Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 14 Issue 5 (2023), Pages 28-38
https://doi.org/10.54379/jma-2023-5-3.

MAXIMAL CONVERGENCE OF FABER SERIES IN WEIGHTED SMIRNOV CLASSES WITH VARIABLE EXPONENT ON THE DOMAINS BOUNDED BY SMOOTH CURVES

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Abstract. In this paper, we suppose that the boundary of a domain G in the complex plane $\mathbb{C}$ belongs to a special subclass of smooth curves and that the canonical domain G_R, $R > 1$ is the largest domain where a function f is analytic. We investigate the rate of convergence to the function f by the partial sums of Faber series of the function f on the domain $\overline{G}$. Under the boundary conditions of the domain G, we obtain some results which characterize the maximal convergence of the Faber expansion of the function f which belongs to the weighted Smirnov class with variable exponent $E_{\omega}^{p(\cdot)}(G_R)$.

1. Introduction and new results

Let G be a simply connected domain bounded by a rectifiable curve Γ in the complex plane $\mathbb{C}$, and let also the complement of the closed domain $\overline{G}$ be a simply connected domain G' containing the point of infinity $z = \infty$. By the Riemann conformal mapping theorem there exists a unique function $w = \varphi(z)$ meromorphic in G' which maps the domain G' conformally and univalently onto the domain $|w| > 1$ and satisfies the conditions

$$\varphi(\infty) = \infty, \varphi'(\infty) = \gamma > 0,$$

where γ is the capacity of G. Let ψ be the inverse to φ and let ψ_0 be the mapping which maps the unit disk onto the domain G under the conditions $\psi_0(0) = 0$ and $\psi_0'(0) > 0$. Let Γ_r be the image of the circle $|w| = r, \quad 0 < r < 1$, under the mapping ψ_0. If a function $f(z)$, analytic on a domain G, satisfies the inequality

$$\int_{\Gamma_r} |f(z)|^p \, |dz| \leq M, \quad p > 0$$

for any r such that $0 < r < 1$, then f belongs to the Smirnov class $E^p(G)$. In this definition one can replace the set of the images of the circles $\{\Gamma_r\}$ by an arbitrary sequence of rectifiable curves $\{\Gamma_n\}$, which converge from inside the domain G to the curve Γ (see, e.g., [26, p. 77]). If $f \in E^p(G)$, then it has a nontangential limit

2010 *Mathematics Subject Classification.* 30A10, 30E10, 41A10, 41A25, 41A58.
Key words and phrases. Faber Series; Conformal Mappings; Smooth Curves; Maximal Convergence; Weighted Smirnov Class with Variable Exponent.

Submitted August 30, 2022. Published October 18, 2023.
Communicated by L. E. Persson.

almost everywhere on Γ and the boundary function belongs to $L^p(\Gamma)$ (see, e.g., [8, pp. 438-453]).

Definition 1. *Let f be a Lebesque measurable function. For a given Lebesgue measurable variable exponent function $p(\cdot) : \Gamma \to [1, \infty]$ and the weight function $\omega(\cdot) : \Gamma \to (0, \infty)$, if the condition*

$$\int_{\Gamma} [\omega(z) |f(z)|]^{p(z)} |dz| \leq \infty.$$

holds, then the set of Lebesgue measurable functions f is defined to be the weighted Lebesgue spaces $L^{p(.)}_{\omega}(\Gamma)$ with variable exponent.

$L^{p(.)}_{\omega}(\Gamma)$, $\underset{z\in\Gamma}{esssup} p(z) < \infty$, becomes a Banach spaces with the norm

$$\|f\|_{L^{p(.)}_{\omega}(\Gamma)} := inf \left\{ \lambda > 0 : \quad \int_{\Gamma} \left[\omega(z) \left| \frac{f(z)}{\lambda} \right| \right]^{p(z)} |dz| \leq 1 \right\}$$

Obviously we have

$$\|f\|_{L^{p(.)}_{\omega}(\Gamma)} := \|f\omega\|_{L^{p(.)}(\Gamma)},$$

(see, [6, p.192]).

If $p(.) := constant$, then $L^{p(.)}_{\omega}(\Gamma)$ coincides with the weighted Lebesque spaces $L^{p}_{\omega}(\Gamma)$.

Definition 2. *Let $p(.) : \Gamma \to [1, \infty]$ be a Lebesque measurable function and $\omega(\cdot) : \Gamma \to (0, \infty)$ be a weight function. For an analytic function f in G, the set*

$$E^{p(.)}_{\omega}(G) := \left\{ f \in E^{1}_{\omega}(G) : f \in L^{p(.)}_{\omega}(\Gamma) \right\}$$

is called weighted Smirnov class with variable exponent $p(.)$.

$E^{p(.)}_{\omega}(G)$, $\underset{z\in F}{esssup} p(z) < \infty$, becomes a Banach space equipped with the norm

$$\|f\|_{E^{p(.)}_{\omega}(G)} := \|f\|_{L^{p(.)}_{\omega}(\Gamma)}$$

Let F be a segment $[0, 2\pi]$ or a Jordan rectifiable curve Γ in the complex plane $\mathbb{C}$ and let $p(.) : F \to [0, \infty)$ be a Lebesque measurable function such that

$$1 \leq p_{-} := \underset{z\in F}{essinf} p(z) \leq \underset{z\in F}{esssup} p(z) = p_{+} < \infty \tag{1.1}$$

Definition 3. *We say that $p(.) \in P^{log}(F)$, if $p(.)$ satisfies the conditions (1.1) and*

$$|p(z_1) - p(z_2)| \, ln \left(\frac{|F|}{|z_1 - z_2|} \right) \leq c, \quad \forall z_1, z_2 \in F,$$

with a positive constant c, where $|F|$ is the Lebesgue measure of F.

If $p(.) \in P^{log}(F)$ and $p_{-} > 1$, then we say that $p(.) \in P^{log}_{0}(F)$. In our investigations we will use the generalized Muckenhoupt weights class $A_{p(\cdot)}(\Gamma)$ defined as follows.

Definition 4. *For a given exponent $p(.)$ defined on Γ, we say that $\omega \in A_{p(\cdot)}(\Gamma)$ if*

$$\sup_{B_j} |B_j|^{-1} \left\| \omega \chi_{B_j} \right\|_{L^{p(\cdot)}(\Gamma)} \left\| \omega^{-1} \chi_{B_j} \right\|_{L^{q(\cdot)}(\Gamma)} < \infty, \quad \frac{1}{p(.)} + \frac{1}{q(.)} = 1,$$

where the supremum is taken over all balls $B_j \subset \Gamma$ with the characteristic functions χ_{B_j}.

Let $\theta(s)$ denote the angle between the positive direction of the real axis and the tangent to the curve Γ at a point M on curve Γ at a distance s along this curve from a fixed point.

Definition 5. *(see* [11]*). If the inequality*

$$\omega(\theta,\delta) := \sup_{|h|\leq\delta} \|\theta(\cdot) - \theta(\cdot + h)\|_{[0,2\pi]} \leq c\delta^{\alpha} \ln^{\beta}\frac{4}{\delta}, \quad \delta \in (0,\pi] \tag{1.2}$$

holds for some parameters $\alpha \in (0,1]$, $\beta \in [0,\infty)$ and for a positive constant c independent of δ, then $\Gamma \in \mathfrak{B}(\alpha,\beta)$.

In this definition, the norm $\|\cdot\|_{[0,2\pi]}$ means the maximum norm on the interval $[0,2\pi]$.

In particular the class $\mathfrak{B}(\alpha,0)$ coincides with the class of Lyapunov curves. Furthermore the class $\mathfrak{B}(\alpha,\beta)$ is a subclass of Dini-smooth curves, i.e. $\int_0^c \frac{\omega(\theta,t)}{t}dt < \infty$, for some $c > 0$. For a proof of this fact and some additional information about the class $\mathfrak{B}(\alpha,\beta)$ see [11].

If Γ belongs to the class $\mathfrak{B}(\alpha,\beta)$, $\alpha \in (0,1]$, $\beta \in [0,\infty)$, then the inequality

$$0 < c_1 \leq |\psi'(w)| \leq c_2 < \infty, \quad |w| \geq 1 \tag{1.3}$$

is valid for some positive constants c_1 and c_2 (see [26, p. 141]).

For $f \in L_{\omega}^{p(\cdot)}(\Gamma)$, $p \in P_0^{log}(\Gamma)$, we set $f_0 := f \circ \psi$, $p_0 := p \circ \psi$ and $\omega_0 := \omega \circ \psi$. By using (1.3) it can be shown that

$$\begin{aligned} f \in L_{\omega}^{p(\cdot)}(\Gamma) &\iff f_0 \in L_{\omega_0}^{p_0(\cdot)}(T) \\ p \in P_0^{log}(\Gamma) &\iff p_0 \in P_0^{log}(T) \\ \omega \in A_{p(\cdot)}(\Gamma) &\iff \omega_0 \in A_{p_0(\cdot)}(T), \end{aligned}$$

where T is the unit circle.

Now we define the best approximation to the function $f \in E_{\omega}^{p(.)}(G)$ as:

$$E_n^{\omega,p(.)}(f,G) := inf\, \|f - p_n\|_{L_{\omega}^{p(\cdot)}(\Gamma)},$$

where inf is taken over the polynomials p_n of degree at most n.

For construction of the approximation aggregates in $E_{\omega}^{p(.)}(G)$, we use the Faber polynomials φ_k, $k = 0,1,2,...$, defined as usual, (see, e.g., [26, p.33]). Since φ is analytic in the domain G' without the point $z = \infty$, it has a simple pole at the point $z = \infty$, Therefore its Laurent expansion in some neighborhood of the point $z = \infty$ has the form

$$\varphi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + ... + \frac{\gamma_n}{z^n} + \dots$$

For a non-negative integer k, we set

$$\varphi^k(z) = \left(\gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + ... + \frac{\gamma_n}{z^n} + \dots\right)^k \tag{1.4}$$

A group of $k+1$ terms containing non-negative powers of z written in (1.4) is called the Faber polynomial of order k for the domain G. For Faber polynomials we use the notation

$$\varphi_k(z) = \gamma^k z^k + a_{k-1}^{(k)} z^{k-1} + a_{k-2}^{(k)} z^{k-2} + \dots + a_1^{(k)} z + a_0^{(k)} \tag{1.5}$$

For the sum of the terms containing negative powers of z in the expansion (1.4) we use the notation

$$-E_k(z) = \frac{b_1^{(k)}}{z} + \frac{b_2^{(k)}}{z^2} + \ldots + \frac{b_n^{(k)}}{z^n} + \ldots$$

Hence the identity

$$\varphi_k(z) = \varphi^k(z) + E_k(z), \quad z \in \overline{G'} \tag{1.6}$$

holds in the sense of convergence. Now we denote

$$\Gamma_R := \{z \in ext\Gamma; \quad |\varphi(z)| = R, \quad R > 1\}, \quad G_R := int\Gamma_R$$

If $R = 1$, the line Γ_1 is the boundary Γ of the domain G. Faber polynomials have the following integral representation:

$$\varphi_k(z) = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{\varphi^k(\varsigma)}{\varsigma - z} d\varsigma, \quad z \in G_R \tag{1.7}$$

If a function f is analytic in the canonical domain G_R, then the following expansion holds

$$f(z) = \sum_{k=0}^{\infty} a_k \varphi_k(z), \quad z \in G_R, R > 1$$

and the series converge absolutely and uniformly on G_R, where

$$a_k := \frac{1}{2\pi i}\int_{|t|=R} \frac{f_0(t)}{t^{k+1}} dt, \quad k = 0, 1, 2, \ldots$$

Now we use the notation

$$R_n(z, f) = f(z) - \sum_{k=0}^{n} a_k \varphi_k(z) = \sum_{k=n+1}^{\infty} a_k \varphi_k(z) \tag{1.8}$$

which is called remaining term. Hence the formulas (1.7) and (1.8) implies that,

$$R_n(z, f) = \frac{1}{2\pi i}\int_{|t|=R} f_0(t)\left[\sum_{k=n+1}^{\infty} \frac{\varphi_k(z)}{t^{k+1}}\right] dt \tag{1.9}$$

In this paper we estimate the remaining term $R_n(z, f)$ when the function f belongs to the class $E_\omega^{p(\cdot)}(G_R)$ in the case when the boundary of the domain G is of the class $\mathfrak{B}(\alpha, \beta)$, $\alpha \in (0, 1]$, $\beta \in [0, \infty)$.

Now we give the main result.

Theorem 1.1. *If G is a domain which is bounded by the curve Γ of the class $\mathfrak{B}(\alpha, \beta)$, $\alpha \in (0, 1]$, $\beta \in [0, \infty)$ and a function $f(\cdot) \in E_\omega^{p(\cdot)}(G_R)$, where $\omega(\cdot) \in A_{p(\cdot)}(\Gamma_R)$ and $p(\cdot) \in P_0^{log}(\Gamma_R)$, $R > 1$, then $R_n(z, f)$ satisfies the inequality*

$$|R_n(z, f)| \le \frac{c(p)}{R^{n+1}(R-1)} E_n^{\omega, p(\cdot)}(f, G_R), \quad z \in \Gamma$$

with some constant $c(p) > 0$ independent of n.

This result is also valid for $z \in \overline{G}$ according to the maximum modulus principle. Theorem 1.1 characterizes the maximal convergence of Faber series of the functions belong to the weighted Smirnov space with variable exponent in the case of domains bounded by the curves of the class $\mathfrak{B}(\alpha, \beta)$, $\alpha \in (0, 1]$ *and* $\beta \in [0, \infty)$. There are some results related to maximal convergence in the literature. Firstly, Bernstein and Walsh studied the maximal convergence of polynomials. They also obtained direct and inverse theorems when the function f is analytic on canonical domain G_R, $R > 1$ (see [26, pp. 54-59] and also [7, p. 27]). Many results about maximal convergence of Faber series were proved by P. K. Suetin in [26, Chapter X]. Suetin obtained some results on maximal convergence of Faber series of the function f in the case when it is analytic on the canonical domain G_R, continuous on $\overline{G}_R$ and in the case that it belongs to the class $E^p(G_R)$. He also proved some results on maximal convergence for the case concerning continuum K. In the case when f is a function in the Smirnov space with variable exponent, for $z \in K$ (continuum) maximal convergence of Faber series was studied in [14]. In [14] there is no assumption on the boundary of the domain G. The result obtained in Theorem 1.1 is an improvement of the result given in [14], in the weighted case and in the case of the domains bounded by smooth curves. The estimation obtained in Theorem 1.1 coincides with the Suetin's result in [26, p. 203, Theorem 3] when $p(.) = p > 1$ is a constant and $\omega(.) = 1$. If f belongs to the Smirnov-Orlicz space, for $z \in \mathfrak{B}(\alpha, \beta)$, $\alpha \in (0, 1]$, $\beta \in [0, \infty)$, maximal convergence of Faber series was studied in [20], in the case $\omega(.) = 1$. If f belongs to the Smirnov-Orlicz space, for $z \in K$ (continuum) maximal convergence of Faber series was studied in [10], in the case $\omega(.) = 1$.

2. Auxiliary Results

Let P_n be the best approximation polynomial to the function $f \in E_\omega^{p(\cdot)}(G_R)$ and $P_n^*(t) := P_n(\psi(t))$. Hence P_n^* is the best approximation polynomial to the function $f_0 \in L_{\omega_0}^{p_0(\cdot)}(|t| = R)$. The formula (1.9) implies that

$$R_n(z, f) = \frac{1}{2\pi i} \int_{|t|=R} \{f_0(t) - P_n^*(t)\} \sum_{k=n+1}^{\infty} \frac{\varphi_k(z)}{t^{k+1}} dt. \tag{2.1}$$

In view of (1.6), we find that

$$\sum_{k=n+1}^{\infty} \frac{\varphi_k(z)}{t^{k+1}} = \sum_{k=n+1}^{\infty} \frac{\varphi^k(z)}{t^{k+1}} + \sum_{k=n+1}^{\infty} \frac{E_k(z)}{t^{k+1}}, \quad z = \psi(w) \tag{2.2}$$

It is seen from [25, p. 12] that $E_k(\psi(\omega))$ is defined as follows

$$E_k(\psi(\omega)) := \frac{1}{2\pi i} \int_{|\tau|=1} \tau^k F(\tau, \omega) d\tau, \quad |\omega| \geq 1, \tag{2.3}$$

where

$$F(\tau, \omega) := \frac{\psi'(\tau)}{\psi(\tau) - \psi(\omega)} - \frac{1}{\tau - \omega} = \sum_{k=0}^{\infty} \frac{E_k(\psi(\omega))}{t^{k+1}}. \tag{2.4}$$

If Γ is sufficiently smooth, then this expansion converges in the closed domain $|\tau| \geq 1, |\omega| \geq 1$ (see [26, p. 156]).

For $|w| \geq 1$ and $|t| = R$, we have that

$$\sum_{k=n+1}^{\infty} \frac{E_k(\psi(\omega))}{t^{k+1}} = \frac{1}{2\pi i} \int_{|\tau|=1} F(\tau, \omega) \sum_{k=n+1}^{\infty} \frac{\tau^k}{t^{k+1}} d\tau. \tag{2.5}$$

It is very important to notice that if one wants to estimate remaining term $R_n(z, f)$, because of (2.1), (2.2) and (2.5), it is necessary to show that the integral

$$\int_{|\tau|=1} |F(\tau, \omega)| \, |d\tau| \tag{2.6}$$

is finite for all $|w| \geq 1$, according to the geometric properties of the boundary of the domain G. We show this according to the properties of the boundary Γ in Theorem 2.2.

We define the modulus of continuity of ψ' by

$$\omega(\psi', \delta) := \sup_{|h| \leq \delta} \left\| \psi'(e^{i(t+h)}) - \psi'(e^{it}) \right\|_T ,$$

where $\|\cdot\|_T$ means the maximum norm over T.

Lemma 2.1. *If the boundary of the domain G is of the class $\mathfrak{B}(\alpha, \beta)$, $\alpha \in (0,1]$, $\beta \in [0, \infty)$, then the inequality*

$$\int_0^1 \frac{\omega(\psi', \tau)}{\tau} d\tau < \infty$$

holds.

Theorem 2.2. *If G is a domain bounded by a curve Γ of the class $\mathfrak{B}(\alpha, \beta)$, $\alpha \in (0,1]$, $\beta \in [0, \infty)$, then there exists a constant $c > 0$ such that for all $|w| \geq 1$ the following inequality holds:*

$$\int_{|\tau|=1} |F(\tau, w)| \, |d\tau| = \int_{|\tau|=1} \left| \frac{\psi'(\tau)}{\psi(\tau) - \psi(w)} - \frac{1}{\tau - w} \right| |d\tau| \leq c < \infty,$$

and this integral converges uniformly with respect to $|w| \geq 1$.

In the proof of the main result Theorem 1.1, we will use the following theorem, which characterizes the Hölder's inequality in the variable exponent Lebesque spaces. See e.g. ([23, p. 24]) or ([4, p. 27]).

Theorem 2.3. *Let $p(.) : \Gamma \to [1, \infty]$ be a measurable function. If $f \in L^{p(.)}(\Gamma)$ and $g \in L^{q(.)}(\Gamma)$, such that $\frac{1}{p(.)} + \frac{1}{q(.)} = 1$, then $fg \in L^1(\Gamma)$ and the inequality*

$$\int_\Gamma |f(z)g(z)| \, dz \leq c(p) \|f\|_{L^{p(.)}} \|g\|_{L^{q(.)}} \tag{2.7}$$

holds.

3. Proofs

3.1. **Proof of Lemma 2.1.** Since Γ is smooth, from the equality (3) in [22, p. 44], we have that

$$arg\psi'(w) = \theta(s) - argw - \frac{\pi}{2}, \quad w = e^{it}.$$

Therefore,

$$\left| arg\psi'(e^{i(t+h)}) - arg\psi'(e^{it}) \right| \leq |\theta(s+h) - \theta(s)| + |h| .$$

By taking maximum norm over $s \in [0, 2\pi]$ and supremum over $|h| \leq \delta$, respectively and using (1.2) since $\Gamma \in \mathfrak{B}(\alpha, \beta)$, $\alpha \in (0,1]$ *and* $\beta \in [0, \infty)$, we have that

$$\begin{aligned} \left| arg\psi'(e^{i(t+h)}) - arg\psi'(e^{it}) \right| &\leq \|\theta(s+h) - \theta(s)\|_{[0,2\pi]} + |h| \\ &\leq \omega(\theta, \delta) + \delta \\ &\leq c_1 \delta^\alpha ln^\beta \frac{4}{\delta} + \delta, \end{aligned}$$

where $\alpha \in (0,1]$ *and* $\beta \in [0,\infty)$. Hence

$$\omega(arg\psi', \delta) \leq c_1 \delta^{\alpha} ln^{\beta} \frac{4}{\delta} + \delta. \tag{3.1}$$

It is known that

$$ln\psi'(w) = ln\,|\psi'(w)| + i arg\psi'(w)$$

for $|w| = 1$. From this formula, we can conclude that

$$\begin{aligned}\left|ln\psi'(e^{i(t+h)}) - ln\psi'(e^{it})\right| &\leq \left|ln \mid \psi'(e^{i(t+h)}) \mid - ln \mid \psi'(e^{it}) \mid\right| \\ &+ \left|arg\psi'(e^{i(t+h)}) - arg\psi'(e^{it})\right|. \end{aligned} \tag{3.2}$$

Taking into account the relation (1.3), we obtain that

$$\left|ln \mid \psi'(e^{i(t+h)}) \mid - ln \mid \psi'(e^{it}) \mid\right| \leq lnc_2 + lnc_3 = c_4 \tag{3.3}$$

On the other hand, the following inequality

$$\left|\psi'(e^{i(t+h)}) - \psi'(e^{it})\right| \leq c_5 \left|ln\psi'(e^{i(t+h)}) - ln\psi'(e^{it})\right| \tag{3.4}$$

holds (see [26, p. 140]). Hence, (3.2), (3.3) and (3.4) imply that

$$\left|\psi'(e^{i(t+h)}) - \psi'(e^{it})\right| \leq c_6 \left|arg\psi'(e^{i(t+h)}) - arg\psi'(e^{it})\right|$$

If we take maximum norm over T and supremum over $|h| \leq \delta$ in the last inequality and use (3.1), we obtain that

$$\omega(\psi', \delta) \leq c_6 \omega(arg\psi', \delta) \leq c_7 \delta^{\alpha} ln^{\beta} \frac{4}{\delta} + c_8 \delta$$

Now we divide by δ the both side of the last inequality and integrate from 0 to 1 and find that

$$\int_0^1 \frac{\omega(\psi', \delta)}{\delta} d\delta \leq c_9 \int_0^1 \delta^{\alpha-1} ln^{\beta} \frac{4}{\delta} d\delta \leq c_{10} \int_0^1 \delta^{\alpha-1-\varepsilon} d\delta$$

since $lim_{\delta\to 0} \delta^{\varepsilon} ln^{\beta} \frac{4}{\delta} = 0$. Hence we obtain that

$$\int_0^1 \frac{\omega(\psi', \delta)}{\delta} d\delta \leq c_{11} < \infty.$$

The proof is complete.

3.2. **Proof of Theorem 2.2.** $F(\tau, w)$ defined in (2.4) can be written in the following way:

$$F(\tau, w) = \left[\frac{\psi'(\tau)(\tau - w) - \psi(\tau) + \psi(w)}{(\tau - w)^2}\right] : \left[\frac{\psi(\tau) - \psi(w)}{\tau - w}\right]. \tag{3.5}$$

By (1.3), the inequality

$$\left[\frac{\psi(\tau) - \psi(w)}{\tau - w}\right] \geq c_{11} > 0, \quad |\tau| \geq 1, |w| \geq 1$$

holds. If we estimate the integral

$$I(w) = \int_{|\tau|=1} \left|\frac{\psi'(\tau)(\tau - w) - \psi(\tau) + \psi(w)}{(\tau - w)^2}\right| |d\tau| \tag{3.6}$$

with the similar prosedure as that in [26, p. 141], by taking into account that Γ belongs to the class $\mathfrak{B}(\alpha,\beta)$, $\alpha\in(0,1]$, $\beta\in[0,\infty)$ and using Lemma 2.1, we have that

$$I(w)\leq\int_{|\tau|=1}\frac{\omega(\psi',|\tau-w|)}{|\tau-w|}\,|d\tau|\leq c_{12}<\infty$$

Finally we obtain that

$$\int_{|\tau|=1}|F(\tau,w)|\,|d\tau|\leq c<\infty$$

The proof is complete.

3.3. **Proof of Theorem 1.1.** Let $z\in\Gamma$. From the relations (2.1) ve (2.2), it follows that

$$|R_n(z,f)|\leq\frac{1}{2\pi}\int_{|t|=R}|f_0(t)-P_n^*(t)|\left|\sum_{k=n+1}^{\infty}\frac{w^k}{t^{k+1}}\right||dt|$$

$$+\frac{1}{2\pi}\int_{|t|=R}|f_0(t)-P_n^*(t)|\left|\sum_{k=n+1}^{\infty}\frac{E_k(\psi(w))}{t^{k+1}}\right||dt|,$$

where $w=\varphi(z)$ and $t=\varphi(\zeta)$.

We find that

$$\begin{aligned}
I_1: &= \frac{1}{2\pi}\int_{|t|=R}|f_0(t)-P_n^*(t)|\left|\sum_{k=n+1}^{\infty}\frac{w^k}{t^{k+1}}\right||dt| \\
&= \frac{1}{2\pi}\int_{\Gamma_R}|f(\zeta)-P_n(\zeta)|\left|\sum_{k=n+1}^{\infty}\frac{[\varphi(z)]^k}{[\varphi(\zeta)]^{k+1}}\right||\varphi'(\zeta)|\,|d\zeta| \\
&\leq \frac{c_{14}}{2\pi}\int_{\Gamma_R}|f(\zeta)-P_n(\zeta)|\,\frac{|\varphi(z)|^{n+1}}{|\varphi(\zeta)|^{n+1}\,|\varphi(\zeta)-\varphi(z)|}\,|d\zeta| \\
&\leq \frac{c_{14}}{2\pi R^{n+1}(R-1)}\int_{\Gamma_R}\left(\omega(\zeta)\,|f(\zeta)-P_n(\zeta)|\right)\omega^{-1}(\zeta)\,|d\zeta|
\end{aligned}$$

Since $z\in\Gamma$, $|\varphi(z)|=1$, using the Hölder's inequality and the fact that $\omega(\cdot)\in A_{p(\cdot)}(\Gamma_R)$, we have that

$$\begin{aligned}
I_1 &\leq \frac{c_{15}(p)}{2\pi R^{n+1}(R-1)}\,\|\omega(f-P_n)\|_{L^{p(\cdot)}(\Gamma_R)}\left\|\omega^{-1}\right\|_{L^{q(\cdot)}(\Gamma_R)} \\
&\leq \frac{c_{16}(p)}{2\pi R^{n+1}(R-1)}\,\|(f-P_n)\|_{L^{p(\cdot)}_{\omega}(\Gamma_R)}
\end{aligned}$$

Hence, the last inequality implies that

$$I_1\leq\frac{c_{17}(p)}{R^{n+1}(R-1)}E_n^{\omega,p(\cdot)}(f,G_R) \tag{3.7}$$

Now we notate

$$I_2:=\frac{1}{2\pi}\int_{|t|=R}|f_0(t)-P_n^*(t)|\left|\sum_{k=n+1}^{\infty}\frac{E_k(\psi(w))}{t^{k+1}}\right||dt|$$

Using the definition of $E_k(\psi(w))$ and assumptions of Theorem 2.2, we find that

$$I_2 \leq \frac{1}{2\pi}\int_{|t|=R}|f_0(t)-P_n^*(t)|\,\frac{1}{2\pi}\int_{|\tau|=1}\left|\sum_{k=n+1}^{\infty}\frac{\tau^k}{t^{k+1}}F(\tau,w)\right||d\tau|\,|dt|$$

$$\leq \frac{1}{4\pi^2}\int_{|\tau|=1}|F(\tau,w)|\left\{\int_{|t|=R}|f_0(t)-P_n^*(t)|\left|\frac{\tau^{n+1}}{t^{n+1}(t-\tau)}\right||dt|\right\}|d\tau|$$

$$\leq \frac{1}{4\pi^2R^{n+1}(R-1)}\int_{|\tau|=1}|F(\tau,w)|\,|d\tau|\left\{\int_{|t|=R}|f_0(t)-P_n^*(t)|\,|dt|\right\}$$

$$\leq \frac{c_{18}}{4\pi^2R^{n+1}(R-1)}\int_{\Gamma_R}|f(\zeta)-P_n(\zeta)|\,|\varphi'(\zeta)|\,|d\zeta|\,.$$

Again using the Hölder's inequality and the fact that $\omega(\cdot)\in A_{p(\cdot)}(\Gamma_R)$, we have that

$$\begin{aligned} I_2 &\leq \frac{c_{19}(p)}{4\pi^2R^{n+1}(R-1)}\left\|\omega[f-P_n]\right\|_{L^{p(\cdot)}(\Gamma_R)}\left\|\omega^{-1}\right\|_{L^{q(\cdot)}(\Gamma_R)} \\ &\leq \frac{c_{20}(p)}{R^{n+1}(R-1)}\left\|(f-P_n)\right\|_{L^{p(\cdot)}_{\omega}(\Gamma_R)}. \end{aligned}$$

The last inequality implies that

$$I_2 \leq \frac{c_{21}(p)}{R^{n+1}(R-1)}E_n^{\omega,p(.)}(f,G_R). \tag{3.8}$$

From (3.7) and (3.8), we can finally conclude that

$$|R_n(z,f)|\leq I_1+I_2\leq \frac{c(p)}{R^{n+1}(R-1)}E_n^{\omega,p(.)}(f,G_R)$$

with some constant $c(p)>0$ independent of n. The proof is complete.

4. Conclusion

In this paper, assuming that a function f belongs to the weighted Smirnov class with variable exponent $E^{p(.)}_{\omega}(G_R)$ where $R>1$ is the largest number such that the function f is analytic inside G_R, we obtained the rate of maximal convergence of the Faber expansion of the function f. The result obtained in Theorem 1.1 improves and generalises the result in Theorem 1 in [14], because of our study on the domains bounded by a curve Γ of the class of smooth curves and the study on the weighted case, respectively. And also, the result obtained in Theorem 1.1 coincides with the Suetin's result in [26, p. 203, Theorem 3] when $p(.)=p>1$ is a constant and $\omega(.)=1$.

Acknowledgments. The author would like to thank the anonymous referee for his/her comments that helped to improve this article.

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