Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 15 Issue 1 (2024), Pages 1-12
https://doi.org/10.54379/jma-2024-1-1.

SOME FIXED POINT THEOREMS FOR MULTIVALUED KRASNOSEL'SKII-TYPE EQUATIONS UNDER WEAK TOPOLOGY

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Abstract. In this paper, using the approximation method and the technique of weak noncompactness, we obtain some results regarding the existence of the solution for the sum of a multivalued operator and a single-valued operator in the weak topology. In addition, an application of integral inclusion is presented that explains our theory. This study extends some previously known single-valued Krasnosel'skii-type theorems to multivalued versions under weak topology.

1. Introduction

In nonlinear analysis, many integral equations and differential equations can be expressed as the sum of operators. Therefore, recently authors have followed with interest the existence of the solution for the sum of operators [5, 6, 13, 23, 24]. In [20], Krasnosel'skii proved that the equation

$$Kx + Lx = x, \qquad x \in C \tag{1.1}$$

has a solution in C under the following conditions:
(i) $K(x) + L(y) \in C$ for all $x, y \in C$,
(ii) K is a contraction with constant $\alpha < 1$,
(iii) L is continuous on C and $\overline{(L(C))}$ is compact set in X,
where C is a convex closed subset of a Banach space X. This result is also known as a mixed fixed point theorem, which includes the Banach contraction principle and Schauder fixed point theorem. Many variants of the Krasnosel'skii theorem have been obtained by improving the above conditions [9, 11, 19, 25]. For instance in [4], Barroso presented a version of Krasnosel'skii theorem for the linearity condition in weak topology and in [30], Xiang et al established a new result for equation (1.1), where K is an expansive mapping. In course of time, multivalued versions of Krasnosel'skii type theorems have been obtained for solving the inclusion

$$x \in Kx + Sx, \qquad x \in C \tag{1.2}$$

2000 *Mathematics Subject Classification.* 34K13, 47H04, 47H10.
Key words and phrases. Multivalued mapping, Krasnosel'skii fixed point theorem, Measure of weak noncompactness, Approximation method.

Submitted October 19, 2023. Published January 2, 2024.
Communicated by Mikail Et.

and some researchers have extended a number of multivalued analogues for the weak topologys [10, 16, 26, 27].

Studies on Krasnosel'skii's theorem have an important place in the literature due to the wide application areas of equations in the form (1.1) and (1.2). In the first studies, the researchers studied with the condition that the L operator has weak compactness property. This thought stemmed from their focus only on Schauder's fixed point theorem. Then, in [30], by the method of measure of noncompactness, the authors studied without the compactness property of the L operator in strong topology. Over time, new multivalued variants of Krasnosel'skii theorem were obtained by using the method of measure of weak noncompactness and the Schauder's fixed point theorem together [8, 18, 28].

Moreover, the multivalued Krasnosel'skii theorems are related to the notion of selection that Cellina [12] has constructed. Multivalued operators admit an approximate continuous selection established for compact and nonconvex multifunctions under certain conditions [15, 17]. One of the important methods related to selections in the multivalued fixed point theory is the approximation method started by J. Von Neumann [29]. This method works as follows: first, through a single-valued map, an approximation is obtained on the graph of a given multivalued map. Then, by applying a certain process to this single-valued approximation, it is investigated to what extent the properties of the approximation are transferred to the original map.

The scope of this study is as follows: Firstly, using the method of measure of weak noncompactness, we give a new result in weak topology without needing the compactness condition of S. Secondly, we establish a new variant of the Krasnosel'skii theorem offered by Taoudi [25, Theorem 2.1] and we model the proof of obtained result with the help of an approximation selection s_ϵ satisfying the condition of Lemma 4.1, also the invertible of $(I-K)$ and the weakly sequentially continuity of $(I-K)^{-1}s_\epsilon$ play an essential role in the proof. Then based on this result we prove a fixed point theorem for the sum of a multivalued weak compact and a nonexpensive mapping. Finally, we present an application that embodies our results for integral inclusion.

2. Preliminaries

In this section, we present the notations, definitions, and useful results that will be needed for this article.

Let $(X, \|.\|)$ be a Banach space and sets $P(X)=\{Y\subset X : Y\neq\emptyset\}$, $P_{cl}(X)=\{Y\subset X : Y\neq\emptyset,\ Y \text{ is closed}\}$, $P_{bd}(X)=\{Y\subset X : Y\neq\emptyset,\ Y \text{ is bounded }\}$, $P_{cv}(X)=\{Y\subset X : Y\neq\emptyset,\ Y \text{ is convex}\}$ and $P_{wcp}(X)=\{Y\subset X : Y\neq\emptyset,\ Y \text{ is weakly compact}\}$. Let $S : X\to P_{cl}(Y)$ be a multivalued map, the range $R(S)$ and the graph $\Gamma(S)$ of S is defined by

$$R(S)=\bigcup_{x\in X} S(x), \qquad \Gamma(S)=\{(x,y)\in X\times X : x\in X,\ y\in S(x)\}.$$

A single-valued map $s : X\to Y$ is called to be a selection of S if $s(x)\in S(x)$ for every $x\in X$. Also, s is called to be weakly sequentially continuous (w.s.c.) if for each sequence $(x_n)_n\subset X$ with $x_n\rightharpoonup x\in X$, there exists $s(x_n)\rightharpoonup s(x)$. The mapping S is called weakly upper semicontinuous (w.u.sc.) if

$$S^{-1}(V)=\{x\in X : S(x)\cap V\neq\emptyset\}$$

is closed for weak topology in X for a weakly closed set $V \subset Y$ and S is called to be weakly sequentially upper semicontinuous (w.s.u.sc.) if $S^{-1}(V)$ is weakly sequentially closed in X. S is called to be weakly compact if the set $R(S)$ is weakly relatively compact in Y. Let $U \subset X$ is a weakly closed set and it is said to S has a weakly sequentially closed graph (w.s.cg.) if for each $(x_n)_n \subset U,\ x_n \rightharpoonup x$ in U and for each $(y_n)_n$ such that $y_n \in S(x_n)$, $\forall n \in N$, $y_n \rightharpoonup y$ in Y implies $y \in S(x)$, where $\rightharpoonup$ denotes weak convergence.

Let $\hat{W}(X)$ be a subset of $P_{bd}(X)$ consisting of all weakly compact subsets of X, and B_r denote the closed ball centered at 0 with radius r.

Definition 2.1. [25] *A function $\omega : P_{bd}(X) \to \mathbb{R}_+$ is said to be a measure of weak noncompactness (m.w.nc.) if it satisfies the following condition:*
(1) $\omega(\overline{co}(C)) = \omega(C)$; $\overline{co}(C)$ is the closed convex hull of $C \in P_{bd}(X)$,
(2) $C_1 \subseteq C_2 \Rightarrow \omega(C_1) \leq \omega(C_2)$, for $C_1, C_2 \in P_{bd}(X)$,
(3) $\omega(\lambda C_1 + (1-\lambda)C_2) \leq \lambda\ \omega(C_1) + (1-\lambda)\ \omega(C_2)$ for $\lambda \in [0,1]$,
(4) If $(C_n)_{n\geq 1}$ is a sequence of nonempty weakly closed subsets of X with C_1 bounded and $C_1 \supseteq C_2 \supseteq ...C_n \supseteq ...$ with $\lim_{n\to\infty} \omega(C_n) = 0$, then $C_\infty := \bigcap_{n=1}^{\infty} C_n$ is nonempty.

It can be easily shown that the measure ω satisfies $\omega(\overline{C}^w) = \omega(C)$, where $\overline{C}^w$ is the weak closure of C. The first significant instance of an m.w.nc. has been introduced by De Blasi as:

$$\Omega(C) = \inf\{r > 0 : \text{there exists } W \in \hat{W}(X) \text{with } C \subseteq W + B_r\} \tag{2.1}$$

for each $C \in P_{bd}(X)$. This m.w.nc. has other useful features such as the subadditivity

$$\Omega(C_1 + C_2) \leq \Omega(C_1) + \Omega(C_2),$$

and the homogeneity

$$\Omega(\lambda C) = |\lambda|\Omega(C).$$

Also, an important feature of De Blasi m.w.nc. is

$$\Omega(C) = 0 \ \text{ if and only if } C \text{ is relatively weakly compact.} \tag{2.2}$$

Using the m.w.nc., we call that a multivalued mapping $S : C \to P(X)$ is Ω-condensing if S is bounded and $\Omega(S(C)) < \Omega(C)$ for all $C \in P_{bd}(X)$ with $\Omega(C) \neq 0$.

Definition 2.2. [25] *Let X be a Banach space. A mapping $K : X \to X$ is called nonexpansive if*

$$\|Kx - Ky\| \leq \|x - y\|$$

for every $x, y \in X$.

Definition 2.3. [14] *Let $S : X \to P(Y)$ be a multivalued map, $M \subset X$ and $\epsilon > 0$. A single-valued map $s_\epsilon : M \to Y$ is called to be an $\epsilon-$approximation (on the graph) of S if*

$$\Gamma(s_\epsilon) \subset O_\epsilon(\Gamma(S))$$

where $O_\epsilon(.)$ is $\epsilon-$neighborhood.

Some significant features related to the approximation of multivalued maps are given in Lemma 4.1.

3. MEASURE OF WEAK NONCOMPACTNESS FOR KRASNOSEL'SKII TYPE THEOREMS

Let us start this section with the two results we will need.

Theorem 3.1. [7] *Let C be a nonempty, convex, closed subset of X. Assume $S : C \to P_{cv}(C)$ has w.s.cg and $S(C)$ is weakly relatively compact. Then S has a fixed point.*

Theorem 3.2. [7] *Let C be a nonempty, convex, closed subset of X and Ω an m.w.nc. on X. Assume $S : C \to P_{cv}(C)$ has w.s.cg., is Ω-condensing and $S(C)$ is bounded. Then S has a fixed point.*

Now using m.w.nc., we present our first result, where $L(X)$ denotes the space of continuous, linear operators on X.

Theorem 3.3. *Let C be a nonempty, closed, convex and bounded subset of X. Assume $K \in L(X)$ is a w.s.c. mapping and $S : C \to P_{cv}(X)$ is a w.s.u.sc. mapping satisfying:*
(i) $Kx + Sy \subset C$, for $x, y \in C$,
(ii) K is a contraction,
(iii) $(K + S)(C)$ is a bounded set of C,
(iv) $\Omega(K(A) + S(A)) < \Omega(A)$ for all $A \subset C$ with $\Omega(A) \neq 0$.
Then there exists $x \in C$ with $x \in Sx + Kx$.

Proof. Let $y \in C$ and $H_y : C \to P_{cv}(X)$ be the multivalued mapping defined by

$$H_y(x) = K(x) + S(y), \quad x \in C.$$

Since $S(y) \in P_{cv}(X)$ and S is w.s.u.sc., then $H_y(x) \in P_{cv}(X)$. Using (i), obtain that $H_y(C) \subset C$. We show that H_y has w.s.cg.. Let $(\alpha_n, \beta_n) \in \Gamma(H_y)$ be a sequence with $\beta_n \in H_y(\alpha_n), \alpha_n \rightharpoonup x$ and $\beta_n \rightharpoonup y^*$. So there exists $\delta_n \in S(y)$ with $\beta_n = K(\alpha_n) + \delta_n$. Since K is w.s.c. then $K(\alpha_n) \rightharpoonup K(x)$. Consequently, we obtain $\delta_n \rightharpoonup y^* - K(x) \in S(y)$. Thus H_y has w.s.cg.. Now let us show that $H_y(C)$ is relatively weakly compact. For this, we take β_n as a subsequence. From (i), we have $\delta_n \rightharpoonup y^* \in Kx + Sy \subset C$ and $\beta_n \rightharpoonup y^* \in C$. Thus $H_y\,(C)$ is relatively weakly compact. Hence, from Theorem 3.1 there exists $x(y) \in K(x(y)) + S(y)$. By (ii), it can be shown that $I - K$ is invertible (see [30, Lemma 2.13]) and $(I - K)^{-1} \in L(X)$. We define $F : C \to P_{cv}(C)$ by

$$y \to F(y) = (I - K)^{-1} S(y).$$

Since $S(.) \in P_{cv}(X)$ and $(I - K)^{-1} \in L(X)$, then $F(.) \in P_{cv}(X)$ and consequently $F(C) \subset C$. Let $a_o \in A$ and we define the set

$$\breve{A} = \{A : a_o \in A \subset C,\ A \text{ is a bounded convex closed set and } F(A) \subset A\}.$$

$\breve{A}$ is nonempty because of $C \in \breve{A}$. In addition, for each $A \in \breve{A}$, we have

$$\begin{aligned} F(A) &= (I - K)^{-1} S(A) && (3.1)\\ &\subset K(I - K)^{-1} S(A) + S(A) \subset K(A) + S(A). && (3.2) \end{aligned}$$

Using (iv), we can write $\Omega(F(A)) < \Omega(K(A) + S(A)) < \Omega(A)$ for all $A \in \breve{A}$ with $\Omega(A) > 0$.This implies that F is Ω-condensing, from (iii) and (3.1) $F(C)$ is bounded. Finally, for F to fulfill the conditions of Theorem 3.2, we must show that it has w.s.cg.. Let sequence $(x_n) \subset C$ with $x_n \rightharpoonup x \in C$ and $y_n \in F(x_n)$ with $y_n \rightharpoonup y$. Using definition of F, we get that

$$(I-K)(y_n) = S(x_n). \tag{3.3}$$

Considering that $(I-K)$ and S are w.s.c. we can write

$$(I-K)(y_n) \rightharpoonup (I-K)(y) \text{ and } S(x_n) \rightharpoonup S(x). \tag{3.4}$$

Combining of (3.2) and (3.3) we have $(I-K)(y) = S(x)$ and so $y \in (I-K)^{-1}S(x) = F(x)$ this means that F has w.s.cg.. By appling Theorem 3.2 to mapping F, we obtain $x \in (I-K)^{-1}S(x)$ for $x \in C$, so the proof is complete. □

Corollary 3.4. *Let C be a nonempty, closed, convex and bounded subset of X. Assume $K \in L(X)$ is a w.s.c. mapping and $S : X \to P_{cv}(X)$ is a w.s.u.sc. mapping satisfies condition of Theorem 3.3, (i)-(ii) and*

(iii) S is weakly compact.

Then $S + K$ has at least one fixed point .

Proof. From proof of Theorem 3.3, we have $F : C \to P_{cv}(C)$ by

$$y \to F(y) = (I-K)^{-1}S(y).$$

By (iii) and the continuity of $(I-K)^{-1}$, for the set A defined in Theorem 3.3, we see that the set

$$M = \overline{co}(((I-K)^{-1}S(A)) \cup \{a_o\})$$

is weakly compact and mapping $F : M \to P_{cv}(M)$ is w.s.u.sc.. Thus from [1, Theorem 2.1.], there exists $y^* \in M$ with $y^* \in F(y^*)$. This completes the proof. □

4. APPROXIMATION METHOD FOR KRASNOSEL'SKII TYPE THEOREMS

In this section, we use a recently introduced approximation technique for our Krasnosel'skii type results. First, we give weakly version of the approximation selection theorem [18, Theorem 4.1]. This important result forms the basis of this part.

Lemma 4.1. *Let $(X, \|.\|)$ be a normed space, $(Y, \|.\|)$ a Banach space, and $S : X \to P_{cv}(Y)$ be a w.u.sc. multivalued mapping. Then, for every $\epsilon > 0$, there exists a weakly continuous function $s_\epsilon : X \to Y$ with*

$$s_\epsilon(X) \subseteq coS(X)$$

and

$$\Gamma(s_\epsilon) \subseteq \Gamma(S) + \epsilon B_*,$$

where B_* is the weak unit ball of $X \times Y$.

More information about the weak unit ball can be found in Banakh [3].

Proof. Fix $\epsilon > 0$, for $\forall x \in X$, there is $\delta(x)$ with for any $x^* \in B(x, \delta(x))$ and we have

$$S(x^*) \subset cl\left(S(x) + \frac{\epsilon}{2}B\right).$$

Further, we can take $\delta(x) < \frac{\epsilon}{2}$. Since X is a paracompact space, the family of weak balls $\{B(x, \delta(x)/4)\}_{x \in X}$ of X admits a partition of unity subordinated to $\{B(x, \delta(x)/4)\}$. Let us take a family $\{\lambda_i\}_{i \in I}$ of weakly continuous Lipschitzian function such that $\{Supp\ \lambda_i\}_{i \in I}$, where

$$Supp\lambda_i = cl\{x \in X : \lambda_i \neq 0\}$$

is a locally finite refinement of $\{B(x, \delta(x)/4)\}$, that is for every $i \in I$, there is

$$W_i \in \{B(x, \delta(x)/4)\}$$

such that

$$Supp\lambda_i \subset W_i.$$

Choose for each j an $x_j \in W_i$ with $y_i \in S(x_j)$ and define

$$s_\epsilon(x) = \sum_{i \in I} \lambda_i(x)\ y_i,\ x \in X.$$

Given that for $i \in I$, λ_i is weak continuous, we get s_ϵ is a weakly continuous Lipschitzian function and its range is in the convex hull of the $S(X)$. For $x_j \in W_i$ and $y_i \in S(x_j)$, we can have

$$y_i \in S(x_j) \subset cl\left(S(x) + \frac{\epsilon}{2}B\right).$$

Since set $S(x) + \frac{\epsilon}{2}B$ is convex, we conclude that

$$s_\epsilon \in S(x) + \frac{\epsilon}{2}B.$$

So, we have $y_j \in S(x_j)$ with

$$\|s_\epsilon(x) - y_j\| \leq \frac{\epsilon}{2}.$$

Then

$$d((x, s_\epsilon(x)), (x_j, y_j)) \leq d(x, x_j) + \|s_\epsilon(x) - y_j\| \leq \epsilon.$$

And so, we have $(x, s_\epsilon(x)) \in Gr(S) + \epsilon B$. □

We now give a result whose proof we will model on the Lemma 4.1. This result is the multivalued version of the Krasnosel'skii type theorem presented by Taoudi [25, Theorem 2.1].

Theorem 4.2. *Let C be a bounded, convex nonempty subset of X. Suppose that $S : C \to P_{cv}(X)$ is a w.s.u.sc. mapping and $K \in L(X)$ is a w.s.c. mapping satisfying:*
(i) S is weakly compact,
(ii) K is a contraction,
(iii) $Sx + Ky \subset C$, for $x, y \in C$,
(iv) S has w.s.cg.
Then $S + K$ has at least one fixed point in C.

Proof. Let $\epsilon > 0$ be given. Then, by Lemma 4.1 $s_\epsilon : C \to X$ weakly continuous function such that

$$\Gamma(s_\epsilon) \subseteq \Gamma(S) + \epsilon B_* \tag{4.1}$$

and $s_\epsilon(C) \subset coS(C)$. By (iii) and the convexity of C, we obtain

$$\begin{aligned} K(C) + s_\epsilon(C) &\subseteq K(C) + coS(C) \\ &\subseteq co(K(C) + S(C)) \\ &\subset co(C) = C. \end{aligned} \tag{4.2}$$

For fixed $y \in C$, we consider $H_y(x) : C \to C$ defined by $H_y(x) = K(x) + s_\epsilon(y)$ and for each $x \in C$, $H_y(x)$ defines a contraction. Thus, according to the Banach fixed point principle, the equation $x_\epsilon(y) = K(x_\epsilon(y)) + s_\epsilon(y)$ has a unique solution

$x_\epsilon(y) \in C$. By (ii), it can be shown that $I - K$ is invertible (see [30, Lemma 2.13]) and we can define the operator

$$F_\epsilon(x) = (I - K)^{-1} s_\epsilon(x).$$

By (iii),

$$F_\epsilon(C) \subset C \tag{4.3}$$

Now $M = \overline{co}(F_\epsilon(C))$ be the closed convex hull of $F_\epsilon(C)$. Since C is a convex closed subset of X, it follows from (4.3) that $M \subseteq C$ and thus $F_\epsilon(M) \subseteq F_\epsilon(C) \subseteq \overline{co}(F_\epsilon(C)) = M$. That is F_ϵ maps M into itself. We argue that M is weakly compact. If this is not so, then from (2.2), $\Omega(M) > 0$. Using the

$$F_\epsilon = s_\epsilon + KF_\epsilon \tag{4.4}$$

with Definition 2.1, (1) and (2) we get

$$\Omega(M) = \Omega(\overline{co}(F_\epsilon(C)) = \Omega(F_\epsilon(C)) \leq \Omega(s_\epsilon(C) + KF_\epsilon(C)). \tag{4.5}$$

Consider that C is a finite set. Then

$$s_\epsilon(C) \subset coS(C) \Rightarrow \overline{s_\epsilon(C)} \subset \overline{coS(C).}$$

Since S weakly compact, so is $\overline{s_\epsilon(C)}$ and by (2.2), $\Omega(s_\epsilon(C)) = 0$.Thus using the subadditivity of the De Blasi m.w.nc. we obtain

$$\Omega(M) \leq \Omega(s_\epsilon(C)) + \Omega(KF_\epsilon(C)) = \Omega(KF_\epsilon(C)). \tag{4.6}$$

Then, using $\Omega(s_\epsilon C) = 0$ in (4.5), for $\alpha \in (0, 1]$, it results

$$\Omega(KF_\epsilon(C)) \leq \alpha\Omega(F_\epsilon(C)). \tag{4.7}$$

Now, combining (4.6) and (4.7) we obtained

$$\Omega(M) \leq \alpha\Omega(F_\epsilon(C)) = \alpha\Omega(\overline{co}(F_\epsilon(C)) = \alpha\Omega(M) < \Omega(M).$$

This is a contradiction. So M is weakly compact. Further s_ϵ is weakly continuous and $(I - K)^{-1} \in L(X)$ is w.s.c..Therefore, it is clear that F_ϵ is w.s.c.. In this case, F_ϵ satisfies condition of Arino et. al. fixed point theorem [2, Theorem 1] and so there exists $x_\epsilon \in M$ with

$$x_\epsilon = K(x_\epsilon) + s_\epsilon(x_\epsilon).$$

Now let $(\epsilon_n)_{n\in\mathbb{N}}$ with $\epsilon_n \rightharpoonup 0$ and for each $n \in N$ we choose $x_{\epsilon_n} \in M$ so that $x_{\epsilon_n} = K(x_{\epsilon_n}) + s_\epsilon(x_{\epsilon_n})$. Since M is weakly compact, there exists a subsequence of x_{ϵ_n}weakly converging to some $x \in M$, and so

$$s_\epsilon(x_{\epsilon_n}) = (I - K)(x_{\epsilon_n}) \rightharpoonup (I - K)(x).$$

Then, from (4.1), we have

$$d(x_{\epsilon_n}, s_\epsilon(x_{\epsilon_n})), \Gamma(S)) \leq \epsilon_n.$$

Since S has w.s.cg. in $X \times X$, $(I - K)(x) \in S(x)$. Thus, $x \in K(x) + S(x)$ and the proof is complete. □

For Theorem 4.2, we can give the following result.

Corollary 4.3. *Let C be a nonempty convex and weakly compact subset of X. Suppose that $S : C \to P_{wcp,cv}(X)$ is a w.s.u.sc. mapping and $K \in L(X)$ is a w.s.c. mapping satisfying:*
(i) K is a contraction,
(ii) $Sx + Ky \subset C$, for $x, y \in C$,
(iii) S has w.s.cg.

Then $S+K$ has at least one fixed point in C.

Using Theorem 4.2 we prove the following fixed point theorem.

Theorem 4.4. *Let C be a nonempty convex, bounded, closed subset of X. Suppose that $S : C \to P_{cv}(X)$ is a w.s.u.sc. mapping and $K \in L(X)$ is a w.s.c. mapping satisfying:*
(i) S is weakly compact,
(ii) K is nonexpensive,
(iii) if there exists a sequence $(x_n)_n \subset C$ with $(I-K)\,x_n \rightharpoonup y$; then $\{x_n\}$ has a weakly convergent subsequence,
(iv) if $\lambda \in (0,1)$ and $\lambda Kx + Sy \subset C$, for $x, y \in C$,
(iiv) S has w.s.cg.
Then there is $x \in C$ such that $x \in Sx + Kx$.

Proof. For each $\lambda \in (0,1)$ the mappings S and K fulfill the assumptions of Theorem 4.2. Hence, we get $x \in Sx + \lambda Kx$, for $x \in C$. Now, take a sequence $(\lambda_n)_n \subset (0,1)$ with $\lambda_n \to 1$ and consider the sequence $\{x_n\}$ in (iii), then we can write

$$s_\epsilon x_n + \lambda_n K x_n = x_n, \tag{4.8}$$

where s_ϵ is as defined in the proof of Theorem 4.2. Also, C is a bounded set and $s_\epsilon(C) \subset coS(C) \Rightarrow \overline{s_\epsilon(C)} \subset \overline{coS(C)}$. Since S is weakly compact, so is $\overline{s_\epsilon(C)}$. Using the weakly compactness of $s_\epsilon(C)$, we may assume that $s_\epsilon x_n \rightharpoonup y \in C$. Therefore,

$$(I - \lambda_n K)x_n \rightharpoonup y. \tag{4.9}$$

Since $(x_n)_n \subset C$, then it is norm bounded and so is $\{Kx_n\}$. Hence,

$$\|(x_n - Kx_n) - (x_n - \lambda_n K x_n)\| = (1-\lambda_n)\,\|Kx_n\| \to 0. \tag{4.10}$$

from (4.9) and (4.10),we have

$$x_n - Kx_n \rightharpoonup y.$$

By (iii) we conclude that $\{x_n\}$ has a subsequence $x_{n_k} \rightharpoonup x \in C$. Using the weak sequentially continuity of s_ϵ and K in equation (4.8) we get $s_\epsilon x + Kx = x$. Hence, the proof is complete due to the proof of Theorem 4.2. □

Remark. *Theorem 4.4 is a multivalued version of* [25, Theorem 2.4] *with linearity condition.*

Corollary 4.5. *Let C be a nonempty bounded convex closed subset of a reflexive Banach space X. Suppose that $S : C \to P_{cv}(X)$ is a w.s.u.sc. mapping and $K \in L(X)$ is a w.s.c. mapping satisfying:*
(i) K is nonexpensive,
(ii) if $\lambda \in (0,1)$ and $\lambda Kx + Sy \subset C$, for $x, y \in C$,
(iii) S has w.s.cg.
Then there is $x \in C$ such that $x \in Sx + Kx$.

5. An Application

In this section, we will present an application under conditions similar to the hypotheses in [8, Theorem 5.2].

Let X is a reflexive Banach space and $J = [0, R]$. Consider the integral inclusion

$$x(r) \in K(x(r)) + \int_0^R T(s, x(s))ds, \qquad r \in J, \ R > 0, \tag{5.1}$$

where $K : X \to X, \ T : J \times X \to P_{cv}(X)$. For all $r \in J$ and $x \in X$,

$$\|T(r, x)\| = \sup \{|w|, \ w \in T(r, x)\}.$$

The integral in inclusion (5.1) is the Pettis integral. Now we suppose the following hypotheses are satisfied:

(H_1) The mapping $K \in L(X)$ is w.s.c., K is a contraction and for fix E, $\|K(x(r))\| \leq E$ holds.

(H_2) There exists a weakly continuous, Pettis integrable function $y : J \to X$ with $y(r) \in T(t, x(r))$, where $r \in J$ and $x : J \to X$ is a continuous function.

(H_3) For any $k > 0$, $\psi_k \in L^1(J)$ with $\|T(r, v)\| \leq \psi_k(r)$ for $r \in J$ and $v \in X$ such that $|v| \leq k$. Further, assume that

$$\int_0^R \psi_r(r)ds < k.$$

(H_4) $T(t, .)$ has w.s.cg.

Theorem 5.1. *Assume hypotheses (H_1)-(H_4) are satisfied. Then (5.1) has a solution in $C(J, X)$.*

Proof. Define a subset M of X by

$$M = \{x \in C(J, X) : \|x\| \leq N\},$$

it is obvious that M is a closed, convex and bounded set. X is a reflexive, so one concludes that M is weakly compact. Now we introduce the nonlinear mappings K and S :

$K : C(J, X) \to C(J, X)$ and $S : M \to P_{cv}(C(J, X))$,

$$Sx(r) = \left\{ w(r) = \int_0^r y(s)ds, \quad y(r) \in T(t, x(r)) \text{ and } y \text{ is Pettis integrable} \right\}.$$

We can formulate the inclusion (5.1) as follows:

$$x(r) \in Kx(r) + Sx(r). \tag{5.2}$$

For the proof, we must show that all the conditions of Theorem 4.2 are satisfied. Now, let us show S is w.s.u.sc. and weakly compact.

First, we first show S has a w.s.cg.. Let $x_n \in M$, $x_n \rightharpoonup x$ and $z_n \in Sx_n$ such that $z_n \rightharpoonup z$. Let us take a sequence $(y_n) : J \to X$ with y_n is Pettis integrable mapping for $\forall n \in \mathbb{N}$, $x_n(s) \rightharpoonup x(s)$ and $y_n(s) \in T(s, x_n(s))$ for each $s, r \in J$ there is

$$z_n(r) = \int_0^r y_n(s)ds.$$

From (H_4), there is $y(s) \in T(s, x(s))$ for every $s \in J$ and

$$z(r) = \int_0^r y(s)ds \in Sx(r).$$

Hence, S has w.s.cg.. For weakly compactness of S we show that $S(x)$ is closed, bounded and convex valued for each $x \in M$. Let $w_1, w_2 \in S(x)$ with

$$z(r) = \int_0^r y(s)ds \in Sx(r).$$

From definition of mapping S; y_1, y_2 are Pettis integrable and $y_1, y_2 \in F(s, x(s))$ for $\forall s \in J$. For all $\alpha \in (0, 1)$,

$$y_1(s) + (1-\alpha)y_2(s))\ ds.$$

From $T(r, x(r)) \in P_{cv}(X)$, we have

$$\alpha w_1 + (1-\alpha)w_2 \in S(x),$$

thus $S(x)$ is convex and S has w.s.cg., so S is closed valued. From (H_3), $T(r, x(r))$ is bounded and thus $S(x)$ is bounded for all $x \in M$. We know that X is reflexive, which proves that $S(x)$ is weakly relatively compact. Thus, considering [9, Lemma 2.2] we obtain S is w.s.u.sc.

We prove that the condition $K(x) + S(z) \subset M$, for $x, z \in M$, is satisfied. Let $w(r) \in Sz(r)$ and suppose that the relation $s(r) = K(x(r)) + w(r)$, for all $r \in J$. Now,

$$\|s(r)\| \leq \|K(x(r))\|$$

for every $r \in J$. If we take the supremum in the last inequality, we have $\|s\| \leq N$ and so, $s \in M$. Consequently,

$$K(x) + S(z) \subset M, \quad \text{for } x, z \in M.$$

Hence, the requirements of Theorem 4.2 are fulfilled and so multivalued operator equation (5.2) has a solution. □

6. Conclusions

Multivalued Krasnosel'skii-type theorems attract much attention because they have a wide range of applications in integral (differantial) inclusions. Different methods have been used in the literature to prove such theorems. In the third part of this study, we prove a new multivalued version of the Krasnosell'ski-type theorem using the measure of weak noncompactness technique under appropriate conditions for weak topology in Banach spaces. In the next section, we introduce the approximation method under weak topology properties and prove with this method that (1.2) has a solution in Banach spaces. This method, especially based on an $\epsilon-$approximation, is very useful in solving equations involving multivalued mappings with a single-valued selection. Finally, an application of integral inclusion is presented that explains our theory.

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