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THE STABILITY ANALYSIS OF THE SYSTEM OF INTEGRODIFFERENTIAL EQUATIONS

SEBAHEDDIN SEVGIN, JARGEES ABDULWAHID ABDULLAH, SADEQ TAHA ABDULAZEEZ

Abstract. In this study, we use the fixed-point theorem of Margolis and Diaz to investigate the Ulam-Hyers-Rassias stability of a linear system of Volterra integrodifferential equations. We also extended this finding to a n-th order linear Volterra integrodifferential problem. In addition, we present examples to highlight the relevance of our findings. The discovered conclusions are theoretically significant and have possible applications in a variety of mathematical and scientific domains.

1. Introduction

In 1940, Stanislaw Marcin Ulam addressed the question, "Under what conditions is there a linear transformation near an approximation linear transformation?" in his comprehensive lecture at the University of Wisconsin, where he explored key previously unsolved topics (see [1]). Hyers published the first response to Ulam's challenge in 1941, demonstrating that the aggregate Cauchy functional equation for any pair of Banach spaces is stable [2]. Rassias proved the generalization of Hyers' theorem in 1978 by bringing the Cauchy difference to infinity. The Ulam-Hyers-Rassias stability is the name given to this event. The proofs were made by Hyers and Rassias by creating an additive function from a direct function. This is known as the direct technique, and it is a useful tool for studying the stability of numerous functional equations. There are numerous more viable approaches for investigating Ulam-Hyers stability. The fixed point approach is one of them. Baker was the first to adopt this strategy, according to Baker [3]. Radu then established the stability of the additive Cauchy functional equation using Diaz and Margolis' fixed point theorem [9], [4] Cadariu and Radu ([5], [6]) achieved some conclusions for the generalized Ulam-Hyers stability of the Cauchy and Jensen functional equations in the same way.

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The Ulam-Hyers stability of Volterra equation was first proven by Jung using the method built by Cadariu and Radu [5] on Margolis and Diaz's alternative fixed-point theorem in [7]. Later, some researchers used the same method to demonstrate the stability of different versions of Volterra integral equations (see [10], [11], [12], [13]).

Integro-differential equations are an important theoretical topic with many applications. Many studies have examined the Ulam-Hyers stability of Volterra integrodifferential equations in recent years. In Şevgin and Şevli [15], the Ulam-Hyers stability and Ulam-Hyers-Rassias stability of a nonlinear Volterra integrodifferential equation with an initial condition were investigated using Diaz and Margolis' fixed-point approach. More studies can be found at [8], [16], [17], [18], [19], [20].

Let Ω be any interval and $\gamma \in \Omega$. Now, we consider linear system of Volterra integrodifferential equation the form

$$\mathbf{x}'(\xi) = \mathbf{P}(\xi)\mathbf{x}(\xi) + \mathbf{q}(\xi) + \int_{\gamma}^{\xi} \mathbf{K}(\xi,\eta)\mathbf{x}(\eta)d\eta, \quad \xi \in \Omega \tag{1.1}$$

where $\mathbf{q} \in \mathbb{R}^n$ is a vector of continuous functions on Ω, $\mathbf{P}$ is an $n \times n$ matrix of continuous functions on Ω, $\mathbf{K}$ is an $n \times n$ matrix of continuous functions on $\Omega \times \Omega$. We introduce the following notations that we will use in this section:

$$\|\mathbf{w}(\xi)\| = \max_{1\le i\le n} |w_i(\xi)|\,,\ \mathbf{w} \in \mathbb{R}^n,$$

$$\|\mathbf{A}(\xi,\eta)\| = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}(\xi,\eta)|\,,\ \mathbf{A} \in \mathbb{R}^{n\times n}.$$

Definition 1.1. The system (1.1) is said to be Ulam-Hyers-Rassias stable if for each continuously differentiable function $\mathbf{x}(\xi)$ satisfying

$$\left\|\mathbf{x}'(\xi) - \mathbf{P}(\xi)\mathbf{x}(\xi) - \mathbf{q}(\xi) - \int_{\gamma}^{\xi} \mathbf{K}(\xi,\eta)\mathbf{x}(\eta)d\eta\right\| \le \varphi(\xi), \quad \xi \in \Omega$$

where $\varphi(\xi) \ge 0$ for all ξ, there exists a solution $\mathbf{x}_0(\xi)$ of system (1.1) and a constant $C > 0$ such that

$$\|\mathbf{x}(\xi) - \mathbf{x}_0(\xi)\| \le C\varphi(\xi),$$

for all ξ, where C is independent of $\mathbf{x}(\xi)$ and $\mathbf{x}_0(\xi)$.

The focus of this paper is on the Ulam-Hyers-Rassias stability of the system of linear first-order Volterra integrodifferential equation (1.1), which is discussed using the fixed point method of Diaz and Margolis. These results are then applied to a linear Volterra integrodifferential equation of n-th order defined on three different sets: the set of real numbers, a left-closed right-open interval of the set of real numbers, and a left-open right-closed interval of a set of real numbers.

2. Basic Information

This section defines a generalized metric and offers a generalization of fixed point theory, both of which are necessary for proving our main conclusion.

Definition 2.1. Let Y be a nonempty set. A function $\rho : Y \times Y \to [0,\infty]$ is called a generalized metric on Y if and only if ρ satisfies (i) $\rho(y,z) = 0$ if and only if $y = z$; (ii) $\rho(y,z) = \rho(z,y)$ for all $y,z \in Y$; (iii) $\rho(y,w) \le \rho(y,z) + \rho(z,w)$ for all $y,z,w \in Y$.

Theorem 2.1. *[Diaz-Margolis]* [9] *Let (Y,ρ) be a generalized complete metric space. Assume that $\mathcal{A}: Y \to Y$ a strictly contractive operator with the Lipschitz constant $L<1$. If there exists a nonnegative integer k such that $\rho(\mathcal{A}^{k+1}y, \mathcal{A}^k y) < \infty$ for some $y \in Y$, then the followings are true:*

(a) The sequence $\{\mathcal{A}^n y\}$ is convergent, and its limit y^* is a fixed point of $\mathcal{A}$;

(b) y^* is the unique fixed point of $\mathcal{A}$ in

$$Y^* = \left\{ y \in Y \,\middle|\, \rho(\mathcal{A}^k y, z) < \infty \right\};$$

(c) If $z \in Y^*$, then

$$\rho(z, y^*) \leq \frac{1}{1-L}\rho(\mathcal{A}z, z).$$

3. The Stability of a Linear Volterra Integrodifferential Equation System

The fixed-point method was used by Alqifiary [14] to demonstrate the generalized Ulam-Hyers stability of a system of first order linear differential equations. A second-order linear differential equation was then treated with the results that had been obtained. We will use the same method to demonstrate a system of linear Volterra integrodifferential equations' Ulam-Hyers-Rassias stability. This conclusion will also be applied to a linear Volterra integrodifferential equation of nth order.

Firstly, we show that equation (1.1) have the Ulam-Hyers-Rassias stability on $\Omega = [\alpha, \beta)$, where $-\infty < \alpha < \beta \leq \infty$.

Theorem 3.1. *Let $\mathbf{P}: \Omega \to \mathbb{R}^{n\times n}$, $\mathbf{q}: \Omega \to \mathbb{R}^n$ and $\mathbf{K}: \Omega \times \Omega \to \mathbb{R}^{n\times n}$ be a continuous function and let M be a constant such that $\|\mathbf{P}(\xi)\| \geq M$ for all $t \in \Omega$. Let L and N be positive constants with $0 < L + N < 1$. Suppose that φ is an integrable positive valued function on Ω such that*

$$\int_\gamma^\xi \|\mathbf{P}(\xi)\|\varphi(\tau)d\tau \leq L\varphi(\xi), \quad \forall \xi \in \Omega \tag{3.1}$$

and

$$\int_\gamma^\xi \int_\gamma^\eta \|\mathbf{K}(\eta,\tau)\|\, \varphi(\tau)d\tau d\eta \leq N\varphi(\xi),\ \forall(\xi,\eta) \in \Omega\times\Omega. \tag{3.2}$$

If a continuously differentiable function $\mathbf{x}: \Omega \to \mathbb{R}^n$ satisfies

$$\left\| \mathbf{x}'(\xi) - \mathbf{P}(\xi)\mathbf{x}(\xi) - \mathbf{q}(\xi) - \int_\gamma^\xi \mathbf{K}(\xi,\eta)\mathbf{x}(\eta)d\eta \right\| \leq \varphi(\xi) \tag{3.3}$$

for all $\xi \in \Omega$, then there exists a unique solution $\mathbf{x}_0: \Omega \to \mathbb{R}^n$ of the equation (1.1) such that

$$\|\mathbf{x}(\xi) - \mathbf{x}_0(\xi)\| \leq \frac{L}{M - M(L+N)}\varphi(\xi) \tag{3.4}$$

for all $\xi \in \Omega$ and $\mathbf{x}_0(\gamma) = \mathbf{x}(\gamma)$.

Proof. Let us consider the set

$$Y = \{\mathbf{f}: \Omega \to \mathbb{R}^n \mid \ \mathbf{f} \text{ is continuous and } \mathbf{f}(\gamma) = \mathbf{x}(\gamma)\}$$

and the the generalized metric $\rho(\mathbf{f},\mathbf{g})$ defined on Y as

$$\rho(\mathbf{f},\mathbf{g}) = \inf\left\{C \in [0,\infty] \mid \|\mathbf{f}(\xi) - \mathbf{g}(\xi)\| \leq C\varphi(\xi),\ \forall \xi \in \Omega\right\}. \tag{3.5}$$

Then (Y,ρ) is a generalized complete metric space (see [7]). We define the operator $\mathcal{A}: Y \to Y$,

$$(\mathcal{A}\mathbf{f})(\xi) = \mathbf{x}(\gamma) - \int_\gamma^\xi (\mathbf{P}(\tau)\mathbf{f}(\tau) + \mathbf{q}(\tau))\, d\tau - \int_\gamma^\xi \int_\gamma^\eta \mathbf{K}(\eta,\tau)\mathbf{f}(\tau) d\tau d\eta,\ \forall \xi \in \Omega \quad (3.6)$$

for all $\mathbf{f} \in Y$. Indeed $\mathcal{A}\mathbf{f}$ is a continuously differentiable function on Ω, since $\mathbf{P}$, $\mathbf{q}$ and $\mathbf{K}(\xi,\eta)$ are continuous function and $\mathcal{A}\mathbf{f}(\gamma) = \mathbf{x}(\gamma)$. Now let $\mathbf{f}, \mathbf{g} \in Y$. Then we have

$$\begin{aligned}
\|\mathcal{A}\mathbf{f}(\xi) - \mathcal{A}\mathbf{g}(\xi)\| &\leq \left\| \int_\gamma^\xi \mathbf{P}(\tau)\left(\mathbf{f}(\tau) - \mathbf{g}(\tau)\right) d\tau + \int_\gamma^\xi \int_\gamma^\eta \mathbf{K}(\eta,\tau)\left(\mathbf{f}(\tau) - \mathbf{g}(\tau)\right) d\tau d\eta \right\| \\
&\leq \rho(\mathbf{f},\mathbf{g}) \int_\gamma^\xi \|\mathbf{P}(\tau)\| \varphi(\tau) d\tau + \rho(\mathbf{f},\mathbf{g}) \int_\gamma^\xi \int_\gamma^\eta \|\mathbf{K}(\eta,\tau)\| \varphi(\tau) d\tau d\eta \\
&\leq \rho(\mathbf{f},\mathbf{g}) L\varphi(\xi) + \rho(\mathbf{f},\mathbf{g}) N\varphi(\xi) \\
&= (L+N)\rho(\mathbf{f},\mathbf{g})\varphi(\xi)
\end{aligned}$$

for all $\xi \in \Omega$. Therefore

$$\rho(\mathcal{A}\mathbf{f}, \mathcal{A}\mathbf{g}) \leq (L+N)\rho(\mathbf{f},\mathbf{g}).$$

Given that $0 < L+N < 1$, the operator $\mathcal{A}$ is a contraction of the constant $L+N$. So, by integrating both sides of the relation (3.3) on $[\gamma,\xi]$, we get

$$\left\| \mathbf{x}(\xi) - \mathbf{x}(\gamma) - \int_\gamma^\xi (\mathbf{P}(\tau)\mathbf{x}(\tau) + \mathbf{q}(\tau))\, d\tau - \int_\gamma^\xi \int_\gamma^\eta \mathbf{K}(\eta,\tau)\mathbf{x}(\tau) d\tau d\eta \right\| \leq \frac{L}{M}\varphi(\xi),$$

for each $\xi \in \Omega$, it means that $\rho(\mathbf{x}, \mathcal{A}\mathbf{x}) \leq L/M < \infty$. According to Theorem 2.1, there exists continuously differentiable function $\mathbf{x}_0 : \Omega \to \mathbb{R}^n$ such that $\mathbf{x}_0 = \lim_{n\to\infty} \mathcal{A}^n \mathbf{x}$. In this case, $\mathbf{x}_0$ is unique fixed-point of $\mathcal{A}$ over

$$Y^* = \{\mathbf{f} \in Y : \rho(\mathcal{A}^{n_0}x, \mathbf{f}) < \infty\}.$$

It may be proved that

$$Y^* = \{\mathbf{f} \in Y \mid \rho(\mathbf{x}, \mathbf{f}) < \infty\}.$$

As a result, the set Y is unaffected by n_0. To show that the function $\mathbf{x}_0$ is a solution to the equation (1.1), we differentiate both sides of the relation with regard to ξ.

$$\mathbf{x}_0(\xi) = \mathcal{A}\mathbf{x}_0(\xi), \quad \xi \in \Omega.$$

Thus

$$\mathbf{x}_0'(\xi) = \mathbf{P}(\xi)\mathbf{x}_0(\xi) + \mathbf{q}(\xi) + \int_\gamma^\xi \mathbf{K}(\xi,\eta)\mathbf{x}_0(\eta) d\eta$$

the function $\mathbf{x}_0$ is a solution to the equation (1.1) and supports the relation for all $\xi \in \Omega$, which means that the relationship is supported by:
$\mathbf{x}_0(\gamma) = \mathbf{x}(\gamma)$.

Applying again Theorem 2.1, we obtain

$$\rho(\mathbf{f}, \mathbf{x}_0) \leq \frac{1}{1-(L+N)} \rho(\mathbf{f}, \mathcal{A}\mathbf{f}),$$

for all $\mathbf{f} \in Y^*$. Since $\mathbf{x} \in Y^*$, we have

$$\rho(\mathbf{x}, \mathbf{x}_0) \leq \frac{1}{1-(L+N)} \rho(\mathbf{x}, \mathcal{A}\mathbf{x}) \leq \frac{L}{M - M(L+N)}.$$

Hence

$$|\mathbf{x}(\xi)-\mathbf{x}_0(\xi)| \leq \frac{L}{M-M(L+N)}\varphi(\xi)$$

for all $\xi \in \Omega$. This inequality proves the relation (3.4). □

Similarly, we show that equation (1.1) have the Ulam-Hyers-Rassias stability on the interval $\Omega = (\alpha, \beta]$, where $-\infty \leq \alpha < \beta < \infty$.

Theorem 3.2. *Let $\mathbf{P} : \Omega \to \mathbb{R}^{n\times n}$, $\mathbf{q} : \Omega \to \mathbb{R}^n$ and $\mathbf{K} : \Omega \times \Omega \to \mathbb{R}^{n\times n}$ be continuous function and suppose that M be a constant such that $\|\mathbf{P}(\xi)\| \geq M$ for each $\xi \in \Omega$. Let L and N be positive constants with $0 < L+N < 1$ and $\gamma \in \Omega$. Suppose that φ is an integrable positive valued function on Ω such that*

$$\int_\gamma^\xi \|\mathbf{P}(\xi)\|\varphi(\tau)d\tau \leq L\varphi(\xi), \quad \forall \xi \in \Omega$$

and

$$\int_\gamma^\xi \int_\gamma^\eta \|\mathbf{K}(\eta,\tau)\|\, \varphi(\tau)d\tau d\eta \leq N\varphi(\xi),\, \forall(\xi,\eta) \in \Omega\times\Omega.$$

If a continuously differentiable function $\mathbf{x} : \Omega \to \mathbb{R}^n$ satisfies

$$\left\|\mathbf{x}'(\xi) - \mathbf{P}(\xi)\mathbf{x}(\xi) - \mathbf{q}(\xi) - \int_\gamma^\xi \mathbf{K}(\xi,\eta)\mathbf{x}(\eta)d\eta\right\| \leq \varphi(\xi)$$

for all $\xi \in \Omega$, then there exists a unique solution $\mathbf{x}_0(\xi) : \Omega \to \mathbb{R}^n$ of the equation (1.1) such that

$$\|\mathbf{x}(\xi)-\mathbf{x}_0(\xi)\| \leq \frac{L}{M-M(L+N)}\varphi(\xi) \tag{3.7}$$

for all $\xi \in \Omega$ and $\mathbf{x}_0(\gamma) = \mathbf{x}(\gamma)$.

Proof. It can be proved by following the way used in Theorem 3.1 on the interval Ω. □

Using Theorem 3.1 and Theorem 3.2, we will show that the equation (1.1) has the Ulam-Hyers-Rassias stability on $\mathbb{R}$ as follows.

Corollary 3.3. *Let $\mathbf{P} : \mathbb{R} \to \mathbb{R}^{n\times n}$, $\mathbf{q} : \mathbb{R} \to \mathbb{R}^n$ and $\mathbf{K} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{n\times n}$ be continuous functions and let M be a positive constant such that $\|\mathbf{P}(\xi)\| \geq M$ for all $\xi \in \mathbb{R}$. Let L and N be positive constants with $0 < L+N < 1$. Suppose that φ is an integrable positive valued function on $\mathbb{R}$ such that*

$$\int_0^\xi \|\mathbf{P}(\xi)\|\varphi(\tau)d\tau \leq L\varphi(\xi), \quad \forall \xi \in \mathbb{R} \tag{3.8}$$

and

$$\int_0^\xi \int_0^\eta \|\mathbf{K}(s,\tau)\|\, \varphi(\tau)d\tau ds \leq N\varphi(\xi),\, \forall(\xi,\eta) \in \mathbb{R}\times\mathbb{R}.$$

If a continuously differentiable function $\mathbf{x} : \mathbb{R} \to \mathbb{R}^n$ satisfies

$$\left\|\mathbf{x}'(\xi) - \mathbf{P}(\xi)\mathbf{x}(\xi) - \mathbf{q}(\xi) - \int_0^\xi \mathbf{K}(\xi,\eta)\mathbf{x}(\eta)d\eta\right\| \leq \varphi(\xi)$$

for all $\xi \in \mathbb{R}$*, then there exists a unique solution* $\mathbf{x}_0(\xi) : \mathbb{R} \to \mathbb{R}^n$ *of the equation (1.1) such that*

$$\|\mathbf{x}(\xi) - \mathbf{x}_0(\xi)\| \leq \frac{L}{M - M(L+N)}\varphi(\xi) \tag{3.9}$$

for all $\xi \in \mathbb{R}$ *and* $\mathbf{x}_0(0) = \mathbf{x}(0)$*.*

Proof. By the relation (3.8) we have

$$\int_0^{\xi} \|\mathbf{P}(\tau)\| \varphi(\tau)d\tau \leq L\varphi(\xi), \tag{3.10}$$

for all $\xi \geq 0$. Applying Theorem 3.1, there exists a solution of equation (1.1), $\mathbf{x}_1 : [0, \infty) \to \mathbb{R}^n$ which satisfies the inequality (3.4) and $\mathbf{x}_1(0) = \mathbf{x}(0)$.

From (3.8) we also obtain

$$\int_{\xi}^{0} \|\mathbf{P}(\tau)\| \varphi(\tau)d\tau \leq L\varphi(\xi), \tag{3.11}$$

for all $\xi < 0$. Applying Theorem 3.2, there exists a solution of equation (1.1), $\mathbf{x}_2 : (-\infty, 0] \to \mathbb{R}^n$ which satisfies the inequality (3.7) and $\mathbf{x}_2(0) = \mathbf{x}(0)$. The function

$$\mathbf{x}_0(\xi) = \begin{cases} \mathbf{x}_1(\xi), & \xi \geq 0 \\ \mathbf{x}_2(\xi), & \xi < 0, \end{cases}$$

is a continuously differentiable function on $\mathbb{R}$. It can be easily checked that it is a solution of equation (1.1) on $\mathbb{R}$ and it satisfies inequality (3.9). □

4. Stability of a n-th Order Integrodifferential Equation

In this section we will prove the Ulam-Hyers-Rassias stability for the following n-th order linear Volterra integrodifferential equation

$$y^{(n)}(\xi) = \sum_{k=0}^{n-1} a_k(\xi)y^{(k)}(\xi) + f(\xi) + \int_{\gamma}^{\xi} K(\xi, \eta)y(\eta)d\eta, \tag{4.1}$$

where $a_0, a_1, \cdots, a_{n-1}$ and f are continuous functions on Ω, and K is continuous functions on $\Omega \times \Omega$.

Firstly, we show that equation (4.1) have the Ulam-Hyers-Rassias stability on the interval $\Omega = [\alpha, \beta)$, where $-\infty < \alpha < \beta \leq \infty$.

Theorem 4.1. *Let* $a_0, a_1, \cdots, a_{n-1} : \Omega \to \mathbb{R}$*,* $f : \Omega \to \mathbb{R}$ *and* $K : \Omega \times \Omega \to \mathbb{R}$ *be continuous functions and let* M *be a positive constant such that* $|p_*(\xi)| \geq M$ *for all* $\xi \in \Omega$*. Let* L *and* N *be positive constants with* $0 < L + N < 1$ *and* $\gamma \in \Omega$*. Suppose that* φ *is an integrable positive valued function on* Ω *such that*

$$\int_{\gamma}^{\xi} |p_*(\tau)| \varphi(\tau)d\tau \leq L\varphi(\xi), \quad \forall \xi \in \Omega \tag{4.2}$$

and

$$\int_{\gamma}^{\xi}\int_{\gamma}^{\eta} |K(\eta, \tau)| \varphi(\tau)d\tau d\eta \leq N\varphi(\xi), \quad \forall (\xi, \eta) \in \Omega \times \Omega \tag{4.3}$$

where $p_*(\xi) = \max\left\{1, \sum_{k=0}^{n-1} a_k(\xi)\right\}$. *If a n-times continuously differentiable function $y : \Omega \to \mathbb{R}$ such that*

$$\left| y^{(n)}(\xi) - \sum_{k=0}^{n-1} a_k(\xi) y^{(k)}(\xi) - f(\xi) - \int_\gamma^\xi K(\xi,\eta) y(\eta) d\eta \right| \le \varphi(\xi) \tag{4.4}$$

for all $\xi \in \Omega$, then there exista a unique n-times continuously differentiable solution $y_0(\xi) : \Omega \to \mathbb{R}$ of the equation (4.1) such that

$$|y(\xi) - y_0(\xi)| \le \frac{L}{M - M(L+N)} \varphi(\xi) \tag{4.5}$$

for all $\xi \in \Omega$ and $y_0(\gamma) = y(\gamma)$, $y_0'(\gamma) = y'(\gamma)$, ..., $y_0^{(n-1)}(\gamma) = y^{(n-1)}(\gamma)$.

Proof. The equation (4.1) can be converted into a system of Volterra integrodifferential equations. For this, let us make the following substitutions

$$x_1(\xi) = y(\xi),\ x_2(\xi) = y'(\xi),\ \cdots,\ x_{n-1}(\xi) = y^{(n-2)}(\xi),\ x_n(\xi) = y^{(n-1)}(\xi).$$

From these substitutions, we get the following system of equations:

$$\begin{aligned} x_1'(\xi) &= y'(\xi) = x_2(\xi) \\ x_2'(\xi) &= y''(\xi) = x_3(\xi) \\ &\vdots \\ x_{n-1}'(\xi) &= y^{(n-1)}(\xi) = x_n(\xi) \\ x_n'(\xi) &= y^{(n)}(\xi) = \sum_{k=0}^{n-1} a_k(\xi) y^{(k)}(\xi) + f(\xi) + \int_\gamma^\xi K(\xi,\eta) x_1(\eta) d\eta. \end{aligned}$$

Then the above system can be written in the vector-matrix form as follow, which is also a system of Volterra integrodifferential equations:

$$\mathbf{x}'(\xi) = \mathbf{P}(\xi)\mathbf{x}(\xi) + \mathbf{q}(\xi) + \int_\gamma^\xi \mathbf{K}(\xi,\eta)\mathbf{x}(\eta) d\eta, \tag{4.6}$$

where

$$\mathbf{x}(\xi) = \begin{bmatrix} x_1(\xi) \\ x_2(\xi) \\ \vdots \\ x_n(\xi) \end{bmatrix}, \mathbf{P}(\xi) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_0(\xi) & a_1(\xi) & a_2(\xi) & \cdots & a_{n-1}(\xi) \end{bmatrix},$$

$$\mathbf{q}(\xi) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(\xi) \end{bmatrix}, \mathbf{K}(\xi,\eta) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ K(\xi,\eta) & 0 & \cdots & 0 \end{bmatrix}.$$

By hypotheses, since $p_*(\xi) = \max\left\{1, \sum_{k=0}^{n-1} a_k(\xi)\right\}$, then $\|\mathbf{P}(\xi)\| = \max\left\{1, \sum_{k=0}^{n-1} a_k(\xi)\right\}$ and by relation (4.2) we obtain

$$\int_\gamma^\xi \|\mathbf{P}(\xi)\|\varphi(\tau)d\tau \le L\varphi(\xi).$$

Now, let a function y satisfy the relation (4.4). Since

$$\mathbf{x}(\xi) = \begin{bmatrix} x_1(\xi) \\ x_2(\xi) \\ \vdots \\ x_n(\xi) \end{bmatrix} = \begin{bmatrix} y(\xi) \\ y'(\xi) \\ \vdots \\ y^{(n-1)}(\xi) \end{bmatrix}$$

and

$$\mathbf{x}'(\xi) - \mathbf{P}(\xi)\mathbf{x}(\xi) - \mathbf{q}(\xi) - \int_\gamma^\xi \mathbf{K}(\xi,\eta)\mathbf{x}(\eta)d\eta = \begin{bmatrix} y'(\xi) \\ y''(\xi) \\ \vdots \\ y^{(n)}(\xi) \end{bmatrix}$$

$$- \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_0(\xi) & a_1(\xi) & a_2(\xi) & \cdots & a_{n-1}(\xi) \end{bmatrix} \begin{bmatrix} y(\xi) \\ y'(\xi) \\ \vdots \\ y^{(n-1)}(\xi) \end{bmatrix}$$

$$- \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(\xi) \end{bmatrix} - \int_\gamma^\xi \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ K(\xi,\eta) & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} y(\eta) \\ y'(\eta) \\ \vdots \\ y^{(n-1)}(\eta) \end{bmatrix} d\eta$$

$$= \begin{bmatrix} y'(\xi) - y'(\xi) \\ y''(\xi) - y''(\xi) \\ \vdots \\ y^{(n)}(\xi) - \sum_{k=0}^{n-1} a_k(\xi)y^{(k)}(\xi) - f(\xi) - \int_\gamma^\xi K(\xi,\eta)y(\eta)d\eta \end{bmatrix},$$

we get

$$\left| y^{(n)}(\xi) - \sum_{k=0}^{n-1} a_k(\xi)y^{(k)}(\xi) - f(\xi) - \int_\gamma^\xi K(\xi,\eta)y(\eta)d\eta \right| \le \varphi(\xi)$$

for all $\xi \in \Omega$. Hence, by Theorem 3.1, there exists a solution

$$\mathbf{x}_0(\xi) = \begin{bmatrix} x_1^{(0)}(\xi) \\ x_2^{(0)}(\xi) \\ \vdots \\ x_n^{(0)}(\xi) \end{bmatrix}$$

of equation (4.6) and $\mathbf{x}_0(\gamma) = \mathbf{x}(\gamma)$ such that

$$\|\mathbf{x}(\xi) - \mathbf{x}_0(\xi)\| \leq \frac{L}{M - M(L+N)}\varphi(\xi)$$

for all $\xi \in \Omega$. Therefore, there exists $y_0(\xi)$ satisfying equation (4.1) and $y_0(\gamma) = y(\gamma)$, $y_0'(\gamma) = y'(\gamma)$, ..., $y_0^{(n-1)}(\gamma) = y^{(n-1)}(\gamma)$ such that

$$|y(\xi) - y_0(\xi)| \leq \frac{L}{M - M(L+N)}\varphi(\xi)$$

for all $\xi \in \Omega$. □

Similarly, we show that equation (4.1) have the Ulam-Hyers-Rassias stability on the interval $\Omega = (\alpha, \beta]$, where $-\infty \leq \alpha < \beta < \infty$.

Theorem 4.2. *Let $a_0,\ a_1,\ \cdots,\ a_{n-1} : \Omega \to \mathbb{R}$, $f : \Omega \to \mathbb{R}$ and $K : \Omega \times \Omega \to \mathbb{R}$ be continuous functions and let M be a positive constant such that $|p_*(\xi)| \geq M$ for all $t \in \Omega$. Let L and N be positive constants with $0 < L + N < 1$ and $\gamma \in \Omega$. Suppose that φ is an integrable positive valued function on Ω such that*

$$\int_\gamma^\xi |p_*(\tau)|\, \varphi(\tau) d\tau \leq L\varphi(\xi), \quad \forall \xi \in \Omega$$

and

$$\int_\gamma^\xi \int_\gamma^\eta |K(\eta, \tau)|\, \varphi(\tau) d\tau d\eta \leq N\varphi(\xi), \quad \forall (\xi, \eta) \in \Omega \times \Omega$$

where $p_*(\xi) = \max\left\{1, \sum_{k=0}^{n-1} a_k(\xi)\right\}$. If a n-times continuously differentiable function $y : \Omega \to \mathbb{R}$ satisfies the inequality

$$\left| y^{(n)}(\xi) - \sum_{k=0}^{n-1} a_k(\xi) y^{(k)}(\xi) - f(\xi) - \int_\gamma^\xi K(\xi, \eta) y(\eta) d\eta \right| \leq \varphi(\xi)$$

for all $\xi \in \Omega$, then there exista a unique n-times continuously differentiable solution $y_0(\xi) : \Omega \to \mathbb{R}$ of the equation (4.1) such that

$$|y(\xi) - y_0(\xi)| \leq \frac{L}{M - M(L+N)}\varphi(\xi)$$

for all $\xi \in \Omega$ and $y_0(\gamma) = y(\gamma)$, $y_0'(\gamma) = y'(\gamma)$, ..., $y_0^{(n-1)}(\gamma) = y^{(n-1)}(\gamma)$.

Proof. It can be proved by following the way used in Theorem 3.1 on the interval Ω. □

Using Theorem 4.1 and Theorem 4.2, we will show that the equation (4.1) has the Ulam-Hyers-Rassias stability on $\mathbb{R}$ as follows.

Corollary 4.3. *Let $a_0,\ a_1,\ \cdots,\ a_{n-1} : \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ and $K : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous functions and let M be a constant such that $|p_*(\xi)| \geq M$ for all $t \in \mathbb{R}$. suppose L and N are constants with $0 < L+N < 1$. Suppose that φ is an integrable positive valued function on $\mathbb{R}$ such that*

$$\int_0^\xi |p_*(\tau)|\, \varphi(\tau) d\tau \leq L\varphi(\xi), \quad \forall \xi \in \mathbb{R}$$

and

$$\int_0^{\xi}\int_0^{\eta}|K(\eta,\tau)|\,\varphi(\tau)d\tau d\eta \leq N\varphi(\xi), \quad \forall(\xi,\eta)\in\mathbb{R}\times\mathbb{R}$$

here, $p_*(\xi) = \max\left\{1, \sum_{k=0}^{n-1} a_k(\xi)\right\}$. *If a function* $y:\mathbb{R}\to\mathbb{R}$ *is continuously differentiable* $n-$*times and satisfies the given inequality, then*

$$\left|y^{(n)}(\xi) - \sum_{k=0}^{n-1} a_k(\xi)y^{(k)}(\xi) - f(\xi) - \int_0^{\xi} K(\xi,\eta)y(\eta)d\eta\right| \leq \varphi(\xi)$$

for all $\xi\in\mathbb{R}$, *then there exista a unique n-times continuously differentiable solution* $y_0(\xi):\mathbb{R}\to\mathbb{R}$ *of the Eq. (4.1) such that*

$$|y(\xi)-y_0(\xi)| \leq \frac{L}{M-M(L+N)}\varphi(\xi)$$

for all $\xi\in\mathbb{R}$ *and* $y_0(0)=y(0)$, $y_0'(0)=y'(0)$, ..., $y_0^{(n-1)}(0)=y^{(n-1)}(0)$.

4.1. **Example.** Consider the following Volterra integrodifferential equation of second order with initial conditions:

$$\begin{aligned} y''(\xi) &= y'(\xi)+y(\xi)+2\xi+1+4\int_0^{\xi}(\xi-\eta)y(\eta)d\eta \\ y(0) &= 1,\ y'(0)=2. \end{aligned} \tag{4.7}$$

Equation (4.7) can be transformed into an equivalent system of the following form:

$$\begin{aligned} x_1'(\xi) &= x_2(\xi) \\ x_2'(\xi) &= x_1(\xi)+x_2(\xi)+2\xi+1+4\int_0^{\xi}(\xi-\eta)x_1(\eta)d\eta \end{aligned}$$

Using the vector-matrix notation, we get following system of Volterra integrodifferential equations

$$\mathbf{x}'(\xi) = \mathbf{p}(\xi)\mathbf{x}(\xi)+\mathbf{q}(\xi)+\int_0^{\xi}\mathbf{K}(\xi,\eta)\mathbf{x}(\eta)d\eta,$$

where

$$\mathbf{x}(\xi)=\begin{bmatrix}x_1(\xi)\\x_2(\xi)\end{bmatrix},\ \mathbf{P}(\xi)=\begin{bmatrix}0&1\\1&1\end{bmatrix},\ \mathbf{q}(\xi)=\begin{bmatrix}0\\2\xi+1\end{bmatrix},\ \mathbf{K}(\xi,\eta)=\begin{bmatrix}0&0\\\xi-\eta&0\end{bmatrix}.$$

By hypotheses, $p_*(\xi)=\max\{1,2\}=2$. If we define the integrable function $\varphi:\mathbb{R}\to[0,\infty)$ with $\varphi(\xi)=e^{5\xi}$, then we obtain $L=2/5$ with

$$\int_0^{\xi}2e^{5s}ds = \frac{2}{5}e^{5\xi}-\frac{2}{5}\leq\frac{2}{5}e^{5\xi}, \quad \xi\in[0,1],$$

and $N=1/125$ with

$$\int_0^{\xi}\int_0^{\eta}(\eta-r)e^{5r}drd\eta \leq \frac{1}{125}e^{5\xi}, \quad \xi\in[0,1].$$

From here we see that $L+N=\frac{2}{5}+\frac{1}{125}=\frac{51}{125}\in(0,1)$. If we choose $y(\xi)=\frac{10}{9}e^{2\xi}$, it follows

$$\left|y''(\xi)-y'(\xi)-y(\xi)-2\xi-1-4\int_0^{\xi}(\xi-\eta)y(\eta)d\eta\right|=\frac{1}{9}e^{2\xi}\leq\varphi(\xi):=e^{5\xi},\quad \xi\in[0,1].$$

Therefore, according to Theorem 4.1, we can conclude that the second order Volterra integrodifferential equation (4.7) exhibits Ulam-Hyers-Rassias stability.

The exact solution of equation (4.7) is $y_0(\xi)=e^{2\xi}$. If we take $M=2$, we also see the fact that

$$|y(\xi)-y_0(\xi)|=\left|\frac{10}{9}e^{2\xi}-e^{2\xi}\right|=\frac{1}{9}e^{2\xi}\leq\frac{25}{74}e^{5\xi}=\frac{L}{M-M(L+N)}\varphi(\xi),\quad \xi\in[0,1].$$

5. Conclusions

Our study offered a fresh perspective on the stability of Volterra integrodifferential equations. We proved the Ulam-Hyers-Rassias stability of a system of Volterra integrodifferential equations, under particular conditions on bounded or unbounded intervals, using a fixed-point theorem in a generalized complete metric space. In particular, in cases where the function is continuously differentiable, Ulam-Hyers-Rassias stability is crucial because it enables the identification of an exact solution to the equation that is near to an approximation solution to the integrodifferential equation. In other words, the difference between the perturbed solution and the exact solution, or the distance between the set of all solutions of the integrodifferential equation and the approximation solution, is insignificant. The Hyer-Ulam-Rassias stability notions also serve as a reminder that we don't necessarily need to find exact solutions in an Ulam-Hyers-Rassias stable system; rather, we simply need to find a function that satisfies the essential approximation inequality. In other words, Ulam-Hyers-Rassias stability guarantees the existence of a closed precise solution.

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References

[1] Ulam, S.M. Problems in Modern Mathematics, Science ed.; Wiley, New York, NY, USA, 1940.

[2] Hyers, D.H. On the stability of the linear functional equation. *Proc. Nat. Acad. Sci. USA* **1941**, 27, 222-224.

[3] Baker, J. A. The stability of certain functional equations. *Proceedings of the American Mathematical Society* **1991**, 729-732.

[4] Radu, V. The fixed point alternative and the stability of functional equations. *Fixed Point Theory* **2003**, 4, 91-96.

[5] Cadariu, L.; Radu, V. Fixed points and the stability of Jensen's functional equation. *J. Inequal. Pure Appl. Math.* **2003**, 4, 4.

[6] Cadariu, L.; Radu, V. On the stability of the Cauchy functional equation: a fixed point approach. *Grazer Mathematische Berichte* **2004** 346, 43-52.
[7] Jung, S.M. A fixed point approach to the stability of a Volterra integral equation. *Fixed Point Theory and Applications*, **2007** 2007, 9.
[8] Jung, S.M.; Şevgin, S.; Şevli, H. On the perturbation of Volterra integro-differential equations. *Applied Mathematics Letters* **2013** 2013, 665-669.
[9] Diaz, J.B.; Margolis, B. A fixed point theorem of the alternative, for contractions on a generalized complete metric space. *Bulletin of the AmericanMathematical Society* **1968** 74, 305-309.
[10] Castro, L.P.;Ramos, A. Hyers-Ulam-Rassias stability for a class of nonlinear Volterra integral equations, *Banach J. Math. Anal.* **2009** 3, 36-43.
[11] Castro, L.P.;Ramos, A. Hyers-Ulam and Hyers-Ulam-Rassias stability of Volterra integral equations with delay. In *Integral methods in science and engineering*; Birkhauser Boston, Inc.; Boston, MA, USA, 2010; pp. 85-94.
[12] Akkouchi, M., Hyers-Ulam-Rassias stability of nonlinear Volterra integral equations via a fixed point approach. *Acta Universitatis Apulensis* **2011** 26, 257-266.
[13] Morales, J.R.; Rojas, E.M. Hyers-Ulam and Hyers-Ulam-Rassias stability of nonlinear integral equations with delay. *International Journal of Nonlinear Analysis and Applications*, **2011** 2, 1-6.
[14] Alqifiary, Q. H. Note on the stability for linear systems of differential equations. *International Journal of Applied Mathematical Research* **2013** 3, 15-22.
[15] Şevgin, S.; Şevli, H. Stability of a nonlinear Volterra integro-differential equation via a fixed point approach. *Journal of Nonlinear Sciences and Applications* **2016** 9, 200-207.
[16] Castro, L. P.; Simões, A. M. Hyers-Ulam and Hyers-Ulam-Rassias stability for a class of integro-differential equations. In *Mathematical Methods in Engineering*; Springer: Cham, Switzerland. 2019; pp. 81-94.
[17] Castro, L. P.; Simoes, A. M. Different types of Hyers-Ulam-Rassias stabilities for a class of integro-differential equations. *Filomat* **2017** 31, 5379-5390.
[18] Kucche, K. D.; Shikhare, P. U. Ulam Stabilities for Nonlinear Volterra Delay Integro-differential Equations. *Journal of Contemporary Mathematical Analysis* **2019** 54, 276-287.
[19] Kucche, K. D.; Shikhare, P. U. Ulam-Hyers Stability of Integrodifferential Equations in Banach Spaces via Pachpatte's Inequality. *Asian-European Journal of Mathematics*, **2018** 11, 1850062.
[20] Sousa, J. V. D. C.; De Oliveira, E. C. Ulam–Hyers stability of a nonlinear fractional Volterra integro-differential equation. *Applied Mathematics Letters* **2018** 81, 50-56.

Sebaheddin Sevgin
Yuzuncu Yil University, Faculty of Sciences, Department of Mathematics, 65080 Van, Turkey
E-mail address: ssevgin@yyu.edu.tr

Jargees Abdulwahid Abdullah
University of Duhok, College of Basic Education, Department of Mathematics, 42001 Duhok, Iraq
E-mail address: Jargees76@gmail.com

Sadeq Taha Abdulazeez
University of Duhok, College of Basic Education, Department of Mathematics, 42001 Duhok, Iraq
E-mail address: sadiq.taha@uod.ac