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COMMUTING TUPLES OF (n, m)-POWER NORMAL OPERATORS IN HILBERT SPACES

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ABSTRACT. The purpose of this study is to extend the notion of (n, m)-power normal bounded operator of Hilbert space to a finite set of commuting such operators. Similar studies for other classes of operators exist in the literature. The purpose of this paper is to introduce and study the structure of certain special types of several variable operators on a Hilbert space named joint (n, m)-power normal multioperators. This is a generalization of the classes of joint normal and joint *n*-normal multioperators. We consider a multivariable generalization of these single variable *n*-normal and (n, m)-normal operators and explore some of their basic properties.

1. INTRODUCTION

Let **K** be a complex Hilbert space, $\mathbf{B}[\mathbf{K}]$ be the algebra of all bounded linear operators defined in **K**. For every $R \in \mathbf{B}[\mathbf{K}]$, denote by , ker(R) and R^* the null space and the adjoint of R, respectively.

The success of the theory of normal operators on Hilbert spaces led to several attempts for generalization to a large classes of operators that include normal operators.

For $R, S \in \mathbf{B}[\mathbf{K}]$, we set [R, S] = RS - SR. An operator $R \in \mathbf{B}[\mathbf{K}]$ is called normal if $[R, R^*] = 0$ ([20, 25, 33]), *n*-power normal if $[R^n, R^*] = 0$ ([2, 18, 27, 28]) and (n, m)-power normal if $[R^n, (R^m)^*] = 0$ where n, m be two nonnegative integers ([1, 3, 4, 19]). These concepts have been generalized into what is known as polynomially normal and doubly polynomially normal as follows: An operator $R \in \mathbf{B}[\mathbf{K}]$ is called polynomially normal if there exists a nontrivial polynomial P =

$$\sum_{0 \le k \le n} c_k z^k \in \mathbb{C}([z]) \text{ such that } P(R)R^* - R^*P(R) = \sum_{0 \le k \le n} c_k \left(R^k R^* - R^* R^k \right) = 0$$

([24, 30]) and it is called doubly polynomially normal if there exist two polynomials P and Q where $P(z) = \sum_{0 \le k \le n} a_k z^k \in \mathbb{C}([z])$ and $Q(z) = \sum_{0 \le k \le m} b_k z^k \in \mathbb{C}([z])$ such

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that $P(R)Q(R^*) - Q(R^*)P(R) = 0$, or equivalently

$$\sum_{\substack{0 \le k \le m \\ 0 \le j \le n}} a_k b_j \left(R^k (R^*)^j - (R^*)^j R^k \right) = 0,$$

(see [5]). Referring to the studies relating to the above-mentioned classes, it was found that they have many interesting properties similar to those of normal operators.

Several variables operator theory is a relevant part of functional analysis. Due to the importance of this field, the interest in studying tuples of operators has grown considerably in the recent few years, see for instance [8, 9, 10, 17, 16, 15, 26, 36, 35] and the references therein.

Over the past few years, various aspects of the problem of generalizing the class of normal, hyponormal and *n*-normal operators to multivariables operator theory have appeared in the literature.

Given an *p*-tuple $\mathbf{R} := (R_1, \dots, R_p) \in \mathbf{B}[\mathbf{K}]^p$, we let $[\mathbf{R}^*, \mathbf{R}] \in \mathbf{B}[\mathbf{K} \oplus \dots \oplus \mathbf{K}]$ denote the self-commutator of \mathbf{R} , defined by

$$[\mathbf{R}^*, \mathbf{R}]_{k,l} := [R_l^*, R_k] \ \forall (k, l) \in \{1, \cdots, p\}^2,$$

where $\mathbf{R}^* := (R_1^*, \cdots, R_p^*).$

We shall say ([8, 21]) that **R** is jointly-hyponormal if

$$[\mathbf{R}^*, \mathbf{R}] = \begin{pmatrix} [R_1^*, R_1] & [R_2^*, R_1] & \cdots & [R_p^*, R_1] \\ [R_1^*, R_2] & [R_2^*, R_2] & \cdots & [R_p^*, R_2] \\ \vdots & \vdots & \vdots & \vdots \\ [R_1^*, R_p] & [R_2^*, R_p] & \cdots & [R_p^*, R_p] \end{pmatrix}$$

is a positive operator on $\mathbf{K} \oplus \cdots \oplus \mathbf{K}$, or equivalently

$$\sum_{1 \le i, k \le p} \left\langle \begin{bmatrix} R_i^* & R_k \end{bmatrix} x \mid x \right\rangle \ge 0 \quad \forall \ x \in \mathbf{K}.$$

 \mathbf{R} is said to be jointly normal if \mathbf{R} ([11]) satisfies

$$\begin{cases} [R_k, R_l] = 0, & k, l \in \{1, \cdots, p\} \\ \\ [R_k^*, R_k] = 0, & k = 1, \cdots, p \end{cases}$$

Very recently, the author in [6] has introduced the concept of jointly *n*-normal tuple as follows: $\mathbf{R} = (R_1, \dots, R_p) \in \mathbf{B}[\mathbf{K}]^p$ is said to be jointly *n*-normal multioperators if \mathbf{R} satisfies

$$\begin{cases} [R_k, R_l] = 0, & k, l \in \{1, \cdots, p\} \\ [R_k^n, R_k^*] = 0, & k = 1, \cdots, p \end{cases}$$

,

for some positive integer n.

In the current paper, closely related to this problem of generalization, we introduce a new class of operators, and we investigate numerous properties of this class. Specifically, we introduce the class of jointly (n, m)-power normal for p-tuples of operators and extend some classical theorems on (n, m)-normal, jointly normal and jointly *n*- normal operators [1, 2, 3, 7, 27].

This paper has been organized in two sections. In section two, we introduce a new class of operators named joint (m, n)-power normal multioperators. Our motivation for this study comes from the problem of finding a tuple of operators $\mathbf{R} = (R_1, \cdots, R_p)$ that are jointly (n, m)-power-normal. Some of the basic properties of this class with some examples are studied. Moreover, the product, tensor product of finite numbers of these type are discussed.

2. Commuting Tuples of (n, m)-Power Normal Operators in Hilbert Spaces

In this section we introduce and study the class of jointly (n, m)-power normal.

Deminition 2.1. Let $\mathbf{R} := (R_1, \dots, R_p) \in \mathbf{B}[\mathbf{K}]^p$. We say that \mathbf{R} is jointly (n, m)-power normal tuple or jointly (n, m)-power normal if \mathbf{R} satisfies the following conditions

$$\left[\begin{array}{c} \left[R_k, \ R_l \right] = 0; \quad \forall \ (k,l) \in \{1, \cdots, p\}^2 \\ \left[R_k^n, \ R_k^{*m} \right] = 0 \ \forall \ k = 1, \cdots, p \end{array} \right]$$

for some positive integers n and m.

We observe that when n = m = 1, **R** is jointly normal [11] and when m = 1, **R** is jointly *n*-power normal [6].

The following example shows that there exists a multioperators $\mathbf{R} = (R_1, \dots, R_p) \in \mathbf{B}(\mathbf{K})^p$ such that each R_k is (n, m)-power normal for $k = 1, \dots, p$, however \mathbf{R} is not jointly (n, m)-power normal. Which means that studying these concepts is not trivial.

Example 2.2. Let $\mathbf{R} = (R_1, R_2) \in \mathbf{B}[\mathbb{C}^4]$ where

$R_1 =$	$\left(1 \right)$	0	0	0 `		0	1	0	0	\
	0	0	0	0	and $R_2 =$	1	0	0	0	
	0	0	0	0		0	0	0	0	·
	0	0	0	0		0	0	0	0)

It easy to check that $[R_1, R_2] \neq 0$ and $[R_j^n, R_j^{*m}] = 0$ for j = 1, 2. This means that, each R_j is (n, m)-power normal while that **R** is not jointly (n, m)-power normal.

Remark 1. (1) Let $R \in \mathbf{B}[\mathbf{K}]$ be an (n,m)-power normal operator, then $\mathbf{R} = (R, \dots, R) \in \mathbf{B}[\mathbf{K}]^p$ is jointly (n,m)-power normal.

(2) Let $\mathbf{R} := (R_1, \dots, R_p) \in \mathbf{B}[\mathbf{K}]^p$ be commuting multioperators. If each R_k is (n, m)-power normal single operator, then \mathbf{R} is jointly (n, m)-power normal.

Proposition 2.3. Let $\mathbf{R} = (R_1, \dots, R_p) \in \mathcal{B}[\mathbf{K}]^p$, the following statements hold.

(1) **R** is jointly (n, m)-power normal if and only if **R**^{*} is jointly (m, n)-power normal.

(2) If each R_k for $k = 1, \dots, p$ is invertible, then $\mathbf{R}^{-1} := (R_1^{-1}, \dots, R_p^{-1})$ is jointly (n, m)-power normal if and only if \mathbf{R} is jointly (n, m)-power normal.

(3) If \mathcal{M} is a reducing subspace for each R_k for $k = 1, \dots, p$, then $\mathbf{R} | \mathcal{M} = (R_1 | \mathcal{M}, \dots, R_p | \mathcal{M})$ is jointly (n, m)-power normal.

(4) If V is an unitary operator, then $V^*\mathbf{R}V := (V^*R_1V, \cdots, V^*R_pV)$ is jointly (n,m)-power normal.

Proof. (1) Obviously, $[R_k^*, R_l^*] = [R_l, R_k]^*$ and therefore $[R_k^*, R_l^*] = 0 \iff [R_l, R_k] = 0$.

On the other hand,

$$[R_k^{*m}, R_k^n] = -[R_k^n, R_k^{*m}], \ k = 1, \cdots, p.$$

Thus, **R** is jointly (n, m)-power normal if and only if **R**^{*} is jointly (m, n)-power normal.

(2) Assume that **R** is jointly (n, m)-power normal. Then

$$\begin{bmatrix} R_k, R_l \end{bmatrix} = R_k R_l - R_l R_k = 0 \implies R_k R_l = R_l R_k \quad \text{for all } k, \ l = 1, \cdots, p$$
$$\implies R_l^{-1} R_k^{-1} = R_k^{-1} R_l^{-1} \quad \text{for all } k, \ l = 1, \cdots, p$$
$$\implies \begin{bmatrix} R_l^{-1}, R_k^{-1} \end{bmatrix} = 0 \quad \text{for all } k, \ l = 1, \cdots, p.$$

However, if each R_k is invertible (n, m)-power normal, if follows from [19] that R_k^{-1} is (n, m)-power normal. Hence, $\mathbf{R}^{-1} := (R_1^{-1}, \cdots, R_p^{-1})$ is jointly (n, m)-power normal. The converse follows immediately from the identity $(\mathbf{R}^{-1})^{-1} = \mathbf{R}$.

(3) We have $[R_k | \mathcal{M}, R_l | \mathcal{M})] = 0$ for all $k, l = 1, \dots, p$. On the other hand,

$$\left[\left(R_{k}|\mathcal{M}\right)^{n},\left(R_{k}|\mathcal{M}\right)^{*m}\right]=\left[R_{k}^{n}|\mathcal{M},R_{k}^{*m}|\mathcal{M}\right]=0.$$

(4) We observe that,

$$\begin{bmatrix} V^* R_k V, V^* R_l V \end{bmatrix} = (V^* R_k V) (V^* R_l V) - (V^* R_l V) (V^* R_k V) \\ = V^* R_k R_l V - V^* R_l R_k V \\ = V^* [R_k, R_l] V \\ = 0.$$

Moreover,

$$[(V^*R_kV)^n, (V^*R_kV)^{*m}] = V^*(R_k)^n VV^*R_k^{*m}V - V^*R_k^{*m}VV^*(R_k)^n V = V^*(R_k)^n R_k^{*m}V - V^*R_k^{*m}(R_k)^n V = V^*[(R_k)^n, R_k^{*m}]V = 0.$$

Hence, $V^* \mathbf{R} V$ is jointly (n, m)-power normal.

In the the following theorem we collect some properties of jointly (n, m)-power normal.

Theorem 2.4. Let $\mathbf{R} = (R_1, \dots, R_p) \in \mathbf{B}[\mathbf{K}]^p$ be jointly (n, m)-power normal, then the following properties hold.

(1) **R** is jointly (rn, sm)-power normal for some positive integers r and s.

(2)
$$\mathbf{R}^q := (R_1^{q_1}, \cdots, R_p^{q_p})$$
 is jointly (n, m) -power normal for $q = (q_1, \cdots, q_p) \in \mathbb{N}^p$.

Proof. (1) Under the assumption that **R** is a jointly (n, m)-power normal, it follows that $[R_k, R_l] = 0$ for $k, l = 1, \dots, p$. However

$$\begin{bmatrix} (R_k)^{rn}, R_k^{*(sm)} \end{bmatrix} = (R_k)^{rn} R_k^{*(sm)} - R_k^{*(sm)} R_k \end{bmatrix}^{rn}$$

$$= \underbrace{(R_k)^n \cdots (R_k)^n}_{r-\text{times}} \underbrace{R_k^{*m} \cdots R_k^{*m}}_{s-\text{times}} - \underbrace{R_k^{*m} \cdots R_k^{*m}}_{s-\text{times}} \underbrace{(R_k)^n \cdots (R_k)^n}_{r-\text{times}}$$

$$= \underbrace{R_k^{*m} \cdots R_k^{*m}}_{s-\text{times}} \underbrace{(R_k)^n \cdots (R_k)^n}_{r-\text{times}} - \underbrace{R_k^{*m} \cdots R_k^{*m}}_{s-\text{times}} \underbrace{(R_k)^n \cdots (R_k)^n}_{r-\text{times}}$$

$$= 0.$$

(2) If $q_k = 1$ for all $k \in \{1, \dots, q\}$, then $[R_k^{q_k}, R_l^{q_l}] = 0$.

If $q_k > 1$ for all $k \in \{1, \dots, p\}$, by taking into account [36, Lemma 2.1] we have

$$[R_k^{q_k}, R_l^{q_l}] = \sum_{\substack{\alpha + \alpha' = q_k - 1\\\beta + \beta' = q_l - 1}} R_k^{\alpha} R_l^{\beta} [R_k, R_l] R_l^{\alpha'} R_k^{\beta'}.$$

Now, under the assumption that **R** is jointly (n, m)-power normal, it follows that

$$\begin{bmatrix} R_k^{q_k}, R_l^{q_l} \end{bmatrix} = \sum_{\substack{\alpha + \alpha' = q_k - 1 \\ \beta + \beta' = q_l - 1}} R_k^{\alpha} R_l^{\beta} [R_k, R_l] R_l^{\alpha'} R_k^{\beta'} = 0, \ \forall \ (k, l) \in \{1, \cdots, p\}^2.$$

Moreover, by looking that R_k is a (n, m)-power normal, then from [19, Proposition 2.10], we obtain that $R_k^{q_k}$ is (n, m)-power normal for all $k \in \{1, \dots, q\}$. This means that, $(R_1^{q_1}, \dots, R_p^{q_p})$ is jointly (n, m)-power normal.

Proposition 2.5. Let $\mathbf{R} = (R_1, \dots, R_p) \in \mathbf{B}[\mathbf{K}]^p$. The following statements are true.

- (1) If **R** is jointly (n, n)-power normal, then $\mathbf{R}^n := \left(R_1^n, \cdots, R_p^n\right)$ is jointly normal.
- (2) If $(\mathbf{R})^n$ is jointly normal and $R_k R_l R_l R_k = 0$ for all $k, l = 1, \dots, p$, then \mathbf{R} is jointly (n, m)-power normal.

Proof. (1) If **R** is jointly (n, n)-power normal. Then we get

$$[R_k, R_l] = 0 \Longrightarrow [R_k^n, R_l^n] = 0 \ \forall \ k, l = 1, \cdots, p.$$

However,

$$\left[R_k^n, R_k^{*n}\right] = 0, \quad \forall \ k \in 1, \cdots, p.$$

Therefore \mathbf{R}^n is joint normal.

(2) Since \mathbf{R}^n is jointly normal, we have that

$$[R_k^n, R_k^{*n}] = 0$$
, for each $k = 1, \dots, p$.

Moreover, it is well known that $[R_k^n, R_k] = 0$ for each $k = 1, \dots, p$. By taking into account the Fuglede-Putnam theorem ([37]) we get $[R_k^n, R_k^*] = 0$ and therefore

 $[R_k^n, R_k^{*m}] = 0$ and moreover for each $k = 1, \dots, p$. Therefore **R** is jointly (n, m)-power normal.

Proposition 2.6. Let $\mathbf{R} = (R_1, \cdots, R_p) \in \mathbf{B}[\mathbf{K}]^p$. The following assertions hold.

- (1) If **R** is joint (n, m)-power normal and jointly (n + 1, m)-power normal tuple, then **R** is jointly (n + 2, m)-power normal.
- (2) If **R** is jointly (n, m)-power normal and joint (n, m + 1)-power normal, then **R** is jointly (n, m + 2)-power normal.

Proof. Since **R** is joint (n, m)-power normal and jointly (n + 1, m)-power normal, we have

$$[R_k, R_l] = 0 \ \forall \ k, l = 1, \cdots, p$$
$$[(R_k)^n, R_k^{*m}] = 0 \ k = 1, \cdots, p$$
$$[(R_k)^{n+1}, R_k^{*m}] = 0, \ k = 1, \cdots, p.$$

This implies that

$$\begin{bmatrix} R_k, \ R_l \end{bmatrix} = 0, \quad \forall \quad k, l = 1, \cdots, p$$
$$\begin{pmatrix} R_k \end{pmatrix}^n \begin{bmatrix} R_k R_k^{*m} - R_k^{*m} R_k \end{bmatrix} = 0, \quad k = 1, \cdots, p$$

and therefore,

$$\begin{bmatrix} R_k, R_l \end{bmatrix} = 0 \ \forall \ k, l = 1, \cdots, p$$
$$\begin{bmatrix} (R_k)^{n+2}, R_k^{*m} \end{bmatrix} = 0, \ k = 1, \cdots, p$$

So, **R** is jointly (n+2, m)-power normal tuple.

(2) The proof of the statement (2) follows by similar techniques as in the proof of statement (1), so we omit it. $\hfill \Box$

Proposition 2.7. Let $\mathbf{R} = (R_1, \cdots, R_p) \in \mathbf{B}_{[\mathbf{K}]}^p$, the following statements hold:

- (1) If **R** is joint (n_1, m) -power normal and jointly (n_2, m) -power normal, then **R** is jointly $(n_1 + n_2, m)$ -power normal.
- (2) If **R** is jointly (n, m_1) -power normal and jointly (n, m_2) -power normal, then **R** is jointly $(n, m_1 + m_2)$ -power normal.
- (3) If **R** is jointly (n_1, m) -power normal and jointly (n_2, m) -power normal, then **R** is jointly $(rn_1 + sn_2, m)$ -power normal for $r, s \in \mathbf{N}$.
- (4) If **R** is jointly (n, m_1) -power normal and jointly (n, m_2) -power normal , then **R** is jointly $(n, rm_1 + sm_2,)$ -power normal for $r, s \in \mathbb{N}$.

Proof. (1) We have $[R_k, R_l] = 0$ for $k, l = 1, \dots, p$ and moreover for $k = 1, \dots, p$,

$$\begin{bmatrix} (R_k)^{n_1+n_2}, R_k^{*m} \end{bmatrix} = (R_k)^{n_1+n_2} R_k^{*m} - R_k^{*m} (R_k)^{n_1+n_2} \\ = (R_k)^{n_1} [(R_k)^{n_2}, R_k^{*m}] \\ = 0.$$

(2) We have $[R_k, R_l] = 0$ for $k, l = 1, \dots, p$ and moreover for $k = 1, \dots, p$,

$$[(R_k)^n, R_k^{*(m_1+m_2)}] = (R_k)^n R_k^{*(m_1+m_2)} - R_k^{*(m_1+m_2)} (R_k)^{n_1+n_2} = [(R_k)^n, R_k^{*m_1}] R_k^{*m_2} = 0.$$

Therefore, the required results are satisfied.

Theorem 2.8. Let $\mathbf{R} = (R_1, \cdots, R_p) \in \mathbf{B}[\mathbf{K}]^p$ such that

$$\ker(\mathbf{R}) := \bigcap_{1 \le k \le p} \ker(R_k) = \{0\}.$$

If **R** is joint (n_1, m) -power normal and jointly (n_2, m) -power normal for some positive integer n_1, n_2 and m, then, **R** is jointly $(\max\{n_1, n_2\} - \min\{n_1, n_2\}, m)$ power normal. In particular, if **R** is jointly (n, 1)-power normal and jointly (n + 1, 1)-power normal, then **R** is jointly normal.

Proof. We have, $[R_k, R_l] = 0$ for all $(k, l) \in \{1, \dots, p\}^2$. Moreover for each $k = 1, \dots, p$ we have

$$\left(\left[\left(R_k \right)^{n_1} R_k^{*m} \right] = 0 \\ \left[\left(R_k \right)^{n_2}, R_k^{*m} \right] = 0$$

Considering the case where $n_1 \ge n_2$, so we get

$$[(R_k)^{n_1}, R_k^{*m}] = 0 \implies (R_k)^{n_2} [(R_k))^{n_1 - n_2}, R_k^{*m}] = 0 \implies [(R_k)^{n_1 - n_2}, R_k^{*m}] = 0,$$

and hence **R** is jointly $(n_1 - n_2, m)$ -power normal.

Proposition 2.9. Let $\mathbf{R} = (R_1, \dots, R_p) \in \mathbf{B}[\mathbf{K}]^p$ be commuting tuple. For $n, m \in \mathbb{N}$, set

$$\mathbf{R}' = (R'_1, \cdots, R'_p) = (R_1^n + R_1^{*m}, \cdots, R_p^n + R_p^{*m})$$

and

$$\mathbf{R}'' = (R_1'', \cdots, R_p'') = ((R_1)^n - R_1^{*m}, \cdots, (R_p)^n - R_p^{*m}).$$

Then the following axioms hold.

- (1) **R** is jointly n-power normal if and only if $[R'_k, R''_k] = 0$ for each $k = 1, \dots, p$.
- (2) If **R** is jointly (n, m)-power normal, then $\mathbf{Z} = (R_1^n R_1^{*m}, \cdots, R_p^n R_p^{*m})$ commutes with **R'** and **R''**.
- (3) **R** is jointly (n, m)-power normal, if and only if $(\mathbf{R})^n$ commutes with \mathbf{R}' .
- (4) **R** is jointly (n,m)-power normal if and only if $(\mathbf{R})^n$ commutes with \mathbf{R}'' .

Proof. Obviously, $[R_k, R_l] = 0 \quad \forall (k, l) \in \{1, \dots, p\}^2$. On the other hand,

$$\begin{bmatrix} R'_{k}, R''_{k} \end{bmatrix} = 0 \iff R'_{k}R''_{k} - R''_{k}R'_{k} = 0 \iff ((R_{k})^{n} + R_{k}^{*m})((R_{k})^{n} - R_{k}^{*m}) - ((R_{k})^{n} - R_{k}^{*m})((R_{k})^{n} + R_{k}^{*m}) = 0 \iff (R_{k})^{2n} - (R_{k})^{n}R_{k}^{*m} + R_{k}^{*m}(R_{k})^{n} - R_{k}^{*2m} - ((R_{k})^{2n} + (R_{k})^{n}R_{k}^{*m} - R_{k}^{*m}(R_{k})^{n} - R_{k}^{*2m}) = 0 \iff (R_{k})^{n}R_{k}^{*m} - R_{k}^{*m}(R_{k})^{n} = 0, \ \forall \ k \in \{1, \cdots, p\}.$$

This completes the proof.

Theorem 2.10. Let $\mathbf{R} = (R_1, \dots, R_p) \in \mathbf{B}[\mathbf{K}]^p$ be jointly (n, m)-power normal for $n \geq m$. If each R_k^m is a partial isometry for $k = 1, \dots, p$, then \mathbf{R} is jointly (n + m, m)-power normal.

Proof. Suppose **R** is jointly (n, m)-power normal for $n \ge m$. It is easy to see that each R_k is (n, m)-power normal for $1 \le k \le p$. Under the hypothesis, R_k^m is a partial isometry, it follows from [32, Theorem 2.4] that R_k is (n + m, m)-power normal operator for $k = 1, \dots, p$. Consequently, **R** is jointly (n + m, m)-power normal.

Proposition 2.11. Let $\mathbf{R} = (R_1, \dots, R_p) \in \mathbf{B}[\mathbf{K}]^p$ and $\mathbf{S} = (S_1, \dots, S_p) \in \mathbf{B}[\mathbf{K}]^p$ be two jointly (n, m)-power normal. The following statements hold.

(1) If $[R_i, S_j] = 0, \forall i, j \in \{1, \dots, p\}$ and $R_k S_k = R_k S_k^* = 0$ for all $k \in \{1, \dots, p\}$, then

 $\mathbf{R} + \mathbf{S} = (R_1 + S_1, \cdots, R_p + S_p)$ is jointly (n, m)-power normal.

(2) If $[R_k, S_l] = 0$, $\forall k, l \in \{1, \dots, p\}^2$ and $[R_k, S_k^*] = 0$ for all $k \in \{1, \dots, p\}$, then **RS** = (R_1S_1, \dots, R_pS_p) and **SR** = (S_1R_1, \dots, S_pR_p) are jointly (n, m)-power normal.

Proof. (1) For all $(i, j) \in \{1, \cdots, p\}^2$, we have

$$[R_k + S_k, R_l + S_l] = (R_k + S_k)(R_l + S_l) - (R_l + S_l)(R_k + S_k) = [R_k, R_l] + [S_k, S_l] + [R_k, S_l] + [S_k, R_l] = 0.$$

Besides, for $k \in \{1, 2, \cdots, p\}$, we get

$$(R_{k} + S_{k})^{*m} ((R_{k} + S_{k}))^{n} = (R_{k} + S_{k})^{*m} (R_{k} + S_{k})^{n} = \left(\sum_{j=0}^{m} {m \choose j} R_{k}^{*j} S_{k}^{*m-j}\right) \left(\sum_{j=0}^{n} {n \choose j} (R_{k})^{j} (S_{k})^{n-j}\right) = (R_{k}^{*m} + S_{k}^{*m}) ((R_{k})^{n} + (S_{k})^{n}) = (R_{k}^{*m} (R_{k})^{n} + R_{k}^{*m} (S_{k})^{n} + S_{k}^{*m} (R_{k})^{n} + S_{k}^{*m} (S_{k})^{n} = (R_{k})^{n} R_{k}^{*m} + (S_{k})^{n} S_{k}^{*m} = ((R_{k})^{n} + (S_{k})^{n}) (R_{k} + S_{k})^{*m} = \left(\sum_{j=0}^{n} {n \choose j} (R_{k})^{j} (S_{k})^{n-j}\right) (R_{k} + S_{k})^{*m} = (R_{k} + S_{k})^{n} (R_{k} + S_{k})^{*m}.$$

So, $\mathbf{R} + \mathbf{S}$ is jointly (n, m)-power normal. (2) We have for all $k, l \in \{1, \dots, q\}$,

$$[R_k S_k , R_l S_l] = R_k S_k R_l S_l - R_l S_l R_k S_k$$

= $R_k R_l S_k S_l - R_l R_k S_l S_k$
= $R_k R_l S_k S_l - R_k R_l S_l S_k$
= $R_k R_l (S_k S_l - S_l S_k)$
= $R_k R_l [S_k, S_l] = 0.$

However, let $k \in \{1, \dots, p\}$, we have

$$(R_k S_k)^* (R_k S_k)^n = S_k^* R_k^* (R_k)^n (S_k)^n$$

= $S_k^* (R_k)^n R_k^* (S_k)^n = S_k^* (R_k)^n S_k^n R_k^*$
= $(R_k)^n (S_k)^n S_k^* R_k^*$
= $(R_k S_k)^n (R_k S_k)^*.$

This implies that **RS** is jointly (n, m)-power normal tuple. In same way, we show that **SR** is jointly (n.m)-power normal tuple.

The following proposition shows that the class of jointly (n, m)-power normal is closed subset of $\mathbf{B}[\mathbf{K}]^p$ equipped with the norm,

$$\|\mathbf{R}\| = \|(R_1, \cdots, R_p)\| = \sup_{1 \le j \le p} \|R_j\|.$$

Proposition 2.12. The class of jointly (n, m)-power normal is a closed subset of $\mathbf{B}[\mathbf{K}]^p$.

Proof. Suppose that $(\mathbf{R}_k = (R_1(k), \cdots, R_p(k)))_k \in \mathbf{B}[\mathbf{K}]^p$ be a sequence of jointly (n, m)-power normal for which

$$\|\mathbf{R}_k - \mathbf{R}\| = \sup_{1 \le j \le p} \left(\|R_j(k) - R_j\| \right) \longrightarrow 0, \text{ as } k \longrightarrow \infty,$$

where $\mathbf{R} = (R_1, \dots, R_p) \in \mathbf{B}_d[\mathbf{K}]^p$. Obviously, for each $j \in \{1, \dots, p\}$ we have

$$\lim_{k \to +\infty} \|R_j(k) - R_j\| = 0.$$
(2.1)

Since $(R_j(k))^n R_j(k)^{*m} = R_j(k)^{*m} (R_j(k))$. for each $j = 1, \dots, p$, it follows from [32, Theorem 2.4] that

$$(R_j)^n R_j^{*m} = R_j^{*m} (R_j)^n, \forall j \in \{1, \cdots, p\}.$$

Moreover, for all $i, j \in \{1, \cdots, p\}$ and $k \in \mathbb{N}$, we can see that

$$\begin{aligned} R_{i}(k)R_{j}(k) - R_{i}R_{j} \| &= \|R_{i}(k)\big(R_{j}(k) - R_{j}\big) + \big(R_{i}(k) - R_{i})R_{j}\| \\ &\leq \|R_{i}(k)\|\|R_{j}(k) - R_{j}\| + \|R_{i}(k) - R_{i}\|\|R_{j}\| \\ &\leq (\|R_{i}(k) - R_{i}\| + \|R_{i}\|)\|R_{j}(k) - R_{j}\| + \|R_{i}(k) - R_{i}\|\|R_{j}\| \end{aligned}$$

Hence, in view of (2.1), we obtain

$$||R_i(k)R_j(k) - R_iR_j|| \longrightarrow 0$$
, as $k \to +\infty$, $\forall (i,j) \in \{1, \cdots, q\}^2$.

On the other hand, since $\{\mathbf{R}_k\}_k = \{(R_1(k), \cdots, R_p(k))\}_k$ is a sequence of jointly (n, m)-power normal tuple, then

$$[R_i(k), R_j(k)] = 0 \ \forall \ (i, j) \in \{1, \dots, p\}^2 ; \text{and } k \in \mathbb{N}.$$

Therefore, we immediately get

$$[R_i, R_j] = 0 \quad \forall \ (i, j) \in \{1, 2, \cdots, p\}^2.$$

Therefore, **R** is jointly (n, m)-power normal.

3. Tensor Product and Tensor Sum of Jointly (n, m)-Normal Operators

Given non-zero $R, S \in \mathcal{B}(\mathbf{K})$, let $R \otimes S \in \mathcal{B}(\mathbf{K} \otimes \mathbf{K})$ denote the tensor product on the Hilbert space $\mathbf{K} \otimes \mathbf{K}$. We recall that the tensor product $(R \otimes S)^* (R \otimes S) = R^* R \otimes S^* S$,

$$R \otimes S = (R \otimes I)(I \otimes S) = (I \otimes S)(R \otimes I)$$

 $(R \otimes S)^k = R^k \otimes S^k$. The operation of taking tensor products $R \otimes S$ preserves many properties of R and S, but by no means all of them. Thus, it was proved in [38, Theorem 2.4] that $R \otimes S$ is normal if and only if R and S are normal. Similar result was proved in [7] for *n*-normal operators. However, it was proved in [32] that If $R, S \in \mathbf{B}[\mathbf{K}]$ such that R and S are (n, m)-power normal operators, then $R \otimes S$ is (n, m)-power normal.

In this section, we study the tensor product and tensor sum of two jointly (n, m)power normal. Let $\mathbf{R} = (R_1, \dots, R_p) \in \mathbf{B}[\mathbf{K}]^p$ and $\mathbf{S} = (S_1, \dots, S_p) \in \mathbf{B}[\mathbf{K}]^p$. We denote by

$$\mathbf{R}\otimes\mathbf{S}=\left(R_1\otimes S_1,\cdots,R_p\otimes S_p\right)$$

and

$$\mathbf{R} \boxplus \mathbf{S} = \mathbf{R} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{S} = (R_1 \otimes I + I \otimes S_1, \cdots, R_p \otimes I + I \otimes S_p).$$

Theorem 3.1. Let $\mathbf{R} = (R_1, \dots, R_p) \in \mathbf{B}[\mathbf{K}]^p$ and $\mathbf{S} = (S_1, \dots, S_p) \in \mathbf{B}[\mathbf{K}]^p$ are two jointly (n, m)-power normal, then $\mathbf{R} \otimes \mathbf{S}$ is jointly (n, m)-power normal.

Proof. Since $\mathbf{R} = (R_1, \dots, R_p)$ and $\mathbf{S} = (S_1, \dots, S_p)$ are joint (n, m)-power normal tuples, we have all $(k, l) \in \{1, \dots, p\}^2$

$$[(R_k \otimes S_k), (R_l \otimes S_l)]$$

$$= [(R_k \otimes S_k)(R_l \otimes S_l) - (R_l \otimes S_l)(R_k \otimes S_k)]$$

$$= R_k R_l \otimes S_k S_l - R_l R_k \otimes S_l S_k$$

$$= R_l R_k \otimes S_l S_k - R_l R_k \otimes S_l S_k$$

$$= 0.$$

Moreover, for all $k \in \{1, \dots, p\}$, we have

$$((R_k \otimes S_k))^n (R_k \otimes S_k)^{*m} = (R_k)^n R_k^{*m} \otimes (S_k)^n S_k^{*m}$$

= $R_k^{*m} (R_k)^n \otimes S_k^{*m} (S_k)^n$
= $(R_k \otimes S_k)^{*m} ((R_k \otimes S_k))^n.$

So, $\mathbf{R} \otimes \mathbf{S}$ is joint (n, m)-power normal tuple.

The following example shows that the converse of the above theorem need not hold in general.

Example 3.2. Let
$$R_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{B}[\mathbb{C}^3] \text{ and } R_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathbf{B}[\mathbb{C}^3].$$
 A direct calculation shows that

and

Consider,
$$\mathbf{R} = (R_1, R_2)$$
 and $\mathbf{R} \otimes \mathbf{R} = (R_1 \otimes R_1, R_2 \otimes R_2)$. We observe that \mathbf{R} is not jointly (2, 3)-power normal since $R_1R_2 \neq R_2R_1$. However

$$(R_k \otimes R_k)^{*3} ((R_k \otimes R_k))^2 = ((R_k \otimes R_k))^2 (R_k \otimes R_k)^{*3}, \ k \in \{1, 2\}$$

Hence, $\mathbf{R} \otimes \mathbf{R}$ is jointly (2, 3)-power normal pairs.

In the following theorem we give the conditions under which the converse of Theorem 3.1 is true.

Theorem 3.3. Let $\mathbf{R} = (R_1, \dots, R_p) \in \mathbf{B}[\mathbf{K}]^p$ and $\mathbf{S} = (S_1, \dots, S_p) \in \mathbf{B}[\mathbf{K}]^p$ are commuting multioperators. Then, if $\mathbf{R} \otimes \mathbf{S}$ is jointly (n, n)-power normal tuple if and only if \mathbf{R} and \mathbf{S} are jointly (n, n)-power normal.

Proof. Assume that $\mathbf{T} \otimes \mathbf{S}$ is jointly (n, n)-power normal. By taking into account the statement (1) of Proposition 2.5 it follows that

$$\left(\left(\mathbf{R}\otimes\mathbf{S}\right)\right)^{n}=\left(\left(\left(R_{1}\otimes S_{1}\right)\right)^{n},\cdots,\left(\left(R_{p}\otimes S_{p}\right)\right)^{n}\right)=\left(R_{1}^{n}\otimes S_{1}^{n},\cdots R_{p}^{n}\otimes S_{p}^{n}\right)$$

is jointly normal tuple. From which we deduce that

$$(R_k \otimes S_k)^n = (R_k^n \otimes S_k^n),$$

is normal for each $k = 1, \dots, p$. By [38, Theorem 2.4] it is well known that

 $R^n_k\otimes S^n_k$ is normal if and only if $\,R^n_k$ and S^n_k are normal operators.

However, According to [27, Propositon 2.1] it is well known that R_k^n is normal if and only if that R_k is *n*-power normal and similarly, S_k^n is normal if and only if that S_k is *n*-power normal. Therefore **R** and **S** are jointly (n, n)-power normal.

The converse follows from Theorem 3.1.

Corollary 3.4. Let $\mathbf{R} = (R_1, \dots, R_p)$ and $\mathbf{S} = (S_1, \dots, S_p)$ be jointly (n, m)-power normal. Then $\mathbf{R}^{\alpha} \otimes \mathbf{S}^{\beta}$ is jointly (n, m)-power normal for all $\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbb{N}^p$ and $\beta = (\beta_1, \dots, \beta_p) \in \mathbb{N}^p$ where $\mathbf{R}^{\alpha} = (R_1^{\alpha_1}, \dots, R_p^{\alpha_p})$ and $\mathbf{S}^{\beta} = (S_1^{\beta_1}, \dots, S_p^{\beta_p})$.

Proof. The proof can be easily derived from the statement (2) of Theorem 2.4 and Theorem 3.1. $\hfill \Box$

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