Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 15 Issue 1 (2024), Pages 13-27
https://doi.org/10.54379/jma-2024-1-2.

ON A STUDY OF SUBNORMAL COMPLETION PROBLEM VIA REAL CUBIC MOMENT PROBLEM

ABDALLAH TAIA, ABDELAZIZ EL BOUKILI, AMAR RHAZI, BOUAZZA EL WAHBI

Abstract. In this article, we study the bivariate subnormal completion problem for a collection of bivariate weight cubic data. We provide some techniques for solving this problem. The results obtained have been constantly illustrated by examples.

1. Introduction

The problem of moments has developed significantly since Stieltjes'study [1], with numerous application in a wide range of domains. In particular, the K- truncated moment problem, where $K \subseteq \mathbb{R}_+^2$, plays an important role in subnormal completion for weighted bivariate shifts by studying their subnormal and hyponormal properties, in the way that a solution to the first produces a solution to the second (see for example [5, 6, 7, 11, 18]).

Given a finite collection $\mathcal{C}$ of pairs of positive numbers called weights, the bivariate subnormal completion problem consists in finding necessary and sufficient conditions for the existence of a bivariate subnormal weighted shift whose initial weights are given by $\mathcal{C}$.

Let us now recall some notions of bounded operators and some properties of weighted shifts to weights which will be useful for solving the problems of subnormal completions, especially with two variables. For more details on subnormality and hypnormality, one can consult [2, 3, 4, 9, 10, 12, 16, 17, 19, 20, 21] for instance.

Let $\mathcal{H}$ be a complex separable Hilbert space of infinite dimension, and let $\mathcal{B}(\mathcal{H})$ denotes the algebra of bounded linear operators on $\mathcal{H}$. Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is normal if it commutes with its adjoint T^*, i.e. $T^*T = TT^*$, subnormal if it has a normal extension and T is said to be hyponormal if $[T^*, T] := T^*T - TT^* \geq 0$.

1991 *Mathematics Subject Classification.* Primary:47B20, 47B37, Secondary:44A60.
Key words and phrases. bivariate subnormal completion problem, bivariate weighted shift, cubic moment problem.

Submitted August 8, 2023. Published January 12, 2024.
Communicated by N. Braha.

In [4], the author stated a criterion of subnormality (Halmos' criterion [4, II.1.9]), as follows

$$T \text{ is subnormal } \iff \sum_{i,j=0}^{k} \left\langle T^i x_j, T^j x_i \right\rangle \geq 0.$$

for all $k \geq$ and any finite collection $x_0, x_1, \cdots, x_k$ of elements of $\mathcal{H}$. Or equivalently,

$$\begin{pmatrix} I & T^* & \cdots & T^{*k} \\ T & T^*T & \cdots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \cdots & T^{*k}T^k \end{pmatrix} \succeq 0. \tag{1.1}$$

Clearly, hyponormality is a necessary condition for subnormality.
Given an n-tuples $\mathbf{T} = (T_1, \ldots, T_n)$ of operators on $\mathcal{H}$ with $n \geq 2$, we denote by $[\mathbf{T}^*, \mathbf{T}]$ the self-commutator of $\mathbf{T}$, defined by $[\mathbf{T}^*, \mathbf{T}]_{ij} := \left[T_j^*, T_i\right]_{1 \leq i,j \leq n}$.
For example, if $n = 2$,

$$[\mathbf{T}^*, \mathbf{T}] = \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] \end{pmatrix}.$$

We say that $\mathbf{T} = (T_1, \ldots, T_n)$ is normal if $\mathbf{T}$ is commuting and every T_i is normal, and $\mathbf{T}$ is subnormal if $\mathbf{T}$ is the restriction of a normal n-tuples to a common invariant subspace.
$\mathbf{T} = (T_1, \ldots, T_n)$ is called jointly hyponormal if $[\mathbf{T}^*, \mathbf{T}] \geq 0$, i.e. $\langle [\mathbf{T}^*, \mathbf{T}]\, x, x \rangle \geq 0$ for all $x \in \mathcal{H}^n$.
And an operator $T \in \mathcal{H}$ is called l-hyponormal ($l \geq 1$) if $\left(1, T, T^2, \cdots, T^l\right)$ is jointly hyponormal, that is $M_l(T) \equiv \left(\left[T^{*j}, T^i\right]\right)_{i,j=1}^{l} \geq 0$.

By Definition 2.2 in [9], a commuting pair $\mathbf{T} = (T_1, T_2)$ of operators on $\mathcal{H}$ is called l-hyponormal, if

$$\mathbf{T}(l) := \left(T_1, T_2, T_1^2, T_2T_1, T_2^2, \cdots, T_1^l, T_2T_1^{l-1}, T_2^2T_1^{l-2}, \cdots, T_2^l\right)$$

is hyponormal, or equivalently

$$0 \leq [\mathbf{T}(l)^*, \mathbf{T}(l)] = \left(\left[(T_2^qT_1^p)^*, T_2^mT_1^n\right]\right)_{\substack{0 \leq n+m \leq l \\ 0 \leq p+q \leq l}}.$$

$\ell^2(\mathbb{Z}_+^2)$ denotes the Hilbert space of square summable complex sequences indexed by $\mathbb{Z}_+^2$. Let $\{e_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}_+^2}$ be the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+^2)$.
For a pair of positive real number sequences, called weights $(\alpha_{\mathbf{k}}, \beta_{\mathbf{k}}) \in \ell^\infty(\mathbb{Z}_+^2)$, $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$, we define the bivarite weighted shift $\mathbf{T} \equiv (T_1, T_2)$ associated with $(\alpha_{\mathbf{k}}, \beta_{\mathbf{k}})$ by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1} \quad \text{ and } \quad T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where $\varepsilon_1 := (1, 0)$ and $\varepsilon_2 := (0, 1)$.

One can simply check that

$$T_1T_2 = T_2T_1 \quad \Leftrightarrow \quad \beta_{\mathbf{k}+\varepsilon_1}\alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}} \quad \left(\text{ for all } \mathbf{k} \in \mathbb{Z}_+^2\right). \tag{1.2}$$

The relation (1.2) translates the commutativity condition of $\mathbf{T}$.

By Lemma 1.1 in [5], an operator $\mathbf{T}$ is hyponormal if and only if, for all $\mathbf{k} \in \mathbb{Z}_+^2$, the following three conditions are satisfied.

(i) $\alpha_{\mathbf{k}+\varepsilon_1} \geqslant \alpha_{\mathbf{k}}$,
(ii) $\beta_{\mathbf{k}+\varepsilon_2} \geqslant \beta_{\mathbf{k}}$,
(iii) $\left((\alpha_{\mathbf{k}+\varepsilon_1})^2 - (\alpha_{\mathbf{k}})^2\right)\left((\beta_{\mathbf{k}+\varepsilon_2})^2 - (\beta_{\mathbf{k}})^2\right) \geqslant \left(\alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}}\right)^2$.

For $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$, we define the moment of order $\mathbf{k}$ of $(\alpha_{\mathbf{k}}, \beta_{\mathbf{k}})$ by

$$\gamma_{\mathbf{k}} \equiv \gamma_{(k_1,k_2)} := \begin{cases} 1 & \text{if } (k_1,k_2) = (0,0) \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 \cdot \beta_{(k_1,0)}^2 \cdots \beta_{(k_1,k_2-1)}^2 & \text{otherwise.} \end{cases} \tag{1.3}$$

By virtue of to the commutativity condition (1.2), $\gamma_{\mathbf{k}}$ can be calculated using any non-decreasing path from $(0,0)$ to (k_1, k_2).

According to Berger's theorem, [19, Theorem 3], a bivariate weighted shifts $\mathbf{T} \equiv (T_1, T_2)$ is subnormal if and only if, there exists a probability measure μ defined on the rectangle $R = [0, \| T_1 \|^2] \times [0, \| T_2 \|^2]$ such that

$$\gamma_{\mathbf{k}} = \int_R t_1^{k_1} t_2^{k_2} d\mu(t_1, t_2), \qquad \text{for all } \ \mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2.$$

In [9, Theorem 2.4], the l-hyponormality for bivariate weighted shifts is characterized as follows. $\mathbf{T} = (T_1, T_2)$ is subnormal if and only if

$$0 \leq \mathcal{M}_{\mathbf{k}}(l) := \left(\gamma_{\mathbf{k}+(m,n)+(p,q)}\right)_{\substack{0 \leq m+n \leq l \\ 0 \leq p+q \leq l}} \text{ for all } \mathbf{k} \in \mathbb{Z}_+^2. \tag{1.4}$$

Clearly, the matrix $\mathcal{M}_{\mathbf{k}}(l)$ is a truncation of the moment matrix associated with the Berger measure of $\mathbf{T}$.

The general statement of the problem of subnormal completion can be formulated as follows. Given a finite collection $\mathcal{C}_m := \{(\alpha_{\mathbf{k}}, \beta_{\mathbf{k}})\}_{|\mathbf{k}|=k_1+k_2 \leqslant m}$ of pairs of positive numbers satisfying (1.2) with $|\mathbf{k} + \varepsilon_i| \leqslant m$ $(i = 1, 2)$, find necessary and sufficient conditions for the existence of a bivariate subnormal weighted shift whose initial weights are the elements of $\mathcal{C}_m$.

In this paper, we investigate the case $m = 2$. In Section 2, we recall some tools that will be needed for solving the problem of subnormal completion. Section 3 is devoted to the statement of our main results related to the bivariate subnormal completion problem with cubic data, i.e. $m = 2$. Somme numerical examples, performed by Mathematica software, are also provided to illustrate some statements pointed out through this paper.

2. Needed Tools

Let $m \in \mathbb{Z}_+$ and a collection of pairs of positive numbers $\mathcal{C}_m := \{(\alpha_{\mathbf{k}}, \beta_{\mathbf{k}})\}$ where $|\mathbf{k}| \leq m$ and $|\mathbf{k}| = k_1 + k_2$. By Definition 3.1 in [11], we say that a weighted bivariate shift $\mathbf{T} \equiv (T_1, T_2)$ with weight sequences $\{\tilde{\alpha}_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}_+^2}$ and $\{\tilde{\beta}_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}_+^2}$ is a subnormal completion of $\mathcal{C}_m$ if,

(i) $\mathbf{T}$ is subnormal;
(ii) $\left(\tilde{\alpha}_{\mathbf{k}}, \tilde{\beta}_{\mathbf{k}}\right) = (\alpha_{\mathbf{k}}, \beta_{\mathbf{k}})$ for $|\mathbf{k}| \leq m$.

We denote this subnormal completion by $\mathcal{C}_\infty \equiv \left\{(\tilde{\alpha}_{\mathbf{k}}, \tilde{\beta}_{\mathbf{k}})\right\}_{\mathbf{k} \in \mathbb{Z}_+^2}$.

Definition 3.3 in [11] states that $\tilde{\mathcal{C}}_{m+1} \equiv \left\{\left(\tilde{\alpha}_{\mathbf{k}}, \tilde{\beta}_{\mathbf{k}}\right)\right\}_{|\mathbf{k}| \leq m+1}$ is an extension of $\mathcal{C}_m$ if $\left(\tilde{\alpha}_{\mathbf{k}}, \tilde{\beta}_{\mathbf{k}}\right) = (\alpha_{\mathbf{k}}, \beta_{\mathbf{k}})$ when $|\mathbf{k}| \leq m$.

If $m = 2l$ where $l \in \mathbb{Z}_+^*$, the data of the sequence $\gamma \equiv \gamma^{(m+1)} = \{\gamma_{ij}\}_{0\le i+j\le m+1}$ associated to $\mathcal{C}_m$ by the relation (1.3) will be in the form of matrices $\mathcal{M}(l) \equiv \mathcal{M}_{\mathbf{0}}(l) \equiv \mathcal{M}(\mathcal{C}_m)$ and $B(l+1)$ as follows

$$\mathcal{M}(l) = \begin{pmatrix} M[0,0] & M[0,1] & \dots & M[0,l] \\ M[1,0] & M[1,1] & \dots & M[1,l] \\ \vdots & \vdots & \ddots & \vdots \\ M[l,0] & M[l,1] & \dots & M[l,l] \end{pmatrix} \quad \text{and} \quad B(l+1) = \begin{pmatrix} M[0,l+1] \\ M[1,l+1] \\ \vdots \\ M[l,l+1] \end{pmatrix}. \tag{2.1}$$

Where $\mathcal{M}(l) = (M[i,j])_{0\le i,j\le l}$ is a symmetric matrix of blocks and that each block

$$M[i,j] = \begin{pmatrix} \gamma_{i+j,0} & \gamma_{i+j-1,1} & \cdots & \gamma_{i,j} \\ \gamma_{i+j-1,1} & \gamma_{i+j-2,2} & \cdots & \gamma_{i-1,j+1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{j,i} & \gamma_{j-1,i+1} & \cdots & \gamma_{0,i+j} \end{pmatrix}, \quad 0 \le i,j \le l,$$

has Hankel's property.
For instance, for $m = 2$, the two matrices $\mathcal{M}(1)$ and $B(2)$ are as given by

$$\mathcal{M}(1) = \left(\begin{array}{c|cc} \gamma_{00} & \gamma_{10} & \gamma_{01} \\ \hline \gamma_{10} & \gamma_{20} & \gamma_{11} \\ \gamma_{01} & \gamma_{11} & \gamma_{02} \end{array}\right) \quad \text{and} \quad B(2) = \left(\begin{array}{ccc} \gamma_{20} & \gamma_{11} & \gamma_{02} \\ \hline \gamma_{30} & \gamma_{21} & \gamma_{12} \\ \gamma_{21} & \gamma_{12} & \gamma_{03} \end{array}\right). \tag{2.2}$$

A necessary condition for the existence of a representing measure for γ is that $\mathcal{M}(1)$ is positive semidefinite ($\mathcal{M}(1) \succeq 0$). In this case, we seek to construct a matrix $\mathcal{M}(2)$, an extension of $\mathcal{M}(1)$ which should also be positive semidefinite of the form

$$\mathcal{M}(2) = \begin{pmatrix} \mathcal{M}(1) & B(2) \\ B(2)^T & C(2) \end{pmatrix},$$

where $C(2)$ is a (3×3)-Hankel matrix containing quartic moments (of order 4) that we need to determine. We set,

$$C(2) = \begin{pmatrix} \gamma_{40} & \gamma_{31} & \gamma_{22} \\ \gamma_{31} & \gamma_{22} & \gamma_{13} \\ \gamma_{22} & \gamma_{13} & \gamma_{04} \end{pmatrix}. \tag{2.3}$$

With labeling the columns and rows of $\mathcal{M}(2)$ considering the lexicographic order of the monomials in degree, 1, X, Y, X^2, XY, Y^2, the matrix $\mathcal{M}(2)$ is written as follows

$$\mathcal{M}(2) = \begin{array}{c} \\ 1 \\ \\ X \\ Y \\ \\ X^2 \\ XY \\ Y^2 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & X & Y & X^2 & XY & Y^2 \end{array} \\ \left(\begin{array}{c|cc|ccc} \gamma_{00} & \gamma_{10} & \gamma_{01} & \gamma_{20} & \gamma_{11} & \gamma_{02} \\ \hline \gamma_{10} & \gamma_{20} & \gamma_{11} & \gamma_{30} & \gamma_{21} & \gamma_{12} \\ \gamma_{01} & \gamma_{11} & \gamma_{02} & \gamma_{21} & \gamma_{12} & \gamma_{03} \\ \hline \gamma_{20} & \gamma_{30} & \gamma_{21} & \gamma_{40} & \gamma_{31} & \gamma_{22} \\ \gamma_{11} & \gamma_{21} & \gamma_{12} & \gamma_{31} & \gamma_{22} & \gamma_{13} \\ \gamma_{02} & \gamma_{12} & \gamma_{03} & \gamma_{22} & \gamma_{13} & \gamma_{04} \end{array}\right) \end{array}.$$

If $\operatorname{rank}\mathcal{M}(2) = \operatorname{rank}\mathcal{M}(1)$, we say that $\mathcal{M}(2)$ is a flat extension of $\mathcal{M}(1)$. To test the semidefinite positivity of $\mathcal{M}(2)$ as well as its flatness, we need the following Smul'jan's lemma [22].

Lemma 2.1. *Let A be a symmetric matrix. If the block matrix* $\tilde{A} := \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ *is an extension of A, then*

$$\tilde{A} \succeq 0 \Longleftrightarrow \begin{cases} (i) \quad A \succeq 0. \\ (ii) \quad B = AW \quad \text{for some matrix } W. \\ (iii) \quad C \succeq W^T AW. \end{cases}$$

Moreover, $\tilde{A}$ is a flat extension of A, if only if $C = W^T AW$.

According to Douglas's factorization lemma [13], the condition (ii) in Lemma 2.1 is equivalent to $\operatorname{Ran} B \subseteq \operatorname{Ran} A$. Moreover, if (ii) is satisfied and since A is symmetric, $W^T AW$ is also symmetric and does not depend on W.
So, if $\mathcal{M}(1) \succeq 0$ and $\operatorname{Ran} B(2) \subseteq \operatorname{Ran}\mathcal{M}(1)$, then $W^T\mathcal{M}(1)W$ takes the following form

$$W^T\mathcal{M}(1)W = \begin{pmatrix} x & u & v \\ u & y & w \\ v & w & z \end{pmatrix} \tag{2.4}$$

where u, v, w, x, y and z are real numbers.
The relation between y and v allows us to determine $C(2)$, as in (2.3), such that $C(2) - W^T\mathcal{M}(1)W \succeq 0$. So, $\mathcal{M}(2)$ the extension of $\mathcal{M}(1)$ is positive semidefinite.

The following theorem [14, Theorem 2.3] states a necessary and sufficient condition for the existence of a finite atomic measure representing a finite sequence $\gamma = \gamma^{(2l)}$ where $l \in \mathbb{Z}_+^*$.

Theorem 2.2. *The truncated sequence of moments $\gamma^{(2l)}$ admits a finite representing measure* $\operatorname{rank}\mathcal{M}(l)$*-atomic, if and only if $\mathcal{M}(l) \succeq 0$ and admits a flat extension $\mathcal{M}(l+1)$.*

The bivariate subnormal completion, is closely related to the K-truncated moment problem with $K \subseteq [0, \infty)^2$, i.e. when the Berger measure μ exists, it must verify $\operatorname{supp}\mu \subseteq K$.

In [8], the K-complex truncated moment problem is studied using localization matrices. Its equivalent version for two real variables reads as follows.

Theorem 2.3. ([11, Theorem 4.1]) *Let $\mathcal{P} \equiv \{p_1, \dots, p_N\} \subseteq \mathbb{R}[x, y]$ such that $\deg p_i = 2k_i$ or $\deg p_i = 2k_i - 1$ $(1 \le i \le N)$.*
There is a representing measure $\operatorname{rank}\mathcal{M}(n)$*-atomic for $\gamma = \gamma^{(2n)}$ supported in $K_{\mathcal{P}} := \big\{(x, y) \in \mathbb{R}^2 : p_i(x, y) \ge 0, 1 \le i \le N\big\}$ if and only if $\mathcal{M}(n) \succeq 0$ and there exists a certain flat extension $\mathcal{M}(n+1)$ for which the localization matrices $\mathcal{M}_{p_i}(n+k_i) \succeq 0$ $(1 \le i \le N)$. In this case, the representing measure is* $\operatorname{rank}\mathcal{M}(n)$*-atomic, supported in $K_{\mathcal{P}}$, and with precisely* $\operatorname{rank}\mathcal{M}(n) - \operatorname{rank}\mathcal{M}_{p_i}(n + k_i)$ *atoms in $\mathcal{Z}(p_i) := \big\{(x, y) \in \mathbb{R}^2 : p_i(x, y) = 0\big\}$.*

Let us put $p_1 := x$ and $p_2 := y$ then $k_1 = k_2 = 1$, $K_{\mathcal{P}} = \mathbb{R}_+^2$, $\mathcal{M}_{p_1}(n + k_1) = \mathcal{M}_x(n+1)$ and $\mathcal{M}_{p_2}(n + k_1) = \mathcal{M}_y(n + 1)$. By Theorem 2.3 and for an even m, we deduce from [11, Theorem 4.3] the following useful result.

Theorem 2.4. *For a collection $\mathcal{C}_m$ with $m = 2l$, and let $\mathcal{M}(l)$ and $B(l+1)$ be as in* (2.1)*. The following statements are equivalent*

(i) *$\mathcal{C}_m$ has a subnormal completion $\mathcal{C}_\infty$.*
(ii) *There is a representating measure* rank $\mathcal{M}(l)$*-atomic μ for β supported in $\mathbb{R}^2_+$.*
(iii) *$\mathcal{M}(l) = \mathcal{M}(\mathcal{C}_m) \succeq 0$ and $\mathcal{C}_m$ admits an extension $\tilde{\mathcal{C}}_{m+2}$ verifying the commutativity condition* (1.4) *such that the matrix of moments $\mathcal{M}(\tilde{\mathcal{C}}_{m+2}) = \mathcal{M}(l+1)$ is a flat extension of $\mathcal{M}(l)$, $\mathcal{M}_x(l+1) \succeq 0$ and $\mathcal{M}_y(l+1) \succeq 0$.*

Moreover, Berger measure μ of $\mathcal{C}_\infty$ has rank $\mathcal{M}(l) -$ rank $\mathcal{M}_x(l+1)$ atoms in $\{0\} \times \mathbb{R}_+$ (resp. rank $\mathcal{M}(l) -$ rank $\mathcal{M}_y(l+1)$ atoms in $\mathbb{R}_+ \times \{0\}$).

With the notation used in (1.4), we denote the localizing matrices by

$$\mathcal{M}_x(l+1) = \mathcal{M}_{(1,0)}(l+1) \text{ and } \mathcal{M}_y(l+1) = \mathcal{M}_{(0,1)}(l+1).$$

3. Bivariate subnormal completion with cubic data

In this section, we give a solution to bivariate subnormal completion problem with cubic data, formulated as follows.

(PR): Let $\mathcal{C}_2$ be as defined previously and let $\gamma = \{\gamma_{ij}\}_{0 \le i+j \le 3}$ be the associated sequence given by (1.3). Is there a subnormal completion of $\mathcal{C}_2$?

Let us consider the finite collection of pairs of positive real numbers $\mathcal{C}_2$, by setting

$$\begin{array}{llll} \alpha^2_{(0,0)} = a, & \beta^2_{(0,0)} = b, & \alpha^2_{(1,0)} = c, & \beta^2_{(1,0)} = d \\ \alpha^2_{(0,1)} = e, & \beta^2_{(0,1)} = f, & \alpha^2_{(2,0)} = g, & \beta^2_{(2,0)} = h \\ \alpha^2_{(1,1)} = p, & \beta^2_{(1,1)} = q, & \alpha^2_{(0,2)} = r, & \beta^2_{(0,2)} = s, \end{array}$$

and employing the commutativity condition (1.2), we get

$$ad = be, \quad ch = dp \quad \text{and} \quad eq = fr.$$

According to (1.3), the elements of the sequence $\gamma \equiv \{\gamma_{ij}\}_{0 \le i+j \le 3}$ are given by

$$\begin{array}{llll} \gamma_{00} = 1, & & & \\ \gamma_{10} = a, & \gamma_{01} = b, & & \\ \gamma_{20} = ac, & \gamma_{11} = ad, & \gamma_{02} = bf, & \\ \gamma_{30} = acg, & \gamma_{21} = bep, & \gamma_{12} = beq, & \gamma_{03} = bfs. \end{array}$$

The two matrices associated to the sequence γ are

$$\mathcal{M}(1) = \begin{array}{c} \begin{array}{ccc} 1 & X & Y \end{array} \\ \begin{pmatrix} 1 & a & b \\ a & ac & be \\ b & be & bf \end{pmatrix} \end{array} \quad \text{and} \quad B(2) = \begin{pmatrix} ac & be & bf \\ acg & bep & beq \\ bep & beq & bfs \end{pmatrix}. \tag{3.1}$$

The condition of $\mathcal{M}(1)$ being positive semidefinite is a necessary for the existence of a representing measure μ of γ. In this case, we have

$$c \ge a, \quad f \ge b \quad \text{and} \quad cf - de \ge 0. \tag{3.2}$$

If $\mathcal{M}(2)$ is a flat extension of $\mathcal{M}(1)$ then, the localization matrices $\mathcal{M}_{(1,0)}(2)$ and $\mathcal{M}_{(0,1)}(2)$ are the restrictions of $\mathcal{M}(2)$ to the first three rows and columns indexed by the monomials X, X^2 and XY and Y, XY and Y^2, respectively. We have

$$\mathcal{M}_{(1,0)}(2) = \begin{array}{c} \begin{array}{ccc} X & X^2 & XY \end{array} \\ \begin{pmatrix} a & ac & be \\ ac & acg & bep \\ be & bep & beq \end{pmatrix} \end{array} \quad \text{and} \quad \mathcal{M}_{(0,1)}(2) = \begin{array}{c} \begin{array}{ccc} Y & XY & Y^2 \end{array} \\ \begin{pmatrix} b & be & bf \\ be & bep & beq \\ bf & beq & bfs \end{pmatrix} \end{array}. \tag{3.3}$$

Theorem 3.1. *Let $\mathcal{M}(1)$ and $B(2)$ be as defined in* (3.1). *If $\mathcal{M}(1) \succeq 0$ with $\operatorname{rank}\mathcal{M}(1) = 1$ and $\operatorname{Ran}B(2) \subseteq \operatorname{Ran}\mathcal{M}(1)$, then $\mathcal{M}(1)$ admits a flat extension $\mathcal{M}(2)$, $\mathcal{M}_{(1,0)}(2) \succeq 0$ and $\mathcal{M}_{(0,1)}(2) \succeq 0$. Consequently, $\mathcal{C}_2$ admits a subnormal completion $\mathcal{C}_\infty$.*

Proof. Since $\mathcal{M}(1) \succeq 0$, $\operatorname{rank}\mathcal{M}(1) = 1$ and $\operatorname{Ran}B(2) \subseteq \operatorname{Ran}\mathcal{M}(1)$ then,

$$a = c = e = g = p \text{ and } b = f = q = s.$$

Moreover, by applying Theorem 3.5 in [15], γ admits a finite unique representing measure $\operatorname{rank}\mathcal{M}(1)$-atomic $\mu = \gamma_{00}\delta_{(\gamma_{10},\gamma_{01})} = \delta_{(a,b)}$.
Therefore, $\mathcal{M}(1)$ admits a flat extension $\mathcal{M}(2)$ given by

$$\mathcal{M}(2) = \begin{pmatrix} 1 & a & b & a^2 & ab & b^2 \\ a & a^2 & ab & a^3 & a^2b & ab^2 \\ b & ab & b^2 & a^2b & ab^2 & b^3 \\ a^2 & a^3 & a^2b & a^4 & a^3b & a^2b^2 \\ ab & a^2b & ab^2 & a^3b & a^2b^2 & ab^3 \\ b^2 & ab^2 & b^3 & a^2b^2 & ab^3 & b^4 \end{pmatrix}.$$

Since $\operatorname{supp}\mu \subseteq \mathbb{R}_+^2$, then according to Theorem 2.4, $\mathcal{C}_2$ admits an extension $\tilde{\mathcal{C}_4}$. Based on the quartic moments (the entries of the $C(2)$ matrix) and taking into account the commutativity condition (1.2), we can

$$\tilde{\alpha}^2_{(3,0)} = \tilde{\alpha}^2_{(0,3)} = \tilde{\alpha}^2_{(2,1)} = \tilde{\alpha}^2_{(1,2)} = a \text{ and } \tilde{\beta}^2_{(3,0)} = \tilde{\beta}^2_{(0,3)} = \tilde{\beta}^2_{(2,1)} = \tilde{\beta}^2_{(1,2)} = b.$$

Calculations show that $\mathcal{M}_{(1,0)}(2) \succeq 0$, $\mathcal{M}_{(0,1)}(2) \succeq 0$, $\operatorname{rank}\mathcal{M}_{(1,0)}(2) = 1$ and $\operatorname{rank}\mathcal{M}_{(0,1)}(2) = 1$.
So, using Theorem 2.4 again, $\mathcal{C}_2$ admits a subnormal completion $\mathcal{C}_\infty$ and Berger's measure is $\operatorname{rank}M(1)$-atomic given by $\mu = \delta_{(a,b)}$. □

Theorem 3.2. *Let $\mathcal{M}(1)$ and $B(2)$ be as defined in* (3.1). *If $\mathcal{M}(1) \succeq 0$ with $\operatorname{rank}\mathcal{M}(1) = 2$ and $\operatorname{Ran}B(2) \subseteq \operatorname{Ran}\mathcal{M}(1)$ then, $\mathcal{M}(1)$ admits a flat extension $\mathcal{M}(2)$. In addition, if $\mathcal{M}_{(1,0)}(2) \succeq 0$ and $\mathcal{M}_{(0,1)}(2) \succeq 0$, $\mathcal{C}_2$ has a subnormal completion $\mathcal{C}_\infty$.*

Proof. Let $\mathcal{M}(1)$ be as defined in (3.1), and $\operatorname{rank}\mathcal{M}(1) = 2$. The linear dependency relations between these columns must be as follows

$$\begin{cases} X = a.1 \quad \text{with } f > b > 0. \\ \text{or} \\ Y = \frac{b(c-e)}{c-a}.1 + \frac{b(e-a)}{a(c-a)}.X \text{ with } c > a > 0 \text{ and } f = \frac{ba(c-2e)+be^2}{a(c-a)} \geq b. \end{cases}$$

- Case 1: $X = a.1$ and $f > b > 0$.

Since $\mathcal{M}(1) \succeq 0$, $\operatorname{Ran}B(2) \subseteq \operatorname{Ran}\mathcal{M}(1)$ and $\operatorname{rank}\mathcal{M}(1)$, we must have

$$a = c = g = e = p \text{ and } q = s = f \text{ with } f > b.$$

Thus, by applying Theorem 3.5 in [15], γ admits a finite unique representing measure rank $\mathcal{M}(1)$-atomic $\mu = \left(1-\frac{b}{f}\right)\delta_{(a,0)} + \frac{b}{f}\delta_{(a,f)}$.
Consequently, $\mathcal{M}(1)$ admits a flat extension $\mathcal{M}(2)$,

$$\mathcal{M}(2) = \begin{pmatrix} 1 & a & b & a^2 & ab & bf \\ a & a^2 & ab & a^3 & a^2b & abf \\ b & ab & bf & a^2b & abf & bf^2 \\ a^2 & a^3 & a^2b & a^4 & a^3b & a^2bf \\ ab & a^2b & abf & a^3b & a^2bf & abf^2 \\ bf & abf & bf^2 & a^2bf & abf^2 & bf^3 \end{pmatrix}.$$

With the same arguments used in the proof of Theorem 3.1, $\mathcal{C}_2$ admits an extension $\tilde{\mathcal{C}_4}$ with the choices

$$\tilde{\alpha}^2_{(3,0)} = \tilde{\alpha}^2_{(0,3)} = \tilde{\alpha}^2_{(2,1)} = \tilde{\alpha}^2_{(1,2)} = a, \quad \tilde{\beta}^2_{(3,0)} = b \quad \text{and} \quad \tilde{\beta}^2_{(1,2)} = \tilde{\beta}^2_{(2,1)} = \tilde{\beta}^2_{(0,3)} = f.$$

Some calculations show that $\mathcal{M}_{(1,0)}(2) \succeq 0$, $\mathcal{M}_{(0,1)}(2) \succeq 0$, rank $\mathcal{M}_{(0,1)}(2) = 1$ and rank $\mathcal{M}_{(1,0)}(2) = 0$ if $a = 0$ and rank $\mathcal{M}_{(1,0)}(2) = 2$ if $a > 0$.
Hence, according to Theorem 2.4, $\mathcal{C}_2$ admits a subnormal completion $\mathcal{C}_\infty$ and Berger's measure is $\mu = \left(1-\frac{b}{f}\right)\delta_{(a,0)} + \frac{b}{f}\delta_{(a,f)}$, with one atom in $\mathbb{R}_+ \times \{0\}$ $(a \geq 0)$ and two atoms in $\{0\} \times \mathbb{R}_+$ if $a = 0$ and no atoms otherwise.

- Case 2: $Y = \frac{b(c-e)}{c-a}.1 + \frac{b(e-a)}{a(c-a)}.X$ with $c > a > 0$ and $f = \frac{ba(c-2e)+be^2}{a(c-a)} \geq b$.

For this case there are three sub-cases to consider $e = a$ or $e = c$ or ($e \neq a$ and $e \neq c$).
<u>Subcase 2.1:</u> If $e = a$ then $Y = b.1$ with $c > a > 0$ and $f = b$.
The conditions $\mathcal{M}(1) \succeq 0$ and $\operatorname{Ran} B(2) \subseteq \operatorname{Ran} \mathcal{M}(1)$ require that $c = g = p$ and $q = s = b$.

For the same reasons as in Case 1, γ admits a finite representing measure rank $\mathcal{M}(1)$-atomic $\mu = \left(1-\frac{a}{c}\right)\delta_{(0,b)} + \frac{a}{c}\delta_{(c,b)}$.
Consequently, $\mathcal{M}(1)$ admits a flat extension $\mathcal{M}(2)$ with

$$\mathcal{M}(2) = \begin{pmatrix} 1 & a & b & ac & ab & b^2 \\ a & ac & ab & ac^2 & abc & ab^2 \\ b & ab & b^2 & abc & ab^2 & b^3 \\ ac & ac^2 & abc & ac^3 & abc^2 & ab^2c \\ ab & abc & ab^2 & abc^2 & ab^2c & ab^3 \\ b^2 & ab^2 & b^3 & ab^2c & ab^3 & b^4 \end{pmatrix}.$$

Once again with the choices

$$\tilde{\alpha}^2_{(3,0)} = \tilde{\alpha}^2_{(2,1)} = \tilde{\alpha}^2_{(1,2)} = c, \tilde{\beta}^2_{(3,0)} = \tilde{\beta}^2_{(2,1)} = \tilde{\beta}^2_{(1,2)} = \tilde{\beta}^2_{(0,3)} = b \text{ and } \tilde{\alpha}^2_{(0,3)} = a,$$

$\mathcal{C}_2$ admits an extension $\tilde{\mathcal{C}_4}$.

By calculations, we get $\mathcal{M}_{(1,0)}(2) \succeq 0$, $\mathcal{M}_{(0,1)}(2) \succeq 0$, rank $\mathcal{M}_{(1,0)}(2) = 1$ and rank $\mathcal{M}_{(0,1)}(2) = 0$ if $b = 0$ and rank $\mathcal{M}_{(0,1)}(2) = 2$ otherwise.
Hence, according to Theorem 2.4, $\mathcal{C}_2$ admits a subnormal completion $\mathcal{C}_\infty$ and Berger's measure is given by $\mu = \left(1-\frac{a}{c}\right)\delta_{(0,b)} + \frac{a}{c}\delta_{(c,b)}$ with a single atom in $\{0\} \times \mathbb{R}_+$ and two atoms in $\mathbb{R}_+ \times \{0\}$ if $b \geq 0$ and none otherwise.
<u>Subcase 2.2:</u> If $c = e$, then $Y = \frac{b}{a}.X$ with $c > a > 0$ and $f = \frac{bc}{a}$.
The conditions $\mathcal{M}(1) \succeq 0$ and $\operatorname{Ran} B(2) \subseteq \operatorname{Ran} \mathcal{M}(1)$ require that $c = g = e = p$ and $q = s = \frac{bc}{a}$.

For the same reasons as in case 1, γ admits a unique representing measure rank $\mathcal{M}(1)$-atomic $\mu = \left(1-\frac{a}{c}\right)\delta_{(0,0)} + \frac{a}{c}\delta_{\left(c,\frac{bc}{a}\right)}$ and then $\mathcal{M}(1)$ admits a flat extension $\mathcal{M}(2)$, such that

$$\mathcal{M}(2) = \begin{pmatrix} 1 & a & b & ac & bc & \frac{b^2c}{a} \\ a & ac & bc & ac^2 & bc^2 & \frac{b^2c^2}{a} \\ b & bc & \frac{b^2c}{a} & bc^2 & \frac{b^2c^2}{a} & \frac{b^3c^2}{a^2} \\ ac & ac^2 & bc^2 & ac^3 & bc^3 & \frac{b^2c^3}{a} \\ bc & bc^2 & \frac{b^2c^2}{a} & bc^3 & \frac{b^2c^3}{a} & \frac{b^3c^3}{a^2} \\ \frac{b^2c}{a} & \frac{b^2c^2}{a} & \frac{b^3c^2}{a^2} & \frac{b^2c^3}{a} & \frac{b^3c^3}{a^2} & \frac{b^4c^3}{a^3} \end{pmatrix}.$$

Moreover, with quartic moments (the entries of the matrix $C(2)$) and taking into account the commutativity (1.2), $\mathcal{C}_2$ admits an extension $\tilde{\mathcal{C}_4}$ with the choices

$$\tilde{\alpha}^2_{(3,0)} = \tilde{\alpha}^2_{(2,1)} = \tilde{\alpha}^2_{(1,2)} = \tilde{\alpha}^2_{(0,3)} = c \text{ and } \tilde{\beta}^2_{(3,0)} = \tilde{\beta}^2_{(2,1)} = \tilde{\beta}^2_{(1,2)} = \tilde{\beta}^2_{(0,3)} = \frac{bc}{a}.$$

Some calculations show that $\mathcal{M}_{(1,0)}(2) \succeq 0$, $\mathcal{M}_{(0,1)}(2) \succeq 0$, rank $\mathcal{M}_{(1,0)}(2) = 1$ ($b \geq 0$) and rank $\mathcal{M}_{(0,1)}(2) = 1$ if $b > 0$ and rank $\mathcal{M}_{(0,1)}(2) = 0$ otherwise.
Hence, according to Theorem 2.4, $\mathcal{C}_2$ admits a subnormal completion $\mathcal{C}_\infty$ and Berger's measure is given by $\mu = \left(1-\frac{a}{c}\right)\delta_{(0,0)} + \frac{a}{c}\delta_{\left(c,\frac{bc}{a}\right)}$ with one two atoms belonging to $\mathbb{R}_+ \times \{0\}$ if $b = 0$ and only one atom belonging to both $\{0\} \times \mathbb{R}_+$ and $\mathbb{R}_+ \times \{0\}$ if $b > 0$.

Subcase 2.3: If $c \neq e$ and $a \neq e$ then, $Y = \frac{b(c-e)}{c-a}.1 + \frac{b(e-a)}{a(c-a)}.X$ with $c > a > 0$ and $f = \frac{ba(c-2e)+be^2}{a(c-a)} \geq b > 0$. The conditions $\mathcal{M}(1) \succeq 0$ and $\operatorname{Ran} B(2) \subseteq \operatorname{Ran} \mathcal{M}(1)$ imply that $c = g = p$, $q = \frac{be}{a}$ and $s = \frac{b^2\left(a^2\left(c^2-3ce+3e^2\right)-2ae^3+ce^3\right)}{a(c-a)(ab(c-2e)+e^2)}$ with $2e < c$.

For the same reasons as in case 1, γ admits a finite representing measure rank $\mathcal{M}(1)$-atomic $\mu = \left(1-\frac{a}{c}\right)\delta_{\left(0,\frac{b(c-e)}{c-a}\right)} + \frac{a}{c}\delta_{\left(c,\frac{be}{a}\right)}$ and therefore $\mathcal{M}(1)$ has a flat extension $\mathcal{M}(2)$ defined as follows

$$\mathcal{M}(2) = \begin{pmatrix} 1 & a & b & ac & be & \lambda_1 \\ a & ac & be & ac^2 & bce & \frac{b^2e^2}{a} \\ b & be & \lambda_2 & bce & \frac{b^2e^2}{a} & \lambda_3 \\ ac & ac^2 & bce & ac^3 & bc^2e & \frac{b^2ce^2}{a} \\ be & bce & \frac{b^2e^2}{a} & bc^2e & \frac{b^2ce^2}{a} & \frac{b^3e^3}{a^2} \\ \lambda_4 & \frac{b^2e^2}{a} & \lambda_5 & \frac{b^2ce^2}{a} & \frac{b^3e^3}{a^2} & \lambda_6 \end{pmatrix},$$

where

$$\lambda_1 = \left(\frac{ba(c-2e)+be^2}{a(c-a)}\right), \lambda_2 = b\left(\frac{ba(c-2e)+be^2}{a(c-a)}\right), \lambda_3 = \frac{b^3\left(\frac{e^3}{a^2}+\frac{(c-e)^3}{(c-a)^2}\right)}{c},$$

$$\lambda_4 = b\left(\frac{ba(c-2e)+be^2}{a(c-a)}\right), \lambda_5 = \frac{b^3\left(\frac{e^3}{a^2}+\frac{(c-e)^3}{(c-a)^2}\right)}{c} \text{ and } \lambda_6 = \frac{b^4\left(\frac{e^4}{a^3}+\frac{(c-e)^4}{(c-a)^3}\right)}{c}.$$

With the choices $\tilde{\alpha}^2_{(3,0)} = \tilde{\alpha}^2_{(2,1)} = \tilde{\alpha}^2_{(1,2)} = c$, $\tilde{\beta}^2_{(3,0)} = \tilde{\beta}^2_{(1,2)} = \tilde{\beta}^2_{(2,1)} = \frac{be}{a}$, $\tilde{\alpha}^2_{(3,0)} = \frac{ce^3}{a^2\left(\frac{e^3}{a^2}+\frac{(c-e)^3}{(a-c)^2}\right)}$ and $\tilde{\beta}^2_{03} = \frac{(b(3a^2e^4-3ace^4+c^2e^4+a^3(c-2e)(c^2-2ce+2e^2)))}{(a(-a+c)(-2ae^3+ce^3+a^2(c^2-3ce+3e^2)))}$, we also show that $\mathcal{C}_2$ admits an extension $\tilde{\mathcal{C}}_4$.

By some calculations, $\mathcal{M}_{(1,0)}(2) \succeq 0$, $\mathcal{M}_{(0,1)}(2) \succeq 0$, $\operatorname{rank} \mathcal{M}_{(1,0)}(2) = 1$ and $\operatorname{rank} \mathcal{M}_{(0,1)}(2) = 2$.
Thus, by Theorem 2.4, $\mathcal{C}_2$ admits a subnormal completion $\mathcal{C}_\infty$ and Berger's measure is given by $\mu = \left(1-\frac{a}{c}\right)\delta_{(0,\frac{b(c-e)}{c-a})} + \frac{a}{c}\delta_{(c,\frac{be}{a})}$ with a single atom in $\{0\}\times\mathbb{R}_+$.
Whence, the proof is ended. □

Remark. *In all previous cases, when constructing $\mathcal{M}(2)$, we set $C(2) = W^T\mathcal{M}(1)W$. Indeed, in relation (2.4) we always find $v = y$.*

The following theorem deals with the case where $\mathcal{M}(1)$ is positive definite ($\mathcal{M}(1) > 0$) and $v = y$.

Theorem 3.3. *Let $\mathcal{M}(1)$ and $B(2)$ be as defined in (3.1), y and v be as in (2.4). If $\mathcal{M}(1) > 0$, $\operatorname{Ran} B(2) \subseteq \operatorname{Ran} \mathcal{M}(1)$ and $v = y$, $\mathcal{M}(1)$ admits a flat extension $\mathcal{M}(2)$. In addition, if $\mathcal{M}_{(1,0)}(2) \succeq 0$ and $\mathcal{M}_{(0,1)}(2) \succeq 0$, $\mathcal{C}_2$ admits a subnormal completion $\mathcal{C}_\infty$ and the Berger measure is 3-atomic.*

Proof. Since $\mathcal{M}(1) > 0$, $\operatorname{Ran} B(2) \subseteq \operatorname{Ran} \mathcal{M}(1)$ and $v = y$ then according to [15, Theorem 3. 3], $\mathcal{M}(1)$ admits a flat extension $\mathcal{M}(2)$. Hence, γ admits a unique representing measure μ, $\operatorname{rank} \mathcal{M}(1)$-atomic ($\operatorname{rank} \mathcal{M}(1) = 3$) in $\mathbb{R}^2$ and since $\mathcal{M}_{(1,0)}(2) \succeq 0$ and $\mathcal{M}_{(0,1)}(2) \succeq 0$ then by Theorem 2.4, the support of μ is included in $\mathbb{R}^2_+$.
Thus, $\mathcal{C}_2$ admits a subnormal completion $\mathcal{C}_\infty$ and μ is its Berger measure. □

The following example illustrates this last theorem.

Example 3.4. *Let $\mathcal{C}_2$ be a collection of pairs of positive numbers defined by*

$$\begin{array}{llll}
\alpha_{(0,0)} = \frac{\sqrt{7}}{2}, & \beta_{(0,0)} = \frac{3}{2}, & \alpha_{(1,0)} = \sqrt{\frac{19}{7}}, & \beta_{(1,0)} = \sqrt{2} \\
\alpha_{(0,1)} = \frac{\sqrt{14}}{3}, & \beta_{(0,1)} = \sqrt{\frac{7}{3}}, & \alpha_{(2,0)} = \sqrt{\frac{55}{19}}, & \beta_{(2,0)} = \sqrt{2} \\
\alpha_{(1,1)} = \sqrt{\frac{19}{7}}, & \beta_{(1,1)} = \sqrt{2}, & \alpha_{(0,2)} = \frac{2\sqrt{3}}{3}, & \beta_{(0,2)} = \sqrt{\frac{17}{7}}.
\end{array}$$

The sequence of moments $\gamma = \{\gamma_{ij}\}_{0\le i+j\le 3}$ defined by (1.3) is

$$\begin{array}{llll}
\gamma_{00} = 1, & & & \\
\gamma_{10} = \frac{7}{4}, & \gamma_{01} = \frac{9}{4}, & & \\
\gamma_{20} = \frac{19}{4}, & \gamma_{11} = \frac{7}{2}, & \gamma_{02} = \frac{21}{4}, & \\
\gamma_{30} = \frac{55}{4}, & \gamma_{21} = \frac{19}{2}, & \gamma_{12} = 7, & \gamma_{03} = \frac{51}{4}.
\end{array}$$

The two matrices associated to γ are defined as follows

$$\mathcal{M}(1)=\begin{pmatrix} 1 & \frac{7}{4} & \frac{9}{4} \\ \frac{7}{4} & \frac{19}{4} & \frac{7}{2} \\ \frac{9}{4} & \frac{7}{2} & \frac{21}{4} \end{pmatrix} \quad \text{and} \quad B(2)=\begin{pmatrix} \frac{19}{4} & \frac{7}{2} & \frac{21}{4} \\ \frac{55}{4} & \frac{19}{2} & 7 \\ \frac{19}{2} & 7 & \frac{51}{4} \end{pmatrix}.$$

Straightforward calculations show that $\mathcal{M}(1)>0$, $\operatorname{rank}\mathcal{M}(1)=3$ *and* $\operatorname{Ran}B(2)\subseteq\operatorname{Ran}\mathcal{M}(1)$,

$$\text{with } W=\begin{pmatrix} -9 & 0 & -6 \\ 4 & 2 & 0 \\ 3 & 0 & 5 \end{pmatrix} \text{ and } W^T\mathcal{M}(1)W=\begin{pmatrix} \frac{163}{4} & \frac{55}{2} & 19 \\ \frac{55}{2} & 19 & 14 \\ 19 & 14 & \frac{129}{4} \end{pmatrix}.$$

Noticing that $v=y=19$, $\mathcal{M}(1)$ admits a flat extension $\mathcal{M}(2)$ given by

$$\mathcal{M}(2)=\begin{pmatrix} 1 & \frac{7}{4} & \frac{9}{4} & \frac{19}{4} & \frac{7}{2} & \frac{21}{4} \\ \frac{7}{4} & \frac{19}{4} & \frac{7}{2} & \frac{55}{4} & \frac{19}{2} & 7 \\ \frac{9}{4} & \frac{7}{2} & \frac{21}{4} & \frac{19}{2} & 7 & \frac{51}{4} \\ \frac{19}{4} & \frac{55}{4} & \frac{19}{2} & \frac{163}{4} & \frac{55}{2} & 19 \\ \frac{7}{2} & \frac{19}{2} & 7 & \frac{55}{2} & 19 & 14 \\ \frac{21}{4} & 7 & \frac{51}{4} & 19 & 14 & \frac{129}{4} \end{pmatrix}.$$

With commutativity condition (1.2) *and taking* $\tilde{\alpha}^2_{(3,0)}=\frac{163}{55}$, $\tilde{\alpha}^2_{(0,3)}=\frac{56}{51}$, $\tilde{\alpha}^2_{(2,1)}=\frac{55}{19}$, $\tilde{\alpha}^2_{(1,2)}=\frac{19}{7}$, $\tilde{\beta}^2_{(3,0)}=\tilde{\beta}^2_{(2,1)}=\tilde{\beta}^2_{(1,2)}=2$, *and* $\tilde{\beta}^2_{(0,3)}=\frac{43}{17}$, $\mathcal{C}_2$ *admits an extension* $\tilde{\mathcal{C}}_4$.

The localizing matrices are

$$\mathcal{M}_{(1,0)}(2)=\begin{pmatrix} \frac{7}{4} & \frac{19}{4} & \frac{7}{2} \\ \frac{19}{4} & \frac{55}{4} & \frac{19}{2} \\ \frac{7}{2} & \frac{19}{2} & 7 \end{pmatrix} \quad \text{and} \quad \mathcal{M}_{(0,1)}(2)=\begin{pmatrix} \frac{9}{4} & \frac{7}{2} & \frac{21}{4} \\ \frac{7}{2} & \frac{19}{2} & 7 \\ \frac{21}{4} & 7 & \frac{51}{4} \end{pmatrix}.$$

Calculations show that $\mathcal{M}_{(1,0)}(2)\succeq 0$, $\mathcal{M}_{(0,1)}(2)\succeq 0$, $\operatorname{rank}\mathcal{M}_{(1,0)}(2)=2$ *and* $\operatorname{rank}\mathcal{M}_{(0,1)}(2)=3$. *The linear dependencies between the columns of $\mathcal{M}(2)$ are*

$$X^2=-9+4X+3Y, \quad XY=2X \quad \text{and} \quad Y^2=-6+5Y.$$

The variety cone of $\mathcal{M}(2)$ is $\mathcal{V}(\mathcal{M}(2))=\mathcal{Z}(P)\cap\mathcal{Z}(Q)\cap\mathcal{Z}(R)$ where

$$P(x,y)=x^2-4x-3y+9, Q(x,y)=xy-2x \text{ and } R(x,y)=y^2-5y+6.$$

So, $\mathcal{V}(\mathcal{M}(2))=\{(0,3);(1,2);(3,2)\}$.
The densities ρ_1, ρ_2 and ρ_3 related to the atoms $(0,3)$, $(1,2)$ and $(3,2)$ are solutions

of the system

$$\begin{cases} \rho_1 + \rho_2 + \rho_3 = 1 \\ \rho_2 + 3\rho_3 = \frac{7}{4} \\ 3\rho_1 + \rho_2 + 3\rho_3 = \frac{9}{4} \end{cases}.$$

Hence, Berger measure is given by

$$\mu = \frac{1}{2}\delta_{(0,3)} + \frac{1}{2}\delta_{(1,2)} + \frac{1}{4}\delta_{(3,2)}.$$

Finally, by Theorem 2.4, $\mathcal{C}_2$ admits a subnormal completion $\mathcal{C}_\infty$.
Note that there is only one atom $(0,3) \in \{0\}\mathbb{R}_+^2$ since $\operatorname{rank}\mathcal{M}(1) - \operatorname{rank}\mathcal{M}_{(1,0)}(2) = 1$. However, there is no atom in $\mathbb{R}_+^2 \in \{0\}$ since $\operatorname{rank}\mathcal{M}(1) - \operatorname{rank}\mathcal{M}_{(0,1)}(2) = 0$.

Remark. *For a collection $\mathcal{C}_2$ of pairs of positive numbers and the sequence of moments $\gamma = \{\gamma_{ij}\}_{0\le i+j\le 3}$ given by (1.3) satisfy the conditions of Theorem 3.3 except $v \neq y$, then nothing can be concluded about the existence of a subnormal completion. The following two examples clarify this statement.*

Example 3.5. *Let $\mathcal{C}_2$ be the collection of the following pairs of positive numbers of the following data*

$$\begin{array}{llll} \alpha_{(0,0)} = \sqrt{2}, & \beta_{(0,0)} = 2, & \alpha_{(1,0)} = \sqrt{3}, & \beta_{(1,0)} = 2, \\ \alpha_{(0,1)} = \sqrt{2}, & \beta_{(0,1)} = \sqrt{5}, & \alpha_{(2,0)} = 2, & \beta_{(2,0)} = 2, \\ \alpha_{(1,1)} = \sqrt{3}, & \beta_{(1,1)} = \sqrt{5}, & \alpha_{(0,2)} = \sqrt{2}, & \beta_{(0,2)} = 2\sqrt{5}. \end{array}$$

The sequence of moments $\gamma = \{\gamma_{ij}\}_{0\le i+j\le 3}$ defined by (1.3) is

$$\begin{array}{llll} \gamma_{00} = 1, & & & \\ \gamma_{10} = 2, & \gamma_{01} = 4, & & \\ \gamma_{20} = 6, & \gamma_{11} = 8, & \gamma_{02} = 20, & \\ \gamma_{30} = 24, & \gamma_{21} = 24, & \gamma_{12} = 40, & \gamma_{03} = 400. \end{array}$$

The matrices associated to γ are

$$\mathcal{M}(1) = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 6 & 8 \\ 4 & 8 & 20 \end{pmatrix} \quad \text{and} \quad B(2) = \begin{pmatrix} 6 & 8 & 20 \\ 24 & 24 & 40 \\ 24 & 40 & 400 \end{pmatrix}.$$

With some calculations, we get $\mathcal{M}(1) > 0$ and $\operatorname{Ran} B(2) \subseteq \operatorname{Ran}\mathcal{M}(1)$ with

$$W^T\mathcal{M}(1)W = \begin{pmatrix} 108 & 96 & 120 \\ 96 & 112 & 800 \\ 120 & 800 & 26000 \end{pmatrix}.$$

We have $y = 112 \neq 120 = v$. So, according to Theorem 2.4, γ admits at least one representing measure 4-atomic.
Indeed, for the choice of quartic moments $\gamma_{40} = 108$, $\gamma_{31} = 96$, $\gamma_{22} = 120$, $\gamma_{13} = 800$ and $\gamma_{04} = 26000$, $\mathcal{M}(1)$ admits a positive semidefinite extension $\mathcal{M}(2)$ given by

$$\mathcal{M}(2) = \begin{pmatrix} 1 & 2 & 4 & 6 & 8 & 20 \\ 2 & 6 & 8 & 24 & 24 & 40 \\ 4 & 8 & 20 & 24 & 40 & 400 \\ 6 & 24 & 24 & 108 & 96 & 120 \\ 8 & 24 & 40 & 96 & 120 & 800 \\ 20 & 40 & 400 & 120 & 800 & 26000 \end{pmatrix}.$$

By the techniques used in [15], *we construct the extension* $\mathcal{M}(3)$ *of* $\mathcal{M}(2)$. *We get,*

$$\mathcal{M}(3) = \begin{pmatrix} 1 & 2 & 4 & 6 & 8 & 20 & 24 & 24 & 40 & 400 \\ 2 & 6 & 8 & 24 & 24 & 40 & 108 & 96 & 120 & 800 \\ 4 & 8 & 20 & 24 & 40 & 400 & 96 & 120 & 800 & 26000 \\ 6 & 24 & 24 & 108 & 96 & 120 & 504 & 432 & 480 & 2400 \\ 8 & 24 & 40 & 96 & 120 & 800 & 432 & 480 & 2400 & 52000 \\ 20 & 40 & 400 & 120 & 800 & 26000 & 480 & 2400 & 52000 & 1960000 \\ 24 & 108 & 96 & 504 & 432 & 480 & 2376 & 2016 & 2160 & 9600 \\ 24 & 96 & 120 & 432 & 480 & 2400 & 2016 & 2160 & 9600 & 156000 \\ 40 & 120 & 800 & 480 & 2400 & 52000 & 2160 & 9600 & 156000 & 3920000 \\ 400 & 800 & 26000 & 2400 & 52000 & 1960000 & 9600 & 156000 & 3920000 & 149000000 \end{pmatrix},$$

Calculations, show that $\mathcal{M}(3)$ *is flat. Hence,* γ *admits a representing measure* μ *of 4 atoms.*

The localizing matrices associated to $\mathcal{M}(3)$ *are*

$$\mathcal{M}_{(1,0)}(3) = \begin{pmatrix} 2 & 6 & 8 & 24 & 24 & 40 \\ 6 & 24 & 24 & 108 & 96 & 120 \\ 8 & 24 & 40 & 96 & 120 & 800 \\ 24 & 108 & 96 & 504 & 432 & 480 \\ 24 & 96 & 120 & 432 & 480 & 2400 \\ 40 & 120 & 800 & 480 & 2400 & 52000 \end{pmatrix},$$

and

$$\mathcal{M}_{(0,1)}(3) = \begin{pmatrix} 4 & 8 & 20 & 24 & 40 & 400 \\ 8 & 24 & 40 & 96 & 120 & 800 \\ 20 & 40 & 400 & 120 & 800 & 26000 \\ 24 & 96 & 120 & 432 & 480 & 2400 \\ 40 & 120 & 800 & 480 & 2400 & 52000 \\ 400 & 800 & 26000 & 2400 & 52000 & 1960000 \end{pmatrix}.$$

With some computations, we obtain $\mathcal{M}_{(1,0)}(3) \succeq 0$, $\mathcal{M}_{(0,1)}(3) \succeq 0$ *and the linear dependency relations between the columns of* $\mathcal{M}(2)$ *are*

$$X^2 = 6X - 6 \quad \textit{and} \quad Y^2 = 80Y - 300.$$

Thus, the atoms of the measure μ *are*

$$\begin{array}{ll} (x_1, y_1) = (3 - \sqrt{3}; 40 - 10\sqrt{13}), & (x_2, y_2) = (3 - \sqrt{3}; 40 + 10\sqrt{13}), \\ (x_3, y_3) = (3 + \sqrt{3}; 40 - 10\sqrt{13}), & (x_4, y_4) = (3 + \sqrt{3}; 40 + 10\sqrt{13}), \end{array}$$

which belong to $\mathbb{R}^2_+$ *and the respective weights are*

$$\rho_1 = \tfrac{1}{780}\left(65\sqrt{3} + 54\sqrt{13} + 18\sqrt{39} + 195\right), \qquad \rho_2 = \tfrac{1}{780}\left(65\sqrt{3} - 54\sqrt{13} - 18\sqrt{39} + 195\right),$$

$$\rho_3 = \tfrac{1}{780}\left(-65\sqrt{3} + 54\sqrt{13} - 18\sqrt{39} + 195\right), \quad \rho_4 = \tfrac{1}{780}\left(-65\sqrt{3} - 54\sqrt{13} + 18\sqrt{39} + 195\right).$$

So Berger's measure is $\mu = \sum_{k=1}^{k=4} \rho_k \delta_{(x_k, y_k)}$ *with* $\operatorname{supp} \mu \subseteq \mathbb{R}^2_+$. *Then,* $\mathcal{C}_2$ *admits a subnormal completion* $\mathcal{C}_\infty$.

Example 3.6. *Let* $\mathcal{C}_2$ *be the collection of the following pairs of positive data numbers*

$$\begin{array}{llll} \alpha_{(0,0)} = \frac{\sqrt{3}}{3}, & \beta_{(0,0)} = \frac{\sqrt{3}}{3}, & \alpha_{(1,0)} = \sqrt{5}, & \beta_{(1,0)} = \sqrt{3}, \\ \alpha_{(0,1)} = \sqrt{3}, & \beta_{(0,1)} = 3, & \alpha_{(2,0)} = \frac{3\sqrt{5}}{5}, & \beta_{(2,0)} = \sqrt{\frac{3}{5}}, \\ \alpha_{(1,1)} = 1, & \beta_{(1,1)} = \frac{\sqrt{3}}{3}, & \alpha_{(0,2)} = \frac{\sqrt{3}}{9}, & \beta_{(0,2)} = \sqrt{10}. \end{array}$$

The sequence of moments $\gamma = \{\gamma_{ij}\}_{0\le i+j\le 3}$ *defined by* (1.3) *is*

$$\begin{aligned}
&\gamma_{00} = 1,\\
&\gamma_{10} = \tfrac{1}{3}, \quad \gamma_{01} = \tfrac{1}{3},\\
&\gamma_{20} = \tfrac{5}{3}, \quad \gamma_{11} = 1, \quad \gamma_{02} = 3,\\
&\gamma_{30} = 3, \quad \gamma_{21} = 1, \quad \gamma_{12} = \tfrac{1}{3}, \quad \gamma_{03} = 30.
\end{aligned}$$

The matrices associated with γ *are*

$$\mathcal{M}(1) = \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{5}{3} & 1 \\ \frac{1}{3} & 1 & 3 \end{pmatrix} \quad \textit{and} \quad B(2) = \begin{pmatrix} \frac{5}{3} & 1 & 3 \\ 3 & 1 & \frac{1}{3} \\ 1 & \frac{1}{3} & 30 \end{pmatrix}.$$

Calculations lead to $\mathcal{M}(1) > 0$ *and* $\operatorname{Ran} B(2) \subseteq \operatorname{Ran} \mathcal{M}(1)$ *with*

$$W = \begin{pmatrix} \frac{6}{5} & \frac{22}{25} & \frac{69}{50} \\ \frac{9}{5} & \frac{13}{25} & -\frac{187}{25} \\ -\frac{2}{5} & -\frac{4}{25} & \frac{617}{50} \end{pmatrix} \textit{ and } W^T\mathcal{M}(1)W = \begin{pmatrix} 7 & \frac{43}{15} & -\frac{39}{5} \\ \frac{43}{15} & \frac{101}{75} & -\frac{149}{75} \\ -\frac{39}{5} & -\frac{149}{75} & \frac{55777}{150} \end{pmatrix}.$$

We have $y = \frac{101}{75} \neq -\frac{39}{5} = v$. *Taking* $C(2) = \begin{pmatrix} 8 & \frac{43}{15} & \frac{101}{75} \\ \frac{43}{15} & \frac{101}{75} & -\frac{149}{75} \\ \frac{101}{75} & -\frac{149}{75} & \frac{5124467}{11250} \end{pmatrix}$, *we construct the extension* $\mathcal{M}(2)$ *of* $\mathcal{M}(1)$ *as follows*

$$\mathcal{M}(2) = \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{5}{3} & 1 & 3 \\ \frac{1}{3} & \frac{5}{3} & 1 & 3 & 1 & \frac{1}{3} \\ \frac{1}{3} & 1 & 3 & 1 & \frac{1}{3} & 30 \\ \frac{5}{3} & 3 & 1 & 8 & \frac{43}{15} & \frac{101}{75} \\ 1 & 1 & \frac{1}{3} & \frac{43}{15} & \frac{101}{75} & -\frac{149}{75} \\ 3 & \frac{1}{3} & 30 & \frac{101}{75} & -\frac{149}{75} & \frac{5124467}{11250} \end{pmatrix}.$$

Note that $\gamma_{13} = -\frac{149}{75} < 0$, *which makes impossible the construction of a subnormal completion* $\tilde{\mathcal{C}}_4$ *of* $\mathcal{C}_2$. *According to the relation* (1.3), $\gamma_{13} = \alpha^2_{(0,0)}\beta^2_{(1,0)}\beta^2_{(1,1)}\tilde{\beta}^2_{(1,2)}$. *Whence* $\tilde{\beta}^2_{(1,2)} < 0$, *which is impossible.*

Therefore, $\mathcal{C}_2$ *does not admit a subnormal completion* $\mathcal{C}_\infty$.

Acknowledgments. The authors would like to thank the anonymous referee(s) for the valuable comments and suggestions which improved the quality of this manuscript.

References

[1] N. I. Akhiezer, *The classical moment problem and some related questions in analysis*, Translated by N. Kemmer Hafner Publishing Co., New York (1965).

[2] A. Athavale. *On joint hyponormality of operators*, Proceedings of the American Mathematical Society **103 2** (1988) 417–423.

[3] J. Bram, *Subnormal operators*, Duke Math. J. **22 1** (1955).

[4] J. B. Conway, *The theory of subnormal operators*, American Mathematical Society (1991).

[5] L. I. Chunji, *Two variable subnormal completion problem*, Hokkaido mathematical journal **32 1** (2003) 21–29.

[6] R. E. Curto, *Joint hyponormality: A bridge between hyponormality and subnormality*, Proc. Symposia Pure. Math. **51** (1977) 69–91.

[7] R. E. Curto and L. A. Fialkow, *Solution of the truncated complex moment problem for flat data*, American Mathematical Soc. **568** (1996).
[8] R. E. Curto and L. Fialkow, *The truncated complex K-moment problem*, Trans. Amer. Math. Soc. **352** **6** (2000) 2825–2855.
[9] R. E. Curto, S. H. Lee and J. Yoon, *k-Hyponormality of multivariable weighted shifts*, Journal of Functional Analysis **229** **2** (2005) 462-480.
[10] R. E. Curto, S. H. Lee and J. Yoon, *Hyponormality and subnormality for powers of commuting pairs of subnormal operators*, Journal of Functional Analysis **245** **2** (2007) 390-412.
[11] R. E. Curto, S. H. Lee and J. Yoon, *A new approach to the 2-variable Subnormal Completion Problem*, Journal of Mathematical Analysis and Applications **370** **1** (2010) 270-283.
[12] R. E. Curto, P. S. Muhly and J. Xia, *Hyponormal pairs of commuting operators*, Contributions to Operator Theory and its Applications **35** (1988) 1–22.
[13] R. G. Douglas, *On majorization and range inclusion of operators in Hilbert space*, Proc. Amer. Math. Soc. **17** (1966), 413–416.
[14] A. El boukili, A. Rhazi and B. El Wahbi, *On A class of real quintic moment problem*, Journal of the Indonesian Mathematical Society **29** **1** (2003) 1–23.
[15] A. El boukili, A. Rhazi and B. El Wahbi, *Further results on the real cubic truncated moment problem*, *arXiv preprint arXiv:2307.04058*, (2023).
[16] R. Gellar and L. J. Wallen, *Subnormal weighted shifts and the Halmos-Bram criterion*, Proc. Japan Acad. Ser. A Math. Sci. **46** **4** (1970).
[17] N. P. Jewell and A. R. Lubin, *Commuting weighted shifts and analytic function theory in serveral variables*, Journal of Operator Theory **1** **2** (1979) 207-223.
[18] D. P. Kimsey, The subnormal completion problem in several variables, Journal of Mathematical Analysis and Applications **434** **2** (2016) 1504–1532.
[19] J. I. Lee and S. H. Lee, *On the 2-variable subnormal completion problem*, Journal of the Chungcheong Mathematical Society **22** **3** (2009) 439–450.
[20] S. H. Lee, W. Y. Lee and J. Yoon, *Subnormality of powers of multivariable weighted shifts*, Journal of Function Spaces **2020** (2020) 1–11.
[21] W. Y. Lee, *Lecture notes on operator theory.* Seoul National Univer., (2010).
[22] J. L. Smuljan, *An operator Hellinger integral*, (in Russian), Mat. Sb. (N.S.) **49** (1959) 381-430.

Abdallah Taia
Laboratory of Analysis, Geometry and Applications (LAGA), Departmentof Mathematics, Faculty of Sciences, Ibn Tofail University, Kenitra, B.P. 133, Morocco
E-mail address: abdallah.taia@uit.ac.ma

Abdelaziz El Boukili
Laboratory of Analysis, Geometry and Applications (LAGA), Departmentof Mathematics, Faculty of Sciences, Ibn Tofail University, Kenitra, B.P. 133, Morocco
E-mail address: abdelaziz.elboukili@uit.ac.ma

Amar Rhazi
Laboratory of Analysis, Geometry and Applications (LAGA), Departmentof Mathematics, Faculty of Sciences, Ibn Tofail University, Kenitra, B.P. 133, Morocco
E-mail address: amar.rhazi@uit.ac.ma

And Bouazza El Wahbii
Laboratory of Analysis, Geometry and Applications (LAGA), Departmentof Mathematics, Faculty of Sciences, Ibn Tofail University, Kenitra, B.P. 133, Morocco
E-mail address: bouazza.elwahbi@uit.ac.ma