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# SOME ASPECTS OF FUNDAMENTAL FORMS OF SURFACES AND THEIR INTERPRETATION

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Abstract. In this paper, we study the first and second fundamental forms of surfaces, exploring their properties as they relate to measuring arc lengths and areas, and identifying isometric surfaces. These forms can be used to define the Gaussian curvature, which is, unlike the first and second fundamental forms, independent of the parametrization of the surfaces. Also, we investigate the geometric behaviour of rectifying curves on regular surfaces under conformal and isometric transformation by using the concept of fundamental forms of surfaces.

## 1. Introduction

For many years, surface theory has been a popular topic for many researchers in many aspects. Calculus is used to study the properties of curves and surfaces of all types in differential geometry. It mainly concentrates on the characteristics of a limited subset of geometric surfaces and curves configurations. In differential geometry, various kinds of curves are investigated, but regular curves are the most important ones. The number of continuous derivatives is a trait that reveals the smoothness of the curve. A curve is regarded as smooth if it is differentiable and consequently continuous everywhere. A curve is referred to as being regular if it can be differentiated and has no zero derivative.

The investigation of regular maps is a crucial field of study in differential geometry. For more information on the regular curve, we refer the readers to see [1, 4, 5, 20]. There are many ways to categorize motions, but we'll concentrate on the ones that preserve particular geometrical characteristics. We categorize transformations generally into the following equivalence classes: conformal, isometric, homothetic, and non-conformal or general motion, depending on the varying nature of the mean curvature (M) and the Gaussian curvature (G).

In isometry, lengths, and angles between curves on surfaces are both preserved. In terms of geometry, isometry preserves the Gaussian curvature's invariance, while

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altering the mean curvature. The isometry between a helicoid and a catenoid, which suggests that they have the same G but different M, is one of the best-known examples. The most significant transformation is a conformal transformation, which preserves angles in terms of magnitude and direction but not always in terms of length. Conformal maps play a significant role in cartography. The stereographic projection, which maps a sphere onto a plane, is the most typical example of conformal transformation. To create the renowned Mercator's world map, the first conformal (angle preserving) world map, Gerardus Mercator first employed this conformal map attribute in the year 1569. We suggest that the readers watch an animated movie on conformal maps that was released by Bobenko and Gunn in 2018 along with Springer VideoMATH for additional details regarding the application of conformal maps [8]. Angles and distances between any pair of intersecting curves are not preserved in the context of general motion. The use of motion, transformation, and maps is for same throughout the paper.

Let $P$ and $\tilde{P}$ be two smooth and regular immersed surfaces in the Euclidean space $\mathbb{R}^3$, and $G : P \to \tilde{P}$ be a smooth map. A necessary and sufficient condition for $G$ to be conformal is that the first fundamental form quantities are proportional. In other words, the area element of $P$ and $\tilde{P}$ are proportional to a differentiable function (factor), which is denoted by $\zeta(x, y)$ and is commonly known as the dilation function. For more information on the dilation function, we refer the readers to see [2, 3, 6, 7, 9, 10, 16, 17, 21]. A generalized class of certain motions is the conformal transformation, which is defined in the following way [9, 13, 17, 18, 21]:

- If the dilation factor $\zeta(x, y) = c$, where $c$ is a constant with $c \neq \{0, 1\}$, then G is a homothetic transformation.
- If the dilation function $\zeta(x, y) = 1$, then G becomes isometry.

We begin our study by examining two properties of surfaces in $\mathbb{R}^3$, called the first and second fundamental forms. For more information about these fundamental forms of surfaces, one can refer [19, 20].

The format of this paper is as follows: Section 2 covers some fundamental definitions and information about the dilation function, geodesic curvature, normal curvature, rectifying curves, and normal curves. Moreover, the first and second fundamental forms of surfaces are also included in this section. Section 3 deals with the understanding of rectifying curves on regular surfaces and their conformal image under various transformations. We also examine the major results in this section.

## 2. Preliminaries

This section includes some essential information on rectifying and normal curves, including their first fundamental form, geodesic and normal curvature, and some basic definitions. We present some known definitions and results for ready reference to go through the work presented in this paper.

Let $\delta : I \subset \mathbb{R} \to \mathbb{R}^3$ be a smooth parametrized curve having unit speed, at least a fourth-order continuous derivative, and an arc length (r). Then the tangent, normal, and binormal of the curve $\delta$ is denoted by $\vec{t}$, $\vec{n}$ and $\vec{b}$ respectively. At each point on the curve $\delta(r)$, the vectors $\vec{t}$, $\vec{n}$ and $\vec{b}$ are mutually perpandicular to each other and so the triplet $\{\vec{t}, \vec{n}, \vec{b}\}$ forms an orthonormal frame.

Consider $\vec{t'}(r) \neq 0$, if the unit normal vector $\vec{n}$ along the tangent at a point on the curve $\delta$, then we can write $\vec{t'}(r) = \kappa(r)\vec{n}(r)$, where $\vec{t'}(r)$ is the derivative of $\vec{t}$ with respect to arc length parameter 'r' and $\kappa(r)$ is the curvature of $\delta(r)$. Also the binormal vector field is denoted by $\vec{b}$ and is defined by $\vec{b} = \vec{t} \times \vec{n}$, and we can write $\vec{b'}(r) = \tau(r)\vec{n}(r)$, where $\tau(r)$ is another curvature function known as torsion of the curve $\delta(r)$.
In [3, 4, 11, 15], Serret-Frenet equations are given as follows:

$$\begin{aligned} \vec{t'}(r) &= \kappa(r)\vec{n}(r), \\ \vec{n'}(r) &= -\kappa(r)\vec{t}(r) + \tau(r)\vec{b}(r), \\ \vec{b'}(r) &= -\tau(r)\vec{n}(r), \end{aligned}$$

where the functions $\kappa$ and $\tau$ are respectively called the curvature and torsion of the curve $\delta$, satisfying the following conditions:

$$\vec{t}(r) = \delta'(\mathrm{r}),\ \vec{n}(r) = \frac{\vec{t'}(r)}{\kappa(r)} \text{ and } \vec{b}(r) = \vec{t}(r) \times \vec{n}(r).$$

From the arbitrary point $\delta(r)$ on the curve $\delta$, the plane spanned by $\{\vec{t}, \vec{n}\}$ is called the osculating plane and the plane spanned by $\{\vec{t}, \vec{b}\}$ is called the rectifying plane. In the same way, the plane spanned by $\{\vec{n}, \vec{b}\}$ is called the normal plane. Whenever we talk about the position vector of the curve, which defines the different kinds of curves [12, 16, 17] :

- If a curve's position vector lies in the normal plane, it can be characterized as a normal curve.
- If a curve's position vector lies in the rectifying plane, it can be characterized as a rectifying curve.
- If a curve's position vector is in the osculating plane, then the curve is said to be an osculating curve.

Due to their prevalence, normal, rectifying, and osculating curves are often covered in every standard book on differential geometry of curves and surfaces. For more information, we refer the readers to see [1, 2, 3, 6, 11]. In the Euclidean 3-dimensional space $\mathbb{R}^3$, Chen [5, 11] introduced the concept of the motion of rectifying curves and investigated some of the properties of such curves. Shaikh and Ghosh in [2, 3, 7, 14, 18] investigated the sufficient conditions for osculating and rectifying curves on smooth surfaces to remain invariants under isometry of surfaces. In the year 2003, Chen [5] came across the following query regarding rectifying curves: *What happens if a space curve's position vector is always located in its rectifying plane*, and found that the component of a space curve's position vector along the surface normal stays invariant when surfaces are isometric.

The main goal of this paper is to expand on the work of Lone et al. [5, 12, 13], where they studied the geometric invariants of normal curves under conformal transformation in $\mathbb{E}^3$. In [5], the author investigated the invariant properties of normal curves under conformal transformation and also studied the normal and tangential components of the normal curves under the same motion. By drawing inspiration from Shaikh and Ghosh's work [2, 6], they investigated the geometric invariants characteristics of rectifying curves on smooth surfaces under the isometry of the surfaces.

Moreover in [5], authors also investigated the invariant properties of osculating curves under the isometry of surfaces. But a natural question arises: what happens with the geometric properties of a rectifying curve with respect to the isometry, and properties of first and second fundamental forms of surfaces in the Euclidean space $\mathbb{R}^3$.

In the present article, we introduce the concept of fundamental forms of surfaces and compute some of their geometric properties. Moreover, we also studied the concept of some of the important curves in spaces namely rectifying curves, and also investigated the invariant sufficient condition for the conformal image of such curve on a regular surface under the conformal and homothetic transformations in $\mathbb{R}^3$. Firstly, try to investigate the following:
*What aspects of a rectifying curve on a smooth immersed surface remain unchanged after conformal transformation.*

If the position vector of a curve is located in the rectifying plane, then the curve is said to be a rectifying curve [8, 12, 15], i.e.,

$$\delta(r) \quad = \quad \mu_1(r)\vec{t}(r) + \mu_2(r)\vec{b}(r), \tag{2.1}$$

where $\mu_1$ and $\mu_2$ are two smooth functions.
Let $\sigma \ : \ U \to P$ be the coordinate chart map on the regular surface $P$ and the smooth parametrized unit speed curve $\delta(r) \ : \ I \to P$, where $I \ = \ (a \ , \ b) \subset \mathbb{R}$ and $U \subset \mathbb{R}^2$.
As a result, the curve $\delta(r)$ is given by

$$\delta(r) \quad = \quad \sigma(x(r), y(r)). \tag{2.2}$$

By using the chain rule to differentiate (2.2), with respect to $r$, we get

$$\delta'(r) \quad = \quad \sigma_x x' + \sigma_y y'. \tag{2.3}$$

Now, $\vec{t}(r) = \delta'(r)$. Then, from equation (2.3), we find that

$$\vec{t}(r) \quad = \quad \sigma_x x' + \sigma_y y'. \tag{2.4}$$

When we differentiate equation (2.4) again in terms of $r$, we get

$$\vec{t}'(r) = x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy}.$$

If $\mathbb{N}$ is the normal to the surface $P$ and $\kappa(r)$ is the curvature of the curve $\delta(r)$, then the normal vector $\vec{n}(r)$ can be written as

$$\vec{n}(r) \quad = \quad \frac{1}{\kappa(r)}(x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy}). \tag{2.5}$$

Now the binormal vector $\vec{b}(r)$ can be written as

$$\vec{b}(r) \ = \ \vec{t}(r) \times \vec{n}(r).$$

By substituting the value of $\vec{t}(r)$ and $\vec{n}(r)$ from equation (2.4) and (2.5) we obtained

$$\begin{aligned}\vec{b}(r) &= \frac{1}{\kappa(r)}[(\sigma_x x' + \sigma_y y') \times (x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy})],\\ &= \frac{1}{\kappa(r)}[(y''x' - y'x'')N + x'^3\sigma_x \times \sigma_{xx} + 2x'^2y'\sigma_x \times \sigma_{xy} + x'y'^2\sigma_x \times \sigma_{yy}\\ &\quad + x'^2y'\sigma_y \times \sigma_{xx} + 2x'y'^2\sigma_y \times \sigma_{xy} + y'^3\sigma_y \times \sigma_{yy}]. \qquad (2.6)\end{aligned}$$

**Definition 2.1.** [4] *Let $P$ and $\tilde{P}$ be two regular surfaces in the Euclidean surface $\mathbb{R}^3$ and $\delta(r)$ be an arc length parametrized curve lying on the surface $P$. Then $\delta'(r)$ is perpandicular to the unit surface normal $\mathbb{N}$, and also $\delta'(r)$ and $\delta''(r)$ are perpandicular. Thus, $\delta''$ can be represented as the linear combination of $\mathbb{N}$ and $\mathbb{N}\times\delta'$, i.e.,*

$$\delta'' = \kappa_n\mathbb{N} + \kappa_g\mathbb{N}\times\delta',$$

*where the parameters $\kappa_n$ and $\kappa_g$, which are commonly known as the normal and geodesic curvatures of the curve $\delta$, and are given by*

$$\begin{aligned}\kappa_n &= \delta''\cdot\mathbb{N},\\ \kappa_g &= \delta''\cdot(\mathbb{N}\times\delta').\end{aligned}$$

**Definition 2.2.** [3] *If there exists a neighborhood $V$ of $G(q) \in \tilde{P}$ such that $G : U \to V$ is an isometry, then the diffeomorphism $G : U \subset P \to \tilde{P}$ of the neighborhood $U$ of a point 'q' in $P$ is referred to as a local isometry at 'q'. The surface $P$ and $\tilde{P}$ are said to be locally isometric when the local isometry exists at every point of $P$. In general, $G$ is said to as global isometry if it is local isometry at each point of the surface $P$.*

**Definition 2.3.** [4] *A smooth surface in $\mathbb{R}^3$ is a subset $P \subset \mathbb{R}^3$ such that each point has a neighbourhood $U \subset P$ and a map $\sigma : V \to \mathbb{R}^3$ from an open set $V \subset \mathbb{R}^2$ such that*

- *$\sigma : V \to U$ is a homeomorphism. This means that $\sigma$ is a bijection that continuously maps $V$ into $U$ and that the inverse function $\sigma^{-1}$ exists and is continuous.*
- *$\sigma(x, y) = (u(x, y), v(x, y), w(x, y))$ has derivatives of all orders.*
- *At all points, the first partial derivatives $\sigma_x = \frac{\partial\sigma}{\partial x}$ and $\sigma_y = \frac{\partial\sigma}{\partial y}$ are linearly independent.*

2.1. **First Fundamental Form.** The study and classification of surface shape are made easier with knowledge of a surface's first fundamental form I and second fundamental form II. The first fundamental form of a surface provides information about the local geometry of the surface. It defines the metric properties of the surface and allows us to calculate arc lengths and angles on the surface. We can determine properties such as curvature and geodesic paths by measuring the lengths and angles of curves on the surface. Regarding the investigation of the intrinsic and extrinsic geometric features of the surfaces, the geometry of the second fundamental form II in particular has grown to be a significant concern. In [18, 21], there are numerous results relating to the curvature characteristics related to I, II, and other variational features.

Let $\sigma = \sigma(x, y)$ represent the equation of a surface. Let us define $E = (\sigma_x \cdot \sigma_x)$, $F = (\sigma_x \cdot \sigma_y)$, $G = (\sigma_y \cdot \sigma_y)$. Then the expression $Edx^2 + 2Fdxdy + Gdy^2$ is called the first fundamental form, and $E$, $F$, and $G$ are called the first fundamental form coefficient or the first fundamental form magnitude. Note that in the above expression, $dx$ and $dy$ cannot vanish together. We denote notion of $\sqrt{EG - F^2}$ by $H$. The first fundamental form can be written as the square of the metric. From this expression, we can see very clearly that the first fundamental form is a quadratic form. It is also a bilinear form, though we only examine it as a quadratic form here. Furthermore, its arguments come from the tangent space, so it is an inner product of the tangent space. We can calculate the first fundamental form for different surfaces. For example, in the case of the sphere we have the equation for the sphere given below:

$$\sigma(x, y) = \text{a} \sin(\text{x}) \sin(\text{y})\mathbf{i} + \text{a} \cos(\text{x}) \sin(\text{y})\mathbf{j} + \text{a} \cos(\text{y})\mathbf{k}.$$

Therefore,

$$\sigma_x = \text{a} \cos(\text{x}) \sin(\text{y})\mathbf{i} - \text{a} \sin(\text{x}) \sin(\text{y})\mathbf{j} \text{ and } \sigma_y = \text{a} \sin(\text{x}) \cos(\text{y})\mathbf{i} + \text{a} \cos(\text{x}) \cos(\text{y})\mathbf{j} - \text{a} \sin(\text{y})\mathbf{k}.$$

Now, $E = (\sigma_x \cdot \sigma_x) = \text{a}^2 \sin^2\text{y}$, $F = (\sigma_x \cdot \sigma_y) = 0$ and $G = (\sigma_y \cdot \sigma_y) = \text{a}^2$.
Then the first fundamental form for sphere is $\text{a}^2 \sin^2 dx^2 + \text{a}^2 dy^2$. For more information on its fundamental form, we can refer the readers to see [19, 20]. Some of the main results concerning the first fundamental form are given as follows:

**Theorem 2.1.** *The expression for the ist fundamental form of surfaces is invariant under the transformation of parameters.*

*Proof.* Let $\vec{\sigma} = \vec{\sigma}(x, y)$ be the equation of a surface, where x and y are parameters. Let the parameters $x$ and $y$ be transformed to other parameters $x^*$ and $y^*$ given by the relation $x^* = x^*(x, y)$ and $y^* = y^*(x, y)$.
On solving these we can find the relations $x = x(x^*, y^*)$ and $y = y(x^*, y^*)$. Now,

$$\begin{aligned} \vec{\sigma}^*_x &= \frac{\partial \vec{\sigma}}{\partial x^*} \\ &= \frac{\partial \vec{\sigma}}{\partial x}\frac{\partial x}{\partial x^*} + \frac{\partial \vec{\sigma}}{\partial y}\frac{\partial y}{\partial x^*} \\ &= \vec{\sigma}_x \frac{\partial x}{\partial x^*} + \vec{\sigma}_y \frac{\partial y}{\partial x^*}. \end{aligned} \tag{2.7}$$

On a similar pattern, we obtain

$$\begin{aligned} \vec{\sigma}^*_y &= \frac{\partial \vec{\sigma}}{\partial y^*} \\ &= \frac{\partial \vec{\sigma}}{\partial x}\frac{\partial x}{\partial y^*} + \frac{\partial \vec{\sigma}}{\partial y}\frac{\partial y}{\partial y^*} \\ &= \vec{\sigma}_x \frac{\partial x}{\partial y^*} + \vec{\sigma}_y \frac{\partial y}{\partial y^*}. \end{aligned} \tag{2.8}$$

Also,

$$dx = \frac{\partial x}{\partial x^*}dx^* + \frac{\partial x}{\partial y^*}dy^*$$

and

$$dy = \frac{\partial y}{\partial x^*}dx^* + \frac{\partial y}{\partial y^*}dy^*.$$

Also, in $(x, y)$ parametric system, the first fundamental form is $Edx^2 + 2Fdxdy + Gdy^2$. Now in $(x^*, y^*)$ parametric system, the first fundamental form is given by the expression $E^*dx^{*2} + 2F^*dx^*dy^* + G^*dy^{*2}$. Now,

$$\begin{aligned}
E^*dx^{*2} + 2F^*dx^*dy^* + G^*dy^{*2} &= (\vec{\sigma}_x^* \cdot \vec{\sigma}_x^*)dx^{*2} + 2(\vec{\sigma}_x^* \cdot \vec{\sigma}_y^*)dx^*dy^* + (\vec{\sigma}_y^* \cdot \vec{\sigma}_y^*)dy^{*2}, \\
&= (\vec{\sigma}_x^*dx^* + \vec{\sigma}_y^*dy^*) \cdot (\vec{\sigma}_x^*dx^* + \vec{\sigma}_y^*dy^*), \\
&= (\vec{\sigma}_x^*dx^* + \vec{\sigma}_y^*dy^*)^2, \\
&= [(\vec{\sigma}_x\frac{\partial x}{\partial x^*} + \vec{\sigma}_y\frac{\partial y}{\partial x^*})dx^* + (\vec{\sigma}_x\frac{\partial x}{\partial y^*} + \vec{\sigma}_y\frac{\partial y}{\partial y^*})dy^*]^2, \\
&= [\vec{\sigma}_x(\frac{\partial x}{\partial x^*}dx^* + \frac{\partial x}{\partial y^*}dy^*) + \vec{\sigma}_y(\frac{\partial y}{\partial x^*}dx^* + \frac{\partial y}{\partial y^*}dy^*)]^2, \\
&= [\vec{\sigma}_xdx + \vec{\sigma}_ydy]^2, \\
&= (\vec{\sigma}_xdx + \vec{\sigma}_ydy) \cdot (\vec{\sigma}_xdx + \vec{\sigma}_ydy), \\
&= (\vec{\sigma}_x \cdot \vec{\sigma}_x)dx^2 + 2(\vec{\sigma}_x \cdot \vec{\sigma}_y)dxdy + (\vec{\sigma}_y \cdot \vec{\sigma}_y)dy^2, \\
&= Edx^2 + 2Fdxdy + Gdy^2.
\end{aligned}$$

This shows that the expression for the first fundamental form of surfaces is invariant under the transformation of parameters. □

2.2. **Second Fundamental Form.** The second fundamental form of a surface describes how the surface curves within its ambient space. It provides information about the curvature and shape of the surface. By examining the second fundamental form, we can identify important geometric properties of the surface, such as the presence of umbilical points or regions of positive or negative curvature. One significant advantage of the Gaussian curvature is its independence from the parametrization of the surface. While the first and second fundamental forms may vary depending on how the surface is parametrized, the Gaussian curvature remains invariant. This makes it a valuable tool for understanding the intrinsic geometry of a surface.

Let $\vec{\sigma} = \vec{\sigma}(x, y)$ be the equation of a surface, where x and y are parameters and $\vec{N}$ be unit normal to the surface at point $P(\vec{\sigma})$. Then we can write $\vec{N} = \frac{\vec{\sigma}_x \times \vec{\sigma}_y}{|\vec{\sigma}_x \times \vec{\sigma}_y|} = \frac{\vec{\sigma}_x \times \vec{\sigma}_y}{H}$. Let us define $d_{xx} = L = \vec{\sigma}_{xx} \cdot \vec{N}$, $d_{xy} = M = \vec{\sigma}_{xy} \cdot \vec{N}$, and $d_{yy} = N = \vec{\sigma}_{yy} \cdot \vec{N}$. Then the expression $Ldx^2 + Mdxdy + Ndy^2$ is called the second fundamental form of the surfaces, and $L$, $M$, $N$ are called the second fundamental coefficient or second fundamental magnitudes of the surfaces. Note that both $dx$ and $dy$ cannot vanish together. For more information on it and the second fundamental form of the surfaces, we refer the reader to see [1, 19]. The coefficient $L$, $M$ and $N$ are the projection of $\vec{\sigma}_{xx}$, $\vec{\sigma}_{xy}$ and $\vec{\sigma}_{yy}$ respectively on the normal vector. We denote the notion for $\sqrt{LN - M^2}$ by $T$.

From the geometrical point of view in [19], the expression of the second fundamental form of surfaces is twice the length of perpendicular as far as the terms of second order, on the tangent plane to a surface at two neighboring points. As we know that $\vec{N} = \frac{\vec{\sigma}_x \times \vec{\sigma}_y}{H}$. It means that $\vec{N} \perp \vec{\sigma}_x$ and $\vec{N} \perp \vec{\sigma}_y$. In [1, 20], a vector of constant magnitude is perpendicular to its derivative vector. Being a vector of constant magnitude, we can write $\vec{N} \perp \vec{N}_x$ and $\vec{N} \perp \vec{N}_y$. Thus, we can conclude that $\vec{N}_x$, $\vec{N}_y$ are coplanar with $\vec{\sigma}_x$ and $\vec{\sigma}_y$. By using these concepts we find the values in terms of $\vec{N}_x$, $\vec{N}_y$ in the following results;

**Theorem 2.2.** *Let $L$, $M$ and $N$ be the second fundamental magnitudes of surfaces, and $\vec{N}_x$ and $\vec{N}_y$ be the derivative of unit normal $\vec{N}$ with respect to $x$ and $y$ respectively. Then*

(i) *The values of $L$, $M$ and $N$ in term of $\vec{N}_x$ and $\vec{N}_y$ i.e., $L = -\vec{N}_x \cdot \vec{\sigma}_x$, $M = -\vec{N}_y \cdot \vec{\sigma}_x = -\vec{N}_x \cdot \vec{\sigma}_y$, and $N = -\vec{N}_y \cdot \vec{\sigma}_y$.*

(ii) $H\vec{N} \times \vec{N}_x = M\vec{\sigma}_x - L\vec{\sigma}_y$.

(iii) $H\vec{N} \times \vec{N}_y = N\vec{\sigma}_x - M\vec{\sigma}_y$.

*Proof.* (i) We know that $\vec{N} = \frac{\vec{\sigma}_x \times \vec{\sigma}_y}{H}$. It means that $\vec{N} \perp \vec{\sigma}_x$ and $\vec{N} \perp \vec{\sigma}_y$. Therefore,

$$\vec{N} \cdot \vec{\sigma}_x = 0, \tag{2.9}$$

$$\vec{N} \cdot \vec{\sigma}_y = 0. \tag{2.10}$$

Differentiating (2.9) with respect to $x$ and $y$ respectively, we obtained

$$\begin{aligned} &\vec{N} \cdot \vec{\sigma}_{xx} + \vec{N}_x \cdot \vec{\sigma}_x = 0, \text{ and } \vec{N} \cdot \vec{\sigma}_{xy} + \vec{N}_y \cdot \vec{\sigma}_x = 0 \\ &\Rightarrow L + \vec{N}_x \cdot \vec{\sigma}_x = 0, \text{ and } M + \vec{N}_y \cdot \vec{\sigma}_x = 0, \\ &\Rightarrow L = -\vec{N}_x \cdot \vec{\sigma}_x, \text{ and } M = -\vec{N}_y \cdot \vec{\sigma}_x. \end{aligned}$$

Again differentiating (2.10) with respect to $x$ and $y$ respectively, we obtained

$$\begin{aligned} &\vec{N} \cdot \vec{\sigma}_{yx} + \vec{N}_x \cdot \vec{\sigma}_y = 0, \text{ and } \vec{N} \cdot \vec{\sigma}_{yy} + \vec{N}_y \cdot \vec{\sigma}_y = 0 \\ &\Rightarrow M + \vec{N}_x \cdot \vec{\sigma}_y = 0, \text{ and } N + \vec{N}_y \cdot \vec{\sigma}_y = 0, \\ &\Rightarrow M = -\vec{N}_x \cdot \vec{\sigma}_y, \text{ and } N = -\vec{N}_y \cdot \vec{\sigma}_y. \end{aligned}$$

This proves (i).

(ii) Since, we have $\vec{N} = \frac{\vec{\sigma}_x \times \vec{\sigma}_y}{H}$ implies that

$$H\vec{N} = \vec{\sigma}_x \times \vec{\sigma}_y. \tag{2.11}$$

Differentiating equation (2.11) with respect to $x$, we get

$$H\vec{N}_x + H_x\vec{N} = \vec{\sigma}_x \times \vec{\sigma}_{xy} + \vec{\sigma}_{xx} \times \vec{\sigma}_y.$$

Taking cross product with $\vec{N}$, we get

$$\begin{aligned} \vec{N} \times H\vec{N}_x + \vec{N} \times H_x\vec{N} &= \vec{N} \times (\vec{\sigma}_x \times \vec{\sigma}_{xy}) + \vec{N} \times (\vec{\sigma}_{xx} \times \vec{\sigma}_y), \\ \Rightarrow \quad H(\vec{N} \times \vec{N}_x) + H_x(\vec{N} \times \vec{N}) &= [(\vec{N} \cdot \vec{\sigma}_{xy})\vec{\sigma}_x - (\vec{N} \cdot \vec{\sigma}_x)\vec{\sigma}_{xy}] + [(\vec{N} \cdot \vec{\sigma}_y)\vec{\sigma}_{xx} - (\vec{N} \cdot \vec{\sigma}_{xx})\vec{\sigma}_y], \\ \Rightarrow \quad H(\vec{N} \times \vec{N}_x) &= [(\vec{N} \cdot \vec{\sigma}_{xy})\vec{\sigma}_x - (\vec{N} \cdot \vec{\sigma}_{xx})\vec{\sigma}_y], \\ \Rightarrow \quad H(\vec{N} \times \vec{N}_x) &= M\vec{\sigma}_x - L\vec{\sigma}_y. \end{aligned}$$

This proves (ii).

(iii) Now, differentiating equation (2.11) with respect to $y$, we get

$$H\vec{N}_y + H_y\vec{N} = \vec{\sigma}_x \times \vec{\sigma}_{yy} + \vec{\sigma}_{xy} \times \vec{\sigma}_y.$$

Taking cross product with $\vec{N}$, we get

$$\begin{aligned} \vec{N}\times H\vec{N}_y + \vec{N}\times H_y\vec{N} &= \vec{N}\times(\vec{\sigma}_x\times\vec{\sigma}_{yy}) + \vec{N}\times(\vec{\sigma}_{xy}\times\vec{\sigma}_y), \\ \Rightarrow \quad H(\vec{N}\times\vec{N}_y) + H_y(\vec{N}\times\vec{N}) &= [(\vec{N}\cdot\vec{\sigma}_{yy})\vec{\sigma}_x - (\vec{N}\cdot\vec{\sigma}_x)\vec{\sigma}_{yy}] + [(\vec{N}\cdot\vec{\sigma}_y)\vec{\sigma}_{xy} - (\vec{N}\cdot\vec{\sigma}_{xy})\vec{\sigma}_y], \\ \Rightarrow \quad H(\vec{N}\times\vec{N}_y) &= [(\vec{N}\cdot\vec{\sigma}_{yy})\vec{\sigma}_x - (\vec{N}\cdot\vec{\sigma}_{xy})\vec{\sigma}_y], \\ \Rightarrow \quad H(\vec{N}\times\vec{N}_y) &= N\vec{\sigma}_x - M\vec{\sigma}_y. \end{aligned}$$

This proves (iii).

□

## 3. Conformal image of a rectifying curve

Consider two regular surfaces $P$ and $\tilde{P}$ in the Euclidean surface $\mathbb{R}^3$ and $\delta(r)$ is a rectifying curve that is located on the surface $P$. Then $\delta(r)$ can be written as:

$$\delta(r) = \mu_1(r)\vec{t}(r) + \mu_2(r)\vec{b}(r).$$

Now using equations (2.4) and (2.6) we get,

$$\begin{aligned} \delta(r) = \; & \mu_1(r)(\sigma_x x' + \sigma_y y') + \mu_2(r)\frac{1}{\kappa(r)}\{(x'y'' - x''y')\mathbb{N} + x'^3\sigma_x\times\sigma_{xx} \\ & + 2x'^2y'\sigma_x\times\sigma_{xy} + x'y'^2\sigma_x\times\sigma_{yy} + x'^2y'\sigma_y\times\sigma_{xx} + 2x'y'^2\sigma_y\times\sigma_{xy} \\ & + y'^3\sigma_y\times\sigma_{yy}\}. \end{aligned} \tag{3.1}$$

In the following theorem, we will take into account the equation $G_*(\delta(r))$, which results from the product of the $3\times 3$ matrix $G_*$ and the $3\times 1$ matrix $\delta(r)$[11].

**Theorem 3.1.** *Let $P$ and $\tilde{P}$ be two regular surfaces in the Euclidean surface $\mathbb{R}^3$ and $G : P \to \tilde{P}$ is a conformal map. Let $\delta(r)$ be a rectifying curve on the surface $P$. Then $\tilde{\delta}(r)$ is also a rectifying curve on the surface $\tilde{P}$ if*

$$\begin{aligned} \tilde{\delta}(r) = \; & \zeta G_*(\delta(r)) + \frac{\mu_2(r)}{\kappa(r)}\{x'^3\zeta G_*\sigma_x\times(\zeta G_*)_x\sigma_x + 2x'^2y'\zeta G_*\sigma_x\times(\zeta G_*)_y\sigma_x \\ & + x'y'^2\zeta G_*\sigma_x\times(\zeta G_*)_y\sigma_y + x'^2y'\zeta G_*\sigma_y\times(\zeta G_*)_x\sigma_x + 2x'y'^2\zeta G_*\sigma_y \\ & \times(\zeta G_*)_x\sigma_y + y'^3\zeta G_*\sigma_y\times(\zeta G_*)_y\sigma_y\}. \end{aligned} \tag{3.2}$$

*Proof.* Given that $\tilde{P}$ is the conformal image of $P$ under the map $G$ and $\delta(r)$ is a rectifying curve on $P$. Let $\sigma(x,y)$ and and $\tilde{\sigma}(x,y)$ be the surface patches of $P$ and $\tilde{P}$ respectively, and $\tilde{\sigma}(x,y) = G\circ\sigma(x,y)$. Then, each tangent space vector $T_p(P)$ is sent to a dilated tangent vector of the tangent space of $T_{G(p)}(\tilde{P})$ with the dilation function $\zeta$ by the differential map $dG = G_*$ of $G$. We shall show that $\tilde{\delta}(r)$ is a rectifying curve on $\tilde{P}$.

$$\tilde{\sigma}_x(x,y) = \zeta(x,y)G_*(\sigma(x,y))\sigma_x, \tag{3.3}$$
$$\tilde{\sigma}_y(x,y) = \zeta(x,y)G_*(\sigma(x,y))\sigma_y. \tag{3.4}$$

When we partially differentiate equations (3.3) and (3.4) with respect to both x and y, we obtain

$$\begin{aligned}
\tilde{\sigma}_{xx} &= \zeta_x G_* \sigma_x + \zeta \frac{\partial G_*}{\partial x} \sigma_x + \zeta G_* \sigma_{xx}, \\
\tilde{\sigma}_{yy} &= \zeta_y G_* \sigma_y + \zeta \frac{\partial G_*}{\partial y} \sigma_y + \zeta G_* \sigma_{yy}, \\
\tilde{\sigma}_{xy} &= \zeta_x G_* \sigma_y + \zeta \frac{\partial G_*}{\partial x} \sigma_y + \zeta G_* \sigma_{xy}, \\
\tilde{\sigma}_{yx} &= \zeta_y G_* \sigma_x + \zeta \frac{\partial G_*}{\partial y} \sigma_x + \zeta G_* \sigma_{yx}.
\end{aligned} \tag{3.5}$$

Since $\sigma$ is a differentiable curve, so for the sake of simplicity, we take $\tilde{\sigma}_{xy} = \tilde{\sigma}_{yx}$ and $\sigma_{xy} = \sigma_{yx}$.
Now,

$$\begin{aligned}
\zeta G_* \sigma_x \times (\zeta G_*)_x \sigma_x &= \zeta G_* \sigma_x \times (\zeta_x G_* \sigma_x + \zeta \frac{\partial G_*}{\partial x} \sigma_x), \\
&= \zeta G_* \sigma_x \times (\zeta_x G_* \sigma_x + \zeta \frac{\partial G_*}{\partial x} \sigma_x + \zeta G_* \sigma_{xx}) - \zeta G_* (\sigma_x \times \sigma_{xx}), \\
&= \tilde{\sigma}_x \times \tilde{\sigma}_{xx} - \zeta G_* (\sigma_x \times \sigma_{xx}). \qquad (3.6)
\end{aligned}$$

On a similar pattern, we find

$$\begin{aligned}
\zeta G_* \sigma_x \times (\zeta G_*)_y \sigma_x &= \tilde{\sigma}_x \times \tilde{\sigma}_{xy} - \zeta G_* (\sigma_x \times \sigma_{xy}), \\
\zeta G_* \sigma_x \times (\zeta G_*)_y \sigma_y &= \tilde{\sigma}_x \times \tilde{\sigma}_{yy} - \zeta G_* (\sigma_x \times \sigma_{yy}), \\
\zeta G_* \sigma_y \times (\zeta G_*)_x \sigma_x &= \tilde{\sigma}_y \times \tilde{\sigma}_{xx} - \zeta G_* (\sigma_y \times \sigma_{xx}), \\
\zeta G_* \sigma_y \times (\zeta G_*)_x \sigma_y &= \tilde{\sigma}_y \times \tilde{\sigma}_{xy} - \zeta G_* (\sigma_y \times \sigma_{xy}), \\
\zeta G_* \sigma_y \times (\zeta G_*)_y \sigma_y &= \tilde{\sigma}_y \times \tilde{\sigma}_{yy} - \zeta G_* (\sigma_y \times \sigma_{yy}).
\end{aligned} \tag{3.7}$$

Now, from the equation (3.1), (3.2), (3.3), (3.4), (3.6) and (3.7), we obtained that

$$\begin{aligned}
\tilde{\delta}(r) &= \zeta G_*(\mu_1(r)(\sigma_x x' + \sigma_y y') + \mu_2(r)\frac{1}{\kappa(r)}\{(x'y'' - x''y')\mathbb{N} + x'^3\sigma_x \times \sigma_{xx} \\
&\quad +2x'^2y'\sigma_x \times \sigma_{xy} + x'y'^2\sigma_x \times \sigma_{yy} + x'^2y'\sigma_y \times \sigma_{xx} + 2x'y'^2\sigma_y \times \sigma_{xy} \\
&\quad +y'^3\sigma_y \times \sigma_{yy}\}) + \frac{\mu_2(r)}{\kappa(r)}\{x'^3\zeta G_*\sigma_x \times (\zeta G_*)_x\sigma_x + 2x'^2y'\zeta G_*\sigma_x \\
&\quad \times(\zeta G_*)_y\sigma_x + x'y'^2\zeta G_*\sigma_x \times (\zeta G_*)_y\sigma_y + x'^2y'\zeta G_*\sigma_y \times (\zeta G_*)_x\sigma_x \\
&\quad +2x'y'^2\zeta G_*\sigma_y \times (\zeta G_*)_x\sigma_y + y'^3\zeta G_*\sigma_y \times (\zeta G_*)_y\sigma_y\}, \\
&= \mu_1(r)(\zeta G_*\sigma_x x' + \zeta G_*\sigma_y y') + \frac{\mu_2(r)}{\kappa(r)}\{(x'y'' - x''y')\zeta G_*\mathbb{N} + x'^3\zeta G_*\sigma_x \\
&\quad \times\sigma_{xx} + 2x'^2y'\zeta G_*\sigma_x \times \sigma_{xy} + x'y'^2\zeta G_*\sigma_x \times \sigma_{yy} + x'^2y'\zeta G_*\sigma_y \times \sigma_{xx} \\
&\quad +2x'y'^2\zeta G_*\sigma_y \times \sigma_{xy} + y'^3\zeta G_*\sigma_y \times \sigma_{yy}\} + \frac{\mu_2(r)}{\kappa(r)}\{x'^3\zeta G_*\sigma_x \times (\zeta G_*)_x\sigma_x \\
&\quad +2x'^2y'\zeta G_*\sigma_x \times (\zeta G_*)_y\sigma_x + x'y'^2\zeta G_*\sigma_x \times (\zeta G_*)_y\sigma_y + x'^2y'\zeta G_*\sigma_y \\
&\quad \times(\zeta G_*)_x\sigma_x + 2x'y'^2\zeta G_*\sigma_y \times (\zeta G_*)_x\sigma_y + y'^3\zeta G_*\sigma_y \times (\zeta G_*)_y\sigma_y\}, \\
&= \mu_1(r)(x'\tilde{\sigma}_x + y'\tilde{\sigma}_y) + \frac{\mu_2(r)}{\kappa(r)}\{(x'y'' - x''y')\tilde{\mathbb{N}} + x'^3\tilde{\sigma}_x \times \tilde{\sigma}_{xx} + 2x'^2y'\tilde{\sigma}_x \\
&\quad \times\tilde{\sigma}_{xy} + x'y'^2\tilde{\sigma}_x \times \tilde{\sigma}_{yy} + x'^2y'\tilde{\sigma}_y \times \tilde{\sigma}_{xx} + 2x'y'^2\tilde{\sigma}_y \times \tilde{\sigma}_{xy} + y'^3\tilde{\sigma}_y \times \tilde{\sigma}_{yy}\}, \\
&= \tilde{\mu}_1(r)\tilde{\vec{t}}(r) + \tilde{\mu}_2(r)\tilde{\vec{b}}(r),
\end{aligned}$$

where $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are some smooth functions. If we assume that $\tilde{\mu}_1 = \mu_1$ and $\tilde{\mu}_2 = \mu_1$ then $\tilde{\delta}(r)$ is a rectifying curve on the surface $\tilde{P}$.

□

**Corollary 3.2.** *Let $\delta(r)$ be a rectifying curve on the surface $P$ and $G : P \to \tilde{P}$ be a homothetic map. Then $\tilde{\delta}(r)$ is also a rectifying curve on the surface $\tilde{P}$ if*

$$\begin{aligned}
\tilde{\delta}(r) &= cG_*(\delta(r)) + \frac{\mu_2(r)}{\kappa(r)}\{x'^3cG_*\sigma_x \times (cG_*)_x\sigma_x + 2x'^2y'cG_*\sigma_x \times (cG_*)_y\sigma_x \\
&\quad +x'y'^2cG_*\sigma_x \times (cG_*)_y\sigma_y + x'^2y'cG_*\sigma_y \times (cG_*)_x\sigma_x + 2x'y'^2cG_*\sigma_y \\
&\quad \times(cG_*)_x\sigma_y + y'^3cG_*\sigma_y \times (cG_*)_y\sigma_y\}.
\end{aligned}$$

*Proof.* We know that for a homothetic map the dilation factor $\zeta(x, y) = c \neq \{0, 1\}$. By putting the value of $\zeta = c$ in the equation (13), we get the desired result for the case of homothetic map and hence, we obtained the expression for the case of homothetic image of rectifying curve. □

## 4. Conclusion

In this article, we study the first and second fundamental forms of surfaces. We investigated some geometric properties of these forms. The sufficient condition for the invariance of the conformal image of the rectifying curve under the isometry is also investigated in this study.

In the future, one can discuss some other properties such as the normal and the tangential components of rectifying curves under different transformations, namely, conformal, homothetic, and isometric of the surfaces, and one can also find out that these components are invariant under isometry of the surfaces. These results can also be extended to check the behaviour of the mean, gaussian, and sectional curvature for some surfaces like translation, minimal and slant surfaces under affine transformations.

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