}(\xi)\Big](\omega,\rho) = 2\pi\delta(\omega)\Gamma(\xi), \quad \text{for } \Re(\xi) > 0.$$

Proof. For $\lambda = 1$ in the kernel function (4.4), we have

$$\exp\big(\Delta h(\rho)\big) := \exp\left(\frac{\Delta}{\rho}\right). \tag{4.6}$$

By applying the $\widehat{\mathfrak{F}}$ transform with kernel function (4.6) to Eq. (2.6), we get

$$\widehat{\mathfrak{F}}\Big[\Gamma_{\Delta}(\xi)\Big](\omega,\rho) = 2\pi\delta(\omega)\Gamma(\xi). \qquad \square$$

Example 4.10. *The $\widehat{\mathfrak{F}}$ transform of the generalized beta function is obtained as*

$$\widehat{\mathfrak{F}}\Big[B_{\Delta}^{(\alpha,\alpha;\kappa,\mu)}(\xi,\eta)\Big](\omega,\rho) = 2\pi\delta(\omega)B(\xi,\eta),$$

where $\Re(\xi) > 0$, $\Re(\eta) > 0$.

Proof. Let the kernel function of the $\widehat{\mathfrak{F}}$ transform be chosen as follows

$$\exp\big(\Delta h(\rho)\big) := \exp\left(\frac{\Delta}{\rho^{\kappa}(1-\rho)^{\mu}}\right). \tag{4.7}$$

By applying the $\widehat{\mathfrak{F}}$ transform with kernel function (4.7) to Eq. (2.10) for $\alpha = \beta$, we have

$$\begin{aligned}\widehat{\mathfrak{F}}\Big[B_{\Delta}^{(\alpha,\alpha;\kappa,\mu)}(\xi,\eta)\Big](\omega,\rho) &= \int_{-\infty}^{+\infty}\int_{0}^{1} \exp(i\omega\Delta)\exp\left(\frac{\Delta}{\rho^{\kappa}(1-\rho)^{\mu}}\right) \\ &\quad \cdot \rho^{\xi-1}(1-\rho)^{\eta-1}\exp\left(-\frac{\Delta}{\rho^{\kappa}(1-\rho)^{\mu}}\right) d\rho d\Delta \\ &= \int_{-\infty}^{+\infty} \exp(i\omega\Delta)\, d\Delta \int_{0}^{1} \rho^{\xi-1}(1-\rho)^{\eta-1} d\rho. \end{aligned} \tag{4.8}$$

Using Eq. (2.5) for $c = 1$ in Eq. (4.8), we get

$$\widehat{\mathfrak{F}}\Big[B_{\Delta}^{(\alpha,\alpha;\kappa,\mu)}(\xi,\eta)\Big](\omega,\rho) = 2\pi\delta(\omega)B(\xi,\eta). \qquad \square$$

Example 4.11. *The $\widehat{\mathfrak{F}}$ transform of the generalized beta function is obtained as*

$$\widehat{\mathfrak{F}}\Big[B(\xi,\eta;\Delta;\lambda)\Big](\omega,\rho) = 2\pi\delta(\omega)B(\xi,\eta),$$

where $\Re(\xi) > 0$, $\Re(\eta) > 0$.

Proof. For $\kappa = \mu = \lambda$ in the kernel function (4.7), we have

$$\exp\big(\Delta h(\rho)\big) := \exp\left(\frac{\Delta}{\rho^{\lambda}(1-\rho)^{\lambda}}\right). \tag{4.9}$$

By applying the $\widehat{\mathfrak{F}}$ transform with kernel function (4.9) to Eq. (2.9), we get

$$\widehat{\mathfrak{F}}\Big[B(\xi,\eta;\Delta;\lambda)\Big](\omega,\rho) = 2\pi\delta(\omega)B(\xi,\eta). \qquad \square$$

Example 4.12. *The $\widehat{\mathfrak{F}}$ transform of the generalized beta function is obtained as*

$$\widehat{\mathfrak{F}}\Big[B(\xi,\eta;\Delta)\Big](\omega,\rho) = 2\pi\delta(\omega)B(\xi,\eta),$$

where $\Re(\xi) > 0$, $\Re(\eta) > 0$.

Proof. For $\lambda = 1$ in the kernel function (4.9), we have

$$\exp\big(\Delta h(\rho)\big) := \exp\left(\frac{\Delta}{\rho(1-\rho)}\right). \tag{4.10}$$

By applying the $\widehat{\mathfrak{F}}$ transform with kernel function (4.10) to Eq. (2.8) , we get

$$\widehat{\mathfrak{F}}\Big[B(\xi,\eta;\Delta)\Big](\omega,\rho) = 2\pi\delta(\omega)B(\xi,\eta). \qquad \square$$

Example 4.13. *The $\widehat{\mathfrak{F}}$ transform of the generalized fractional derivative is obtained as*

$$\widehat{\mathfrak{F}}\Big[D_v^{\varepsilon,\Delta;\kappa,\mu} f(v)\Big](\omega,\rho) = 2\pi\delta(\omega)\left(D_{0+}^{\varepsilon} f\right)(v), \quad \text{for } \Re(\varepsilon) > 0.$$

Proof. Let the kernel function of the $\widehat{\mathfrak{F}}$ transform be chosen as follows

$$\exp\big(\Delta h(\rho)\big) := \exp\left(\frac{\Delta v^{\kappa+\mu}}{\rho^{\kappa}(v-\rho)^{\mu}}\right). \tag{4.11}$$

By applying the $\widehat{\mathfrak{F}}$ transform with kernel function (4.11) to Eq. (2.11) for $\alpha=\beta$, we have

$$\widehat{\mathfrak{F}}\Big[D_v^{\varepsilon,\Delta;\kappa,\mu} f(v)\Big](\omega,\rho) = \frac{1}{\Gamma(r-\varepsilon)} \int\limits_{-\infty}^{+\infty} \exp\left(i\omega\Delta\right) \frac{d^r}{dv^r} \int\limits_{0}^{v} (v-\rho)^{r-\varepsilon-1} f(\rho) \cdot \exp\left(\frac{\Delta v^{\kappa+\mu}}{\rho^{\kappa}(1-\rho)^{\mu}}\right) \exp\left(-\frac{\Delta v^{\kappa+\mu}}{\rho^{\kappa}(1-\rho)^{\mu}}\right) d\rho d\Delta.$$

Then,

$$\widehat{\mathfrak{F}}\Big[D_v^{\varepsilon,\Delta;\kappa,\mu} f(v)\Big](\omega,\rho) = \frac{1}{\Gamma(r-\varepsilon)} \frac{d^r}{dv^r} \int\limits_{0}^{v} (v-\rho)^{r-\varepsilon-1} f(\rho) d\rho \int\limits_{-\infty}^{+\infty} \exp\left(i\omega\Delta\right) d\Delta. \tag{4.12}$$

Using Eq. (2.5) for $c=1$ in Eq. (4.12), we get

$$\widehat{\mathfrak{F}}\Big[D_v^{\varepsilon,\Delta;\kappa,\mu} f(v)\Big](\omega,\rho) = 2\pi\delta(\omega)\left(D_{0+}^{\varepsilon} f\right)(v).$$

□

Example 4.14. *The $\widehat{\mathfrak{F}}$ transform of the generalized fractional derivatives are obtained as*

$$\widehat{\mathfrak{F}}\Big[D_v^{\varepsilon,\Delta;\lambda}\left\{f(v)\right\}\Big](\omega,\rho) = 2\pi\delta(\omega)\left(D_{0+}^{\varepsilon} f\right)(v), \quad \text{for } \Re(\varepsilon)>0$$

and

$$\widehat{\mathfrak{F}}\Big[D_v^{\varepsilon,\Delta;\lambda} f(v)\Big](\omega,\rho) = 2\pi\delta(\omega)\left({}^{c}D_{0+}^{\varepsilon} f\right)(v), \quad \text{for } \Re(\varepsilon)>0.$$

Proof. For $\kappa=\mu=\lambda$ in the kernel function (4.11), we have

$$\exp\big(\Delta h(\rho)\big) := \exp\left(\frac{\Delta v^{2\lambda}}{\rho^{\lambda}(v-\rho)^{\lambda}}\right). \tag{4.13}$$

By applying the $\widehat{\mathfrak{F}}$ transform with kernel function (4.13) to Eqs. (2.12) and (2.13) respectively, we get

$$\widehat{\mathfrak{F}}\Big[D_v^{\varepsilon,\Delta;\lambda}\left\{f(v)\right\}\Big](\omega,\rho) = 2\pi\delta(\omega)\left(D_{0+}^{\varepsilon} f\right)(v)$$

and

$$\widehat{\mathfrak{F}}\Big[D_v^{\varepsilon,\Delta;\lambda} f(v)\Big](\omega,\rho) = 2\pi\delta(\omega)\left({}^{c}D_{0+}^{\varepsilon} f\right)(v).$$

□

Example 4.15. *The $\widehat{\mathfrak{F}}$ transform of the generalized fractional derivatives are obtained as*

$$\widehat{\mathfrak{F}}\Big[D_v^{\varepsilon,\Delta}\left\{f(v)\right\}\Big](\omega,\rho) = 2\pi\delta(\omega)\left(D_{0+}^{\varepsilon} f\right)(v), \quad \text{for } \Re(\varepsilon)>0$$

and

$$\widehat{\mathfrak{F}}\Big[D_v^{\varepsilon,\Delta} f(v)\Big](\omega,\rho) = 2\pi\delta(\omega)\left({}^{c}D_{0+}^{\varepsilon} f\right)(v), \quad \text{for } \Re(\varepsilon)>0.$$

Proof. For $\lambda = 1$ in the kernel function (4.13), we have

$$\exp\big(\Delta h(\rho)\big) := \exp\left(\frac{\Delta v^2}{\rho(v-\rho)}\right). \tag{4.14}$$

By applying the $\widehat{\mathfrak{F}}$ transform with kernel function (4.14) to Eqs. (2.14) and (2.15) respectively, we get

$$\widehat{\mathfrak{F}}\Big[D_v^{\varepsilon,\Delta}\left\{f(v)\right\}\Big](\omega,\rho) = 2\pi\delta(\omega)\left(D_{0+}^{\varepsilon}f\right)(v)$$

and

$$\widehat{\mathfrak{F}}\Big[D_v^{\varepsilon,\Delta}f(v)\Big](\omega,\rho) = 2\pi\delta(\omega)\left({}^{c}D_{0+}^{\varepsilon}f\right)(v). \qquad \square$$

5. Application to differential equations

In this section, we obtain the solutions of the ordinary RL electric current and the fractional motion differential equations via the $\widehat{\mathfrak{F}}$ transform.

Application 5.1. *We consider the ordinary RL electric current differential equation*

$$L\frac{dI(\Delta)}{d\Delta} + RI(\Delta) = E(\Delta),$$

where L is inductance, I is current, R is resistance and E is applied electromagnetic force. If we take $E(\Delta) = \delta(\Delta)$ in the ordinary RL electric current differential equation, we get

$$L\frac{dI(\Delta)}{d\Delta} + RI(\Delta) = \delta(\Delta). \tag{5.1}$$

Proof. By applying the $\widehat{\mathfrak{F}}$ transform to Eq. (5.1), we get

$$L\,\widehat{\mathfrak{F}}\Big[I'(\Delta)\Big](\omega,\rho) + R\,\widehat{\mathfrak{F}}\Big[I(\Delta)\Big](\omega,\rho) = \widehat{\mathfrak{F}}\Big[\delta(\Delta)\Big](\omega,\rho). \tag{5.2}$$

In Eq. (5.2), using Eq. (3.3) for $n = 1$ and considering Eq. (4.1), we obtain

$$-L\big(i\omega + h(\rho)\big)\,\widehat{\mathfrak{F}}\Big[I(\Delta)\Big](\omega,\rho) + R\,\widehat{\mathfrak{F}}\Big[I(\Delta)\Big](\omega,\rho) = 1.$$

Then,

$$\widehat{\mathfrak{F}}\Big[I(\Delta)\Big](\omega,\rho) = \frac{1}{iL\left(\frac{R-Lh(\rho)}{iL} - \omega\right)}. \tag{5.3}$$

By applying the $\widehat{\mathfrak{F}}^{-1}$ transform to Eq. (5.3), we have

$$\begin{aligned} I(\Delta) &= \widehat{\mathfrak{F}}^{-1}\left[\frac{1}{iL\left(\frac{R-Lh(\rho)}{iL} - \omega\right)}\right](\Delta,\rho) \\ &= \frac{1}{2\pi}\frac{1}{iL}\int_{-\infty}^{+\infty}\frac{\exp\Big(-\Delta\big(i\omega + h(\rho)\big)\Big)}{\left(\frac{R-Lh(\rho)}{iL} - \omega\right)}d\omega, \end{aligned}$$

where $\omega = \frac{R-Lh(\rho)}{iL}$ for $\forall \rho \in [a,b]$ is the pole point. Therefore, using the Cauchy Residue Theorem [30], we get

$$\begin{aligned} I(\Delta) &= \frac{\exp\big(-\Delta h(\rho)\big)}{2\pi i L}\left(2\pi i \operatorname{Res}\left(\omega = \frac{R-Lh(\rho)}{iL}\right)\right) \\ &= \frac{\exp\big(-\Delta h(\rho)\big)}{L} \lim_{\omega \to \frac{R-Lh(\rho)}{iL}} \frac{\exp(-i\omega\Delta)\left(\frac{R-Lh(\rho)}{iL} - \omega\right)}{\left(\frac{R-Lh(\rho)}{iL} - \omega\right)} \\ &= \frac{\exp\big(-\Delta h(\rho)\big)}{L} \exp\left(-i\Delta\left(\frac{R-Lh(\rho)}{iL}\right)\right) \\ &= \frac{1}{L}\exp\left(-\frac{\Delta R}{L}\right), \end{aligned}$$

which is the solution of the ordinary RL electric current differential equation. Also, this result coincides with the solution of the ordinary RL electric current differential equation of via the classical Fourier transform, see [8]. □

Application 5.2. *We consider the fractional motion differential equation*

$$y''(\Delta) + \lambda_1 \left({}^cD^{\varepsilon}_{+} y\right)(\Delta) + \lambda_2 y(\Delta) = f(\Delta), \tag{5.4}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ *and* $0 < \Re(\varepsilon) < 1$.

Proof. By applying the $\widehat{\mathfrak{F}}$ transform to Eq. (5.4), we have

$$\widehat{\mathfrak{F}}\Big[y''(\Delta)\Big](\omega,\rho) + \lambda_1\, \widehat{\mathfrak{F}}\Big[\left({}^cD^{\varepsilon}_{+} y\right)(\Delta)\Big](\omega,\rho) + \lambda_2\, \widehat{\mathfrak{F}}\Big[y(\Delta)\Big](\omega,\rho) = \widehat{\mathfrak{F}}\Big[f(\Delta)\Big](\omega,\rho).$$

Using Eqs. (3.3) for $n = 1$ and (3.11) for $r = 1$, we get

$$\begin{aligned} \widehat{\mathfrak{F}}\Big[f(\Delta)\Big](\omega,\rho) &= \big(-i\omega - h(\rho)\big)^2\, \widehat{\mathfrak{F}}\Big[y(\Delta)\Big](\omega,\rho) \\ &\quad + \lambda_1\big(-i\omega - h(\rho)\big)^{\varepsilon}\, \widehat{\mathfrak{F}}\Big[y(\Delta)\Big](\omega,\rho) + \lambda_2\, \widehat{\mathfrak{F}}\Big[y(\Delta)\Big](\omega,\rho). \end{aligned}$$

That is,

$$\widehat{\mathfrak{F}}\Big[y(\Delta)\Big](\omega,\rho) = \frac{\widehat{\mathfrak{F}}\Big[f(\Delta)\Big](\omega,\rho)}{\big(-i\omega - h(\rho)\big)^2 + \lambda_1\big(-i\omega - h(\rho)\big)^{\varepsilon} + \lambda_2}.$$

Let

$$C(\omega) := \big(-i\omega - h(\rho)\big)^2 + \lambda_1\big(-i\omega - h(\rho)\big)^{\varepsilon} + \lambda_2.$$

Then,

$$\widehat{\mathfrak{F}}\Big[y(\Delta)\Big](\omega,\rho) = \frac{\widehat{\mathfrak{F}}\Big[f(\Delta)\Big](\omega,\rho)}{C(\omega)}.$$

Let

$$\widehat{\mathfrak{F}}\Big[g(\Delta)\Big](\omega,\rho) := \frac{1}{C(\omega)}. \tag{5.5}$$

Then,

$$\widehat{\mathfrak{F}}\Big[y(\Delta)\Big](\omega,\rho) = \widehat{\mathfrak{F}}\Big[f(\Delta)\Big](\omega,\rho)\ \widehat{\mathfrak{F}}\Big[g(\Delta)\Big](\omega,\rho) \\ = \widehat{\mathfrak{F}}\Big[f(\Delta) * g(\Delta)\Big](\omega,\rho). \tag{5.6}$$

By applying the $\widehat{\mathfrak{F}}^{-1}$ transform to Eqs. (5.5) and (5.6) respectively, we get

$$g(\Delta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\exp\Big(-\Delta\big(i\omega + h(\rho)\big)\Big)}{C(\omega)} d\omega \tag{5.7}$$

and

$$y(\Delta) = f(\Delta) * g(\Delta) = \int_{-\infty}^{+\infty} f(\Delta - x) g(x) dx. \tag{5.8}$$

Using Eq. (5.7) in Eq.(5.8), we have

$$y(\Delta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\exp\Big(-x\big(i\omega + h(\rho)\big)\Big) f(\Delta - x)}{C(\omega)} d\omega dx. \tag{5.9}$$

Substituting $C(\omega)$ in Eq. (5.9), we have

$$y(\Delta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\exp\Big(-x\big(i\omega + h(\rho)\big)\Big) f(\Delta - x)}{\big(-i\omega - h(\rho)\big)^2 + \lambda_1\big(-i\omega - h(\rho)\big)^{\varepsilon} + \lambda_2} d\omega dx, \tag{5.10}$$

which is the solution of fractional motion differential equation. □

Corollary 5.3. *If we take $h(\rho) = 0$ in Eq.* (5.10)*, we have*

$$y(\Delta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\exp\left(-i\omega x\right) f(\Delta - x)}{(-i\omega)^2 + \lambda_1(-i\omega)^{\varepsilon} + \lambda_2} d\omega dx,$$

which is the solution of fractional motion differential equation via classical Fourier transform, see [4, 15].

6. Conclusion and Recommendations

In this paper, we introduced the generalized Fourier ($\widehat{\mathfrak{F}}$) and the generalized inverse Fourier ($\widehat{\mathfrak{F}}^{-1}$) transforms with h-exponential functions in their kernels. Then, we gave some basic properties and illustrative examples. Furthermore, we obtained the solutions of the ordinary RL electric circuit and the fractional motion differential equations via the $\widehat{\mathfrak{F}}$ and $\widehat{\mathfrak{F}}^{-1}$ transforms.

Our main goals in this paper was to make it possible to easily take the Fourier transform of the generalized special functions and fractional derivatives, which involving the exponential functions in their kernels and to extend the field of application of the Fourier transform. As a conclusion, we have demonstrated these main goals by means of the $\widehat{\mathfrak{F}}$ and $\widehat{\mathfrak{F}}^{-1}$ transforms that we introduced in this paper.

As a result of our research, we concluded that the kernel function of the generalized Fourier transform should be an exponential function, because we observed that this kernel function satisfies the fundamental properties and gives successful results in applications.

For future work on the Fourier transform, we suggest that scientists focus on the following problems.

(1) Is it possible that the kernel function of the Fourier transform can be special functions such as the Mittag-Leffler, Wright, Fox-Wright, Horn, Gauss, Kummer, Lauricella, Srivastava, and Appell functions?
(2) Can the Fourier transforms defined with these functions be applied to the other generalized special functions and the generalized fractional derivatives in the literature?
(3) Can a convolution theorem be given for the Fourier transforms defined by these functions?
(4) Can the Fourier transforms defined by these functions be useful in solving ordinary, partial or fractional differential equations?

Acknowledgments. This work was partly presented in the 2nd International E-Conference on Mathematical and Statistical Sciences: A Selcuk Meeting (ICOMSS-2023) which organized by Selcuk University on June 5-7, 2023 in Konya-Turkey. The authors are thankful to the referees for making valuable suggestions leading to the better presentations of this paper.

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Enes Ata
Department of Mathematics, Faculty of Arts and Sciences, Kirsehir Ahi Evran University, 40100 Kirsehir, Turkey.
E-mail address: enesata.tr@gmail.com

İ. Onur Kiymaz
Department of Mathematics, Faculty of Arts and Sciences, Kirsehir Ahi Evran University, 40100 Kirsehir, Turkey.
E-mail address: iokiymaz@ahievran.edu.tr