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FIXED POINT THEOREM FOR GENERALIZED WARDOWSKI CONTRACTION IN M-METRIC SPACE

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Abstract. In this article, we formulate and establish the Wardowski type fixed point theorem in M-metric spaces. It contains an example which supports our result and utilise ours theorems to investigate the existence and uniqueness of fixed point by settling an integral equation.

1. Introduction and Preliminaries

Due to its extensive applications in fields like engineering, computer science, finance, etc., fixed point theory is also well known. Several researchers demonstrated numerous intriguing extensions and generalisations starting from the Banach contraction principle (BCP) [11], one of the core findings of fixed point theory. BCP first appeared in 1922 and It forms the nucleus of this theory. BCP can be generalised in a variety of ways by changing the contraction mapping or the metric space, i.e. [2, 5, 8, 9, 10, 12]-[18] and many more. In 2014, Wardowski [3] stated as a new generalization known as F-contraction and derived a fixed point theorem for it.

Definition 1.1. [3] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying:
(F1) F is strictly increasing, i.e. $\forall\ \alpha, \beta \in \mathbb{R}^+$ such that $\alpha < \beta, F(\alpha) < F(\beta)$,
(F2) For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim\limits_{n\to\infty} \alpha_n = 0$ if and only if $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$,
(F3) $\exists\ k \in (0, 1)$ such that $\lim\limits_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.
Denote $\triangle_F$ by the collection of all those functions which satisfy the conditions (F1-F3).
A mapping $\mathcal{H} : \xi \to \xi$ is said to be an F-contraction if there exists $\tau > 0$ such that

$$d(\mathcal{H}s, \mathcal{H}t) > 0 \Rightarrow \tau + F(d(\mathcal{H}s, \mathcal{H}t)) \leq d(s, t) \quad \forall\ s, t \in \xi. \tag{1.1}$$

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Theorem 1.1. *Let (ξ, d) be a complete metric space and let $\mathcal{H} : \xi \to \xi$ be an F-contraction. Then, $\mathcal{H}$ has a unique fixed point $s^* \in \xi$ and for every $s \in \xi$ a sequence $\{\mathcal{H}^n s\}_{n\in\mathbb{N}}$ is convergent to s^*.*

As an extension of F-contraction, Wardowski and Van Dung [4] proposed the idea of an F-weak contraction and established a fixed point theorem.

Definition 1.2. Let (ξ, d) be a metric space. $\mathcal{H}$ be a self mapping on ξ is said to be a F-weak contraction on (ξ, d) if there exists $F \in \triangle_F$ and $\tau > 0$ such that for all $s, t \in \xi$ satisfying $d(\mathcal{H}s, \mathcal{H}t) > 0$. the following conditions holds:

$$d(\mathcal{H}s, \mathcal{H}t) > 0 \Rightarrow \tau + F(d(\mathcal{H}s, \mathcal{H}t)) \leq F(N(s, t)) \quad \forall\ s, t \in \xi. \tag{1.2}$$

where

$$N(s, t) = \max\left\{d(s, t), d(s, \mathcal{H}s), d(t, \mathcal{H}t), \frac{d(s, \mathcal{H}s) + d(t, \mathcal{H}t)}{2}\right\}.$$

Later, Piri and Kumam [6] proposed a large class of functions by changing the conditions $(F3)$ in the definition of F-contraction with the following:
$(F3')$ F is continous on $(0, \infty)$.
Afterthat in 2017, Fulga et al. [1] generalizes the Wardowski type-contraction as F_E-contraction and obtained fixed point results for it.
Let F_E denote the familly of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ which satisfies the following conditions:
(F_E1) F is strictly increasing, that is, for all $s, t \in \mathbb{R}^+$, if $s < t$ then $F(s) < F(t)$;
(F_E2) There exists $\tau > 0$ such that $\tau + \lim_{x\to x_0} \inf F(x) > \lim_{x\to x_0} \sup F(x)$, for every $x_0 > 0$.

Definition 1.3. Let (ξ, d) be a metric space. A map $\mathcal{H} : \xi \to \xi$ is said to be a F_E-contraction on (ξ, d) if there exists $F \in F_E$ and $\tau > 0$ such that for all $s, t \in \xi$

$$d(\mathcal{H}s, \mathcal{H}t) > 0 \Longrightarrow \tau + F(d(\mathcal{H}s, \mathcal{H}t)) \leq F(E(s, t)) \tag{1.3}$$

where

$$E(s, t) = d(s, t) + |d(s, \mathcal{H}s) - d(t, \mathcal{H}t)|.$$

Theorem 1.2. *Let (ξ, d) be a complete metric space and let $\mathcal{H} : \xi \to \xi$ be an F_E-contraction. Then, $\mathcal{H}$ has a unique fixed point s^* and for all $s_0 \in \xi$ the sequence $\{\mathcal{H}^n s_0\}$ is convergent to s^*.*

In contrast, Asadi et al. [7] demonstrated the concept of M-metric space as a extension of metric space and partial metric space in 2014 and used its topological properties to support several theorems.
Notation: [7] In the sequel of M-metric space, the notations listed below are useful:
(i) $m_{st} := m(s, s) \vee m(t, t) = \min\{m(s, s), m(t, t)\}$,
(ii) $M_{st} := m(s, s) \wedge m(t, t) = \max\{m(s, s), m(t, t)\}$.

Definition 1.4. [7] If $\xi \neq \emptyset$ and a map $m : \xi \times \xi \to \mathbb{R}^+$ is m-metric, when it satisfies the subsequent conditions:

(1) $m(s,s) = m(t,t) = m(s,t) \Longleftrightarrow s = t$,
(2) $m_{st} \leq m(s,t)$,
(3) $m(s,t) = m(t,s)$,
(4) $(m(s,t) - m_{st}) \leq (m(s,z) - m_{sz}) + (m(z,t) - m_{zt})$.
Hence, the pair (ξ, m) is referred to as an M-metric space.

Definition 1.5. [7] Assume that a Sequence $\{s_n\}$ in (ξ, m). Then,
(I) $\{s_n\}$ converges to a point s iff

$$\lim_{n\to\infty}(m(s_n, s) - m_{s_n,s}) = 0. \tag{1.4}$$

(II) $\{s_n\}$ is m-Cauchy sequence iff

$$\lim_{n,m\to\infty}(m(s_n, s_m) - m_{s_n,s_m}) \text{ and } \lim_{n,m\to\infty}(M(s_n, s_m) - m_{s_n,s_m}) \tag{1.5}$$

exist and finite.
(III) If each m-Cauchy sequence $\{s_n\}$ converges to a point s in a manner that

$$\lim_{n\to\infty}(m(s_n, s) - m_{s_n,s}) = 0 \text{ and } \lim_{n\to\infty}(M(s_n, s) - m_{s_n,s}) = 0. \tag{1.6}$$

Then, M-metric space is m-complete.

Lemma 1.3. [7] *If $s_n \to s$ as $n \to \infty$ in (ξ, m). Then,*

$$\lim_{n\to\infty}(m(s_n, t) - m_{s_n,t}) = m(s,t) - m_{s,t}, \forall\ t \in \xi.$$

Lemma 1.4. [7] *If $s_n \to s$ and $t_n \to t$ as $n \to \infty$ in (ξ, m). Then,*

$$\lim_{n\to\infty}(m(s_n, t_n) - m_{s_n,t_n}) = m(s,t) - m_{s,t}.$$

Lemma 1.5. [7] *If $s_n \to s$ and $t_n \to t$ as $n \to \infty$ in (ξ, m). Then, $m(s,t) = m_{s,t}$. Further if $m(s,s) = m(t,t)$, then $s = t$.*

2. Main Result

In this section,we introduce F_E-contraction in M-metric space and prove a fixed point theorem for it. We also give an example with showing the applicability of our result.

Theorem 2.1. *Let (ξ, m) be a M-metric space. A map $\mathcal{H} : \xi \to \xi$ is said to be a F_E-contraction on (ξ, m) if there exists $F \in F_E$ and $\tau > 0$ such that for all $s, t \in \xi$*

$$m(\mathcal{H}s, \mathcal{H}t) > 0 \Longrightarrow \tau + F(m(\mathcal{H}s, \mathcal{H}t)) \leq F(E(s,t)) \tag{2.1}$$

where

$$E(s,t) = m(s,t) + |m(s, \mathcal{H}s) - m(t, \mathcal{H}t)|. \tag{2.2}$$

Theorem 2.2. *Let (ξ, m) be a complete M-metric space and let $\mathcal{H} : \xi \to \xi$ be an F_E-contraction. Then, $\mathcal{H}$ has a unique fixed point $s^* \in \xi$ and for all $s_0 \in \xi$ a sequence $\{\mathcal{H}^n s_0\}_{n\in\mathbb{N}}$ is convergent to s^*.*

Proof. Let $s_0 \in \xi$ be arbitrary and fixed. We define a sequence $s_{n+1} = \mathcal{H}s_n = \mathcal{H}^n s_0$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that $s_{n_0+1} = s_{n_0}$, because $s_{n_0+1} = \mathcal{H}s_{n_0}$, we obtain that $\mathcal{H}s_{n_0} = s_{n_0}$, so s_{n_0} is a fixed point of $\mathcal{H}$.
Now, we suppose that $s_{n+1} \neq s_n$ for all $n \in \mathbb{N} \cup \{0\}$. So, $m(s_n, s_{n+1}) > 0, \ \forall \ n_0 \in \mathbb{N} \cup \{0\}$ from (2.1) it follows that, for all $n \in \mathbb{N}$

$$\begin{aligned}
m(s_n, s_{n+1}) &= m(\mathcal{H}s_{n-1}, \mathcal{H}s_n) > 0 \\
&\Longrightarrow \tau + F(m(\mathcal{H}s_{n-1}, \mathcal{H}s_n)) \leq F(E(s_{n-1}, s_n)) \\
&\Longleftrightarrow \tau + F(m(s_n, s_{n+1})) \leq \\
&\leq F(m(s_{n-1}, s_n)) + |m(s_{n-1}, \mathcal{H}s_{n-1}) - m(s_n, \mathcal{H}s_n)| \\
&\Longleftrightarrow \tau + F(m(s_n, s_{n+1})) \leq \\
&\leq F(m(s_{n-1}, s_n)) + |m(s_{n-1}, s_n) - m(s_n, s_{n+1})| \qquad (2.3)
\end{aligned}$$

or if we denote by $r_n = m(s_{n-1}, s_n)$, we have

$$\tau + F(r_{n+1}) \leq F(r_n + |r_n - r_{n+1}|).$$

If there exists $n \in \mathbb{N}$ such that $r_{n+1} \geq r_n$, then (5) becomes

$$\tau + F(r_{n+1}) \leq F(r_{n+1}) \implies \tau \leq 0.$$

But, this is a contradiction, so, for $r_{n+1} < r_n$ we have

$$\begin{gathered}
\tau + F(r_{n+1}) \leq F(2r_n - r_{n+1}) \qquad (2.4) \\
\Longleftrightarrow F(r_{n+1}) \leq F(2r_n - r_{n+1}) - \tau < F(2r_n - r_{n+1})
\end{gathered}$$

and using $(F_E 1)$

$$r_{n+1} \leq 2r_n - r_{n+1}.$$

Therefore, the sequence $\{r_n\}$ is strictly increasing and bounded.
Now, let $r = \lim_{n\to\infty} r_n$ and we suppose that $r > 0$. Because $r_n \searrow r$ it results that $(2r_n - r_{n+1}) \searrow r$ and taking the limit as $n \to \infty$ in (6), we get

$$\tau + F(r + 0) \leq F(r + 0) \implies \tau \leq 0.$$

It is a contradiction, so

$$r = \lim_{n\to\infty} r_n = \lim_{n\to\infty} m(s_{n-1}, s_n) = 0. \qquad (2.5)$$

In order to prove that $\{s_n\}$ is a m-Cauchy sequence in (ξ, m), we suppose the contrary, that is there exists $\epsilon > 0$ and the sequences $\{p(k)\}, \{q(k)\}$ of positive integers, with $p(k) > q(k) > k$ such that

$$m(s_{q(k)}, s_{p(k)}) \geq \epsilon \text{ and } m(s_{q(k)-1}, s_{p(k)}) < \epsilon, \text{for any } k \in \mathbb{N}.$$

Then we have

$$\begin{aligned}
\epsilon \leq m(s_{q(k)}, s_{p(k)}) &\leq m(s_{q(k)}, s_{q(k)-1}) + m(s_{q(k)-1}, s_{q(k)}) \\
&< m(s_{q(k)}, s_{q(k)-1}) + \epsilon.
\end{aligned}$$

Letting $k \to \infty$ and using (2.5)

$$\lim_{k\to\infty} m(s_{q(k)}, s_{p(k)}) = \epsilon.$$

Furthermore, using the triangle inequality, we obtain that

$$\begin{aligned} 0 &\leq |m(s_{q(k)+1}, s_{p(k)+1}) - m(s_{q(k)}, s_{p(k)})| \\ &= m(s_{q(k)+1}, s_{q(k)}) + m(s_{p(k)}, s_{p(k)+1}) \end{aligned}$$

and

$$\lim_{k\to\infty} |m(s_{q(k)+1}, s_{p(k)+1}) - m(s_{q(k)}, s_{p(k)})| = 0$$

$$\lim_{k\to\infty} m(s_{q(k)+1}, s_{p(k)+1}) = \lim_{k\to\infty} m(s_{q(k)}, s_{p(k)}) = \epsilon. \tag{2.6}$$

On the other hand, because from (2.5)

$$\lim_{n\to\infty} m(s_n, \mathcal{H}s_n) = m(s_n, s_{n+1}) = 0.$$

There exists $n \in \mathbb{N}$ such that

$$m(s_{q(k)}, \mathcal{H}s_{q(k)}) < \frac{\epsilon}{4}$$

and

$$m(s_{p(k)}, \mathcal{H}s_{p(k)}) < \frac{\epsilon}{4}, \text{ for all } k \in \mathbb{N}. \tag{2.7}$$

Assuming by contradiction, that $\exists\ l \in \mathbb{N}$ such that

$$m(s_{q(l)+1}, s_{p(l)+1}) = 0.$$

From (2.6) and (2.5), it follows that

$$\begin{aligned} \epsilon &\leq m(s_{q(l)}, s_{p(l)}) \\ &\leq \\ &\leq \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}. \end{aligned}$$

This is a contradiction. Also,

$$m(\mathcal{H}s_{q(k)}, \mathcal{H}s_{p(k)}) = m(s_{q(k)+1}, s_{p(k)+1}) > 0$$

for all $k \in \mathbb{N}$ on using (3), we can set $\tau > 0$ such that

$$\tau + F(m(\mathcal{H}s_{q(k)}, \mathcal{H}s_{p(k)})) \leq F(E(s_{q(k)}, s_{p(k)})).$$

For any k, where

$$\begin{aligned} E(s_{q(k)}, s_{p(k)}) &\leq m(s_{q(k)}, s_{p(k)}) + |m(s_{q(k)}, \mathcal{H}s_{q(k)}) - m(s_{p(k)}, \mathcal{H}s_{p(k)})| \\ &= m(s_{q(k)}, s_{p(k)}) + |m(s_{q(k)}, s_{q(k)+1}) - m(s_{p(k)}, s_{p(k)+1})|. \end{aligned}$$

Hence, $\lim_{k\to\infty} E(s_{q(K)}, s_{p(k)}) = \epsilon$ and by (2.6) we have

$$\begin{aligned} \tau + \lim_{k\to\infty} \inf F(m(\mathcal{H}s_{q(k)}, \mathcal{H}s_{p(k)}) &\leq \lim_{k\to\infty} \inf F(E(s_{q(K)}, s_{p(k)})) \\ &\leq \lim_{k\to\infty} \sup F(E(s_{q(K)}, s_{p(k)})) \end{aligned}$$

$$\iff \quad \tau + F(\epsilon^+) \leq F(\epsilon^+).$$

a contradiction. It means that $\lim_{p,q\to\infty}(m(s_{q(k)}, s_{p(k)}) - m_{s_{q(k)},s_{p(k)}}) = 0$ and $\lim_{p,q\to\infty}(M(s_{q(k)}, s_{p(k)}) - m_{s_{q(k)},s_{p(k)}}) = 0$. This shows that $\{s_n\}$ is a m-Cauchy sequence and by completeness of ξ, it converges to point $s^* \in \xi$.

Next, we show that s^* is a fixed point of T. We have consider two cases:

(1) For any $s \in \mathbb{N}$, $\exists\ k_n > k_{n-1}, k_0 = 1$ and $s_{k_{n+1}} = \mathcal{H}s^*$. Then $s^* = \lim_{n\to\infty} s_{k_{n+1}} = \mathcal{H}s^*$, so s^* is a fixed point of $\mathcal{H}$.

(2) There exists $m \in \mathbb{N}$ such that for all $n \geq m, m(Ts_n, Ts^*) > 0$. Subsitituting $s = s_n$ and $t = s^*$ in (2.1) $\exists\ t > 0$ such that,

$$\begin{aligned}
\tau + F(m(\mathcal{H}s_n, \mathcal{H}s^*)) &\leq F(E(s_n, s^*)) \\
\tau + F(m(s_{n+1}, \mathcal{H}s^*)) &\leq F(m(s_n, s^*) + |m(s_n, \mathcal{H}s_n) - m(s^*, \mathcal{H}s^*)|) \\
\tau + F(m(s_{n+1}, \mathcal{H}s^*)) &\leq F(m(s_n, s^*) + |m(s_n, s_{n+1}) - m(s^*, \mathcal{H}s^*)|)
\end{aligned}$$

We suppose that $s^* \neq \mathcal{H}s^*$, letting $n \to \infty$ from (2.6), we obtain

$$\tau + \lim_{x\to m(s^*,\mathcal{H}s^*)} F(x) < \lim_{x\to m(s^*,\mathcal{H}s^*)} \inf F(x) < \lim_{x\to m(s^*,\mathcal{H}s^*)} \sup F(x).$$

Which contradicts (F_E2) of the hypothesis, Hence $\mathcal{H}s^* = s^*$.

Now, let us show that $\mathcal{H}$ must have only one fixed point, if there exists another point $t^* \in \xi, s^* = t^*$ such that $\mathcal{H}t^* = t^*$, then

$$m(s^*, t^*) = m(\mathcal{H}s^*, \mathcal{H}t^*) > 0,$$

we get

$$\begin{aligned}
\tau + F(m(\mathcal{H}s^*, \mathcal{H}t^*)) &\leq F(E(s^*, t^*)) \\
\tau + F(m(s^*, t^*)) &\leq F(m(s^*, t^*) - |m(s^*, \mathcal{H}s^*) - m(t^*, \mathcal{H}t^*)|) \\
&\leq F(m(s^*, t^*) - |m(s^*, s^*) - m(t^*, t^*)|) \\
&\leq F(m(s^*, t^*)
\end{aligned}$$

a contradiction. □

Example: Let $\xi = [0, 1]$ and $m(s, t) = \frac{|s+t|}{2}$, for all $s, t \in \xi$. Then, (ξ, m) is complete M-metric space. Define a self map $\mathcal{H} : \xi \to \xi$ such that $\mathcal{H}(s) = \frac{s}{2}$, for all $s \in \xi$.

Define $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(r) = \ln(r)$ a function, for all $s, t \in \xi$ such that

$$m(\mathcal{H}s, \mathcal{H}t) > 0 \implies \tau + F(m(\mathcal{H}s, \mathcal{H}t)) = \tau + \ln(\frac{|s+t|}{4}).$$

$$E(s, t) = m(s, t) + |m(s, \mathcal{H}s) - m(t, \mathcal{H}t)| = \frac{|s+t|}{2} + \left|\frac{|s + \frac{s}{2}|}{2} - \frac{|t + \frac{t}{2}|}{2}\right|$$

i.e., $E(s, t) = \frac{|5s-t|}{4}$ and suppose $\tau \leq \ln 2$.

Therefore,

$$\tau + \ln\left(\frac{|s+t|}{4}\right) \leq \ln 2 + \ln\left(\frac{|s+t|}{4}\right) \leq \ln\left(\frac{|5s-t|}{4}\right) = F(E(s, t)).$$

Then, the contractive condition is satisfied for all $s, t \in \xi$. Hence, all criterion of the Theorem (2.1) are satisfied and $\mathcal{H}$ has a unique fixed point.

Corollary 2.1. *Let $\mathcal{H} : \xi \to \xi$ be a self mapping of complete p-metric space be an F_E-contraction . Then, $\mathcal{H}$ has a unique fixed point $s^* \in \xi$ and for all $s_0 \in \xi$ a sequence $\{\mathcal{H}^n s_0\}_{n\in\mathbb{N}}$ is convergent to s^*.*

3. Application

In this section, we utilise Theorem (2.2) to examine the existence and uniqueness of the Fredholm integral equation's solution.
Let $X = C([a, b], \mathbb{R})$. Next, we assume the following Fredholm type integral equation:

$$s(p) = \int_a^b \mathcal{S}(p, q, s(p))dq, \text{ for } p, q \in [a, b] \tag{3.1}$$

where $\mathcal{H}, h \in C([a, b], \mathbb{R})$. Define $m : \xi \times \xi \to \mathbb{R}^+$ by

$$m(s(p), t(p)) = \sup_{p\in[a,b]} \frac{(|s(p) + t(p)|)}{2}, \quad \forall\ s, t \in \xi. \tag{3.2}$$

Then, (ξ, m) is an m-complete in M-metric space.

Theorem 3.1. *Suppose that there exist $\tau > 0$ and for all $s, t \in C([a, b], \mathbb{R})$*

$$|\mathcal{H}(p, q, s(p)) + \mathcal{H}(p, q, t(p))| \ \leq\ \frac{e^{-\tau}}{(b-a)}[|s(p) + t(p)| + 2\mathcal{R}(s, t)],\ \forall\ p, q \in [a, b].$$

Where, $\mathcal{R}(s, t) = |m(s, \mathcal{H}s) - m(t, \mathcal{H}t)|$.
Then, the nonlinear integral equation (3.1) has a unique solution.

Proof. Define $\mathcal{H} : \xi \to \xi$ by,

$$\mathcal{H}(s(p)) = \int_a^b \mathcal{S}(p, q, s(p))dq,\ \forall\ p, q \in [a, b].$$

Observe that existence of a fixed point of the operator $\mathcal{H}$ is equivalent to the existence of a solution of the integral equation (3.1). Now, for all $s, t \in \xi$. We have

$$\begin{aligned}
m(Ts, Tt) &= \left|\frac{\mathcal{H}(s(p)) + \mathcal{H}(t(p))}{2}\right| \\
&= \left|\int_a^b \frac{\mathcal{S}(p, q, s(p)) + \mathcal{S}(p, q, t(p))}{2} dq\right| \\
&\leq \int_a^b \left|\left(\frac{\mathcal{S}(p, q, s(p)) + \mathcal{S}(p, q, t(p))}{2}\right) dq\right| \\
&\leq \frac{1}{2}\frac{e^{-\tau}}{(b-a)}\int_a^b [|s(p) + t(p)| + 2\mathcal{R}(s, t)]dq \\
&\leq \frac{e^{-\tau}}{(b-a)}\int_a^b \left(\frac{|s(p)| + |t(p)|}{2} + \mathcal{R}(s, t)\right)dq
\end{aligned}$$

$$\leq \frac{e^{-\tau}}{(b-a)}\left[\sup_{p\in[a,b]}\frac{|s(p)|+|t(p)|}{2}+\mathcal{R}(s,t)\right]\left(\int_a^b dq\right)$$
$$\leq \frac{e^{-\tau}}{(b-a)}[m(s,t)+\mathcal{R}(s,t)](b-a)$$
$$\leq e^{-\tau}(m(s,t)+|m(s,\mathcal{H}s)-m(t,\mathcal{H}t)|)$$
$$\leq e^{-\tau}E(s,t).$$

Thus, the Condition (2.1) is satisfied with $F(\alpha)=\ln(\alpha)$. Therefore, all the parameters of Theorem (2.2) are fulfilled. Hence the operator $\mathcal{H}$ has a unique fixed point, it means that the nonlinear integral equation (3.1) has a unique solution. This completes the proof. □

4. Conclusion

In this article, we established a fixed point theorem for new type of Wardowski contraction in M-metric space and provide a corollary for p-metric space. We also take a suitable example and application to show the existence of the solution of Fredholm integral equation which supported our fixed point theorem.

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