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EXISTENCE OF SOLUTION TO A CLASS OF INTEGRAL EQUATIONS VIA F- CONTRACTIONS

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Abstract. The notion of generalized F-contractions is presented in this article within the context of metric-like spaces. The presence of a fixed point of such contractive mappings is established. An example is also provided to support the correctness of the acquired results. Moreover, our results are used to prove the existence of a solution to an integral equation.

1. Introduction

In 1992, Matthews [12] introduced the concept of a partial metric space which is a generalized metric space. Further, Matthews showed that the Banach contraction principle is valid in partial metric spaces and can be applied in program verification. In this space, the usual metric is replaced by a partial metric with an interesting property that the self-distance of any point of space may not be zero. After that, fixed point results in partial metric spaces were studied by many other authors.
Furthermore, ÓNeill [13] coined the idea of dualistic partial metric by extending the range $\mathbb{R}_0^+$ to $\mathbb{R}$.
Heckmann [9] extended it by omitting the small self-distance axiom. The partial metric defined by Heckmann is called a weak partial metric.
Very recently, Hitzler and Seda [10] generalized the partial metric spaces by introducing dislocated space and projected their generalization of Banach-Caccioppoli's theorem to obtain a unique supported model for acceptable logic programs.
Recently, applications based discussion on new contractions, providing young researchers with fresh ideas, you may refer M. Younis et al. [18, 19, 20, 21, 22]. In 2012, Amini-Harandi [2] introduced a new generalization of a partial metric space which is called a metric-like space.

Definition 1.1. [12] *A partial metric on a nonempty set X is a function $p : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$*
(p1) $x = y$ iff $p(x, x) = p(x, y) = p(y, y)$;
(p2) $p(x, x) \leq p(x, y)$;
(p3) $p(x, y) = p(y, x)$;

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(p4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.
A partial metric space is a pair (X,p) *such that* X *is a nonempty set and* p *is a partial metric on* X.

Every partial metric space is a metric-like space. Below we give another example of metric-like spaces.

Example 1.1. [12] *Let* $X = [0,1]$ *then the mapping* $\sigma_1 : X \times X \to \mathbb{R}$ *defined by* $\sigma_1(x,y) = x + y - xy$ *is a metric like space on* X.

Note that every partial metric space is a metric-like space, but the converse may not be true.

Example 1.2. [11] *Let* $X = \mathbb{R}, k \geq 0$ *and* $\sigma : X \times X \to \mathbb{R}^+$ *be defined by*

$$\sigma(x,y) = \begin{cases} 2k, & \text{if } x = y = 0, \\ k, & \text{otherwise.} \end{cases}$$

Then (X,σ) *is a metric-like space, but for* $k > 0$ *it is not a partial metric space, as* $\sigma(0,0) \nleq \sigma(0,1)$.

Definition 1.2. [2] *Let* (X,σ) *be a metric-like space. Then*
(1) a sequence $\{x_n\}$ *in a metric-like space* (X,σ) *converges to* $x \in X$ *if and only if* $\lim_{n\to\infty}(x_n, x) = \sigma(x,x)$;
(2) a sequence $\{x_n\}$ *in a metric-like space* (X,σ) *is called a* σ*- Cauchy sequence if and only if* $\lim_{n,m\to\infty}(x_n, x_m)$ *exists and is finite;*
(3) a metric-like space (X,σ) *is said to be complete if every* σ*- Cauchy sequence* $\{x_n\}$ *in* X *converges, with respect to* τ_σ, *to a point* $x \in X$ *such that*

$$\sigma(x,x) = \lim_{n\to\infty}(x_n, x) = \lim_{n,m\to\infty}(x_n, x_m).$$

Lemma 1.1. *Let* (X,σ) *be a metric-like space and* $\{x_n\}$ *be a sequence in* X. *If the sequence* $\{x_n\}$ *converges to some* $x \in X$ *with* $\sigma(x,x) = 0$ *then* $\lim_{n\to\infty}(x_n, y) = \sigma(x,y)$ *for all* $x \in X$.

2. Fixed point results for partially ordered metric like spaces

Berinde initiated some new mappings, called weak contraction mappings in a metric space ([6],[7]). He demonstrated that Banach's, Kannan's, and Chatterjee's mappings are weak contractions. Afterward, many generalizations of these results in several spaces appeared in the literature. A detailed synthesis of fixed point problems and their applications can be found in the noteworthy manuscripts [3, 4, 5, 18, 20, 14, 15, 16, 17] Berinde-type weak contractions are usually called almost contractions. Clubbing the ideas of Berinde, ψ, ϕ and the notion of F-contraction, subsequent ψ, ϕ-Berinde-type F- contractive mapping is defined in the framework of partially ordered metric-like spaces.

2.1. Results via (ψ, ϕ) Berinde-type F- contraction.

Definition 2.1. *Let* $(X, \sigma, \precsim)$ *be a complete partially ordered metric-like space. Let* $f : X \to X$ *be mapping. Suppose* $\tau \geq 0$ *and* $F \in \Delta_F$ *are such that for all* $x, y \in X$ *with* $x \neq y$,

$$\begin{aligned} &\sigma(fx, fy) > 0 \Rightarrow \\ &\tau + F(\psi(\sigma(fx, fy))) \leq F(\psi(M(x,y))) - \phi(M(x,y)) + L(N(x,y)) \end{aligned} \tag{2.1}$$

where

$$M(x,y)=\max\Big\{\sigma(x,y),\sigma(x,fx),\sigma_b(y,fy),\frac{\sigma(x,fy)+\sigma(fx,y)}{2}\Big\}, \tag{2.2}$$

and

$$N(x,y)=\min\{\sigma^s(x,fx),\sigma^s(y,fy),\sigma^s(x,fy),\sigma^s(y,fx)\},$$

with $L\geq 0$.
Then mapping f *is called partially ordered* (ψ,ϕ) *Berinde-type* F*- contraction.*

Theorem 2.1. Let $(X,\sigma,\precsim)$ be a complete partially ordered metric like space and $f:X\to X$ be a continuous and non-decreasing (ψ,ϕ) Berinde-type F contraction. If there exist $x_0\in X$ with $x_0\precsim fx_0$, then f has a fixed point.

Proof. Let $x_0\in X$ be such that $x_0\precsim fx_0$ and let $\{x_n\}$ be the sequence of initial point x_0 that is $x_n=f^nx_0=fx_{n-1}$. If $x_n=x_{n-1}$ for some $n\in N$, then x_n is a fixed point of f.
Now let $d_n=d(x_n,x_{n+1})$ for all $n\in N\cup\{0\}$. Assume that $x_n\neq x_{n-1}$ for all $n\in N$. As f is non-decreasing and $x_0\precsim fx_0$, we deduce that

$$x_0\prec x_1\prec x_2\prec\cdots\prec x_n\cdots \tag{2.3}$$

that is x_n and x_{n+1} are comparable and $fx_{n-1}\neq fx_n$ for all $n\in N\cup\{0\}$.
Now we construct a sequence $\{x_n\}$ in X in such a way that $x_n=fx_{n-1}$ for all $n\in N\cup\{0\}$.
Suppose that $\sigma(x_{n_0},x_{n_0+1})=\sigma(x_{n_0},fx_{n_0})=0$, for some $n_0\geq 0$. Then one can get $x_{n_0}=x_{n_0+1}=fx_{n_0}$ then x_{n_0} is a required fixed point, and we are done in this case. Thus, for now, assume that $\sigma(x_n,fx_n)>0$, for all $n\in N$. Consequently, we have

$$\sigma(fx_n,fx_{n+1})=\sigma(x_{n+1},x_{n+2}),\qquad\forall\ n\in N.$$

Then by the Definition 2.1 with $x=x_n$ and $y=x_{n+1}$, we have

$$\begin{aligned}\tau+F\Big(\psi\big(\sigma(fx_n,fx_{n+1})\big)\Big)\leq{}&F\Big(\psi\big(M(x_n,x_{n+1})\big)\Big)-\phi\big(M(x_n,x_{n+1})\big)\\&+L\big(N(x_n,x_{n+1})\big)\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}N(x_n,x_{n+1})&=\min\{\sigma^s(x_n,fx_n),\sigma^s(x_{n+1},fx_{n+1}),\sigma^s(x_n,fx_{n+1}),\sigma^s(x_{n+1},fx_n)\\&=\min\{\sigma^s(x_n,x_{n+1}),\sigma^s(x_{n+1},x_{n+2}),\sigma^s(x_n,x_{n+2}),\sigma^s(x_{n+1},x_{n+1})\\&=0\end{aligned}$$

and

$$\begin{aligned}M(x_n,x_{n+1})&=\max\Big\{\sigma(x_n,x_{n+1}),\sigma(x_n,fx_n),\sigma(x_{n+1},fx_{n+1}),\frac{\sigma(x_n,fx_{n+1})+\sigma(fx_n,x_{n+1})}{2}\Big\}\\&=\max\Big\{\sigma(x_n,x_{n+1}),\sigma(x_n,x_{n+1}),\sigma(x_{n+1},x_{n+2}),\frac{\sigma(x_n,x_{n+2})+\sigma(x_{n+1},x_{n+1})}{2}\Big\}\\&=\max\Big\{\sigma(x_n,x_{n+1}),\sigma(x_{n+1},x_{n+2}),\frac{\sigma(x_n,x_{n+1})+\sigma(x_{n+1},x_{n+2})}{2}\Big\}\\&=\max\Big\{\sigma(x_n,x_{n+1}),\sigma(x_{n+1},x_{n+2})\Big\}.\end{aligned}$$

If $M(x_n,x_{n+1})=\sigma(x_{n+1},x_{n+2})$ then from (2.4),

$$\tau+F\Big(\psi\big(\sigma(x_{n+1},x_{n+2})\big)\Big)\leq F\Big(\psi\big(\sigma(x_{n+1},x_{n+2})\big)\Big)-\phi\big(\sigma(x_{n+1},x_{n+2})\big) \tag{2.5}$$

Which leads to a contradiction, in view of $F1$ and the hypothesis of ψ, ϕ as $\phi\big(\sigma(x_{n+1}, x_{n+2})\big) > 0$. Then we arrive at

$$\begin{aligned} \tau + F\Big(\psi\big(\sigma(x_{n+1}, x_{n+2})\big)\Big) &\leq F\Big(\psi\big(\sigma(x_n, x_{n+1})\big)\Big) - \phi\big(\sigma(x_n, x_{n+1})\big) \\ &< F\Big(\psi\big(\sigma(x_n, x_{n+1})\big)\Big). \end{aligned} \tag{2.6}$$

Thus from (2.6) and $F1$, we get

$$\tau + \psi\big(\sigma(x_{n+1}, x_{n+2})\big) < \psi\big(\sigma(x_n, x_{n+1})\big) \tag{2.7}$$

or equivalently

$$\tau + \sigma(x_n, x_{n+1}) < \sigma(x_{n-1}, x_n), \quad \forall\ n \in N.$$

Therefore $\{\sigma(x_n, x_{n+1})\}_{n \in N}$ is a non negative decreasing sequence of real numbers and is bounded below at 0, consequently convergent to some point $p \in \mathbb{R}^+$, now we claim that $p = 0$. Now suppose $p > 0$.
Letting $n \to \infty$ in (2.6), we have $\tau + F\big(\psi(p)\big) \leq F\big(\psi(p)\big) - \phi(p)$,
which is a contradiction in view of $F1$ and ϕ. Thus we have $p = 0$.
Consequently, we have

$$\lim_{n\to\infty} \sigma(fx_n, fx_{n+1}) = \lim_{n\to\infty} \sigma(x_{n+1}, x_{n+2}) = 0. \tag{2.8}$$

Now we will show that $\{x_n\}$ is a σ-Cauchy sequence.
Suppose that $\{x_n\}$ is not a σ-Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ with $n_k > m_k > k$ such that

$$\sigma(x_{n_k}, x_{m_k}) \geq \epsilon. \tag{2.9}$$

Further, corresponding to m_k, we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ satisfying (2.9). Then

$$\sigma(x_{n_k-1}, x_{m_k}) < \epsilon. \tag{2.10}$$

Using (2.9), (2.10) and the triangle inequality, we have

$$\epsilon \leq \sigma(x_{n_k}, x_{m_k}) \leq \sigma(x_{n_k}, x_{n_k-1}) + \sigma(x_{n_k-1}, x_{m_k}) < \sigma(x_{n_k}, x_{n_k-1}) + \epsilon.$$

Letting $k \to \infty$ and using (2.8), we obtain

$$\lim_{k\to\infty} \sigma(x_{n_k}, x_{m_k}) = \epsilon. \tag{2.11}$$

Again, the triangle inequality gives us

$$\sigma(x_{n_k-1}, x_{m_k}) \leq \sigma(x_{n_k-1}, x_{n_k}) + \sigma(x_{n_k}, x_{m_k}),$$

$$\sigma(x_{n_k}, x_{m_k}) \leq \sigma(x_{n_k}, x_{n_k-1}) + \sigma(x_{n_k-1}, x_{m_k}).$$

Then we have $|\sigma(x_{n_k-1}, x_{m_k}) - \sigma(x_{n_k}, x_{m_k})| \leq \sigma(x_{n_k}, x_{m_k-1})$. Letting $k \to \infty$ in the above inequality and using (2.8) and (2.11), we get

$$\lim_{k\to\infty} \sigma(x_{n_k-1}, x_{m_k}) = \epsilon. \tag{2.12}$$

Similarly, we can show that

$$\begin{aligned} \lim_{k\to\infty} \sigma(x_{n_k}, x_{m_k-1}) &= \lim_{k\to\infty} \sigma(x_{n_k-1}, x_{m_k-1}) \\ &= \lim_{k\to\infty} \sigma(x_{n_k}, x_{m_k+1}) \\ &= \lim_{k\to\infty} \sigma(x_{n_k+1}, x_{m_k}) = \epsilon \end{aligned} \tag{2.13}$$

As

$$\begin{aligned} M(x_{n_k-1}, x_{m_k-1}) &= \max\Big\{\sigma(x_{n_k-1}, x_{m_k-1}), \sigma(x_{n_k-1}, fx_{n_k-1}), \sigma(x_{m_k-1}, fx_{m_k-1}), \\ &\qquad \frac{\sigma(x_{n_k-1}, fx_{m_k-1}) + \sigma(fx_{n_k-1}, x_{m_k-1})}{2}\Big\} \\ &= \max\Big\{\sigma(x_{n_k-1}, x_{m_k-1}), \sigma(x_{n_k-1}, x_{n_k}), \sigma(x_{m_k-1}, x_{m_k}), \\ &\qquad \frac{\sigma(x_{n_k-1}, x_{m_k}) + \sigma(x_{n_k}, x_{m_k-1})}{2}\Big\}, \end{aligned}$$

using (2.8) and (2.11)-(2.13), we have

$$\lim_{k\to\infty} \max\{\epsilon, 0, 0, \epsilon\} = \epsilon. \tag{2.14}$$

As $n_k > m_k$ and $x_{n_k-1}$and x_{m_k-1} are comparable, setting $x = x_{n_k-1}$ and $y = x_{m_k-1}$ in (2.1), we obtain

$$\begin{aligned} \tau + F\Big(\psi\big(\sigma(x_{n_k}, x_{m_k})\big)\Big) &= \tau + F\Big(\psi\big(\sigma(fx_{n_k-1}, fx_{m_k-1})\big)\Big) \\ &\le F\Big(\psi\big(M(x_{n_k-1}, x_{m_k-1})\big)\Big) - \phi\big(M(x_{n_k-1}, x_{m_k-1})\big). \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (2.11) and (2.14), we get

$$\tau + F\big(\psi(\epsilon)\big) \le F\big(\psi(\epsilon)\big) - \phi(\epsilon),$$

which is a contradiction as $\epsilon > 0$. Hence $\{x_n\}$ is a σ-Cauchy sequence. By the completeness of X, there exists $z \in X$ such that $\lim\limits_{n\to\infty} x_n = z$, that is,

$$\lim_{n\to\infty} \sigma(x_n, z) = \sigma(z, z) = \lim_{m,n\to\infty} \sigma(x_m, x_n) = 0 \tag{2.15}$$

Moreover, the continuity of F implies that

$$\lim_{n\to\infty} \sigma(x_{n+1}, z) = \lim_{n\to\infty} \sigma(fx_n, z) = \sigma(fz, z) = 0$$

and this proves that z is a fixed point. □

Notice that the continuity of f in Theorem 2.1 is not necessary and can be dropped.

Theorem 2.2. *Under the same hypotheses of Theorem 2.1 and without assuming the continuity of f, assume that whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x \in X$ implies $x_n \precsim x$ for all $n \in N$, then f has a fixed point in X.*

Proof. Following similar arguments to those given in Theorem 2.1, we construct a nondecreasing sequence $\{x_n\}$ in X such that $x_n \to z$ for some $z \in X$. Using the assumption of X, we have $x_n \precsim z$ for every $n \in N$. Now, we show that $fz = z$. Suppose

$$F(fz, z) = \lim_{n\to\infty} F(fz, x_{n+1}) = \lim_{n\to\infty} F(fz, fx_n) > 0,$$

then from (2.1), we have

$$\begin{aligned} \tau + F\Big(\psi\big(\sigma(fz, x_{n+1})\big)\Big) &= \tau + F\Big(\psi\big(\sigma(fz, fx_n)\big)\Big) \\ &\le F\Big(\psi\big(M(z, x_m)\big)\Big) - \phi\big(M(z, x_n)\big) + L\big(N(z, x_n)\big). \end{aligned} \tag{2.16}$$

where

$$
\begin{aligned}
\sigma(fz,z) &\le M(z,x_n)\\
&= \max\Big\{\sigma(z,x_n),\sigma(fz,z),\sigma(x_n,x_{n+1}),\frac{\sigma(z,x_{n+1})+\sigma(fz,x_n)}{2}\Big\}\\
&= \max\Big\{\sigma(z,x_n),\sigma(fz,z),\sigma(x_n,x_{n+1}),\frac{\sigma(z,x_{n+1})+\sigma(fz,z)+\sigma(z,x_n)}{2}\Big\}.
\end{aligned}
$$

Taking limit as $n\to\infty$, by (2.15), we obtain

$$
\begin{aligned}
\sigma(fz,z) \le \lim_{n\to\infty} M(z,x_n) \le \sigma(z,fz)\\
\Rightarrow \quad M(z,x_n) = \sigma(z,fz)
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma(fz,z) &\le N(z,x_n)\\
&= \min\Big\{\sigma^s(z,fz),\sigma^s(x_n,fx_n),\sigma^s(z,fx_n),\sigma^s(x_n,fz)\Big\}\\
&= \min\Big\{\sigma^s(z,fz),\sigma^s(x_n,x_{n+1}),\sigma^s(z,x_{n+1}),\sigma^s(x_n,fz)\Big\}
\end{aligned}
$$

Taking limit as $n\to\infty$, by (2.15), we obtain

$$\lim_{n\to\infty} N(z,x_n) = 0.$$

Therefore, letting $n\to\infty$ in (2.16), we get

$$\tau + F\Big(\psi\big(\sigma(fz,z)\big)\Big) \le F\Big(\psi\big(\sigma(fz,z)\big)\Big) - \phi\big(\sigma(fz,z)\big),$$

which is a contradiction in view of $F1,\psi$ and ϕ.
Then $\sigma(fz,z)=0$. Thus $fz=z$. □

Next theorem gives a sufficient condition for the uniqueness of the fixed point.

Theorem 2.3. Let all the conditions of Theorem 2.1 (resp. Theorem 2.2) be fulfilled and let the following condition be satisfied: For arbitrary two points $x,y\in X$, there exists $z\in X$ which is comparable with both x and y. Then the fixed point of f is unique.

Proof. Suppose that there exist $z,x\in X$ which are fixed points. We distinguish two cases.
Case I. If x is comparable to z, then $F^n x = x$ is comparable to $F^n z = z$ for $n=1,2,3,\cdots$ and

$$
\begin{aligned}
\tau + F\Big(\psi\big(\sigma(z,x)\big)\Big) &= \tau + F\Big(\psi\big(\sigma(f^n z,f^n x)\big)\Big)\\
&\le F\Big(\psi\big(M(f^{n-1}z,f^{n-1}x)\big)\Big) - \phi\big(M(f^{n-1}z,f^{n-1}x)\big) \qquad (2.17)\\
&\quad + L\big(N(f^{n-1}z,f^{n-1}x)\big).
\end{aligned}
$$

where

$$
\begin{aligned}
M(z,x) &= \max\Big\{\sigma(z,x),\sigma(z,fz),\sigma(x,fx),\frac{\sigma(z,fx)+\sigma(fz,x)}{2}\Big\}\\
&= \Big\{\sigma(z,x),\sigma(z,z),\sigma(x,x),\frac{\sigma(z,x)+\sigma(z,x)}{2}\Big\} \qquad (2.18)\\
&= \sigma(z,x)
\end{aligned}
$$

and

$$\begin{aligned} N(z,x) &= \min\Big\{\sigma^s(z,fz),\sigma^s(x,fx),\sigma^s(z,fx),\sigma^s(x,fz)\Big\} \\ &= \min\Big\{\sigma^s(z,z),\sigma^s(x,x),\sigma^s(z,x),\sigma^s(x,z)\Big\} \\ &= 0 \end{aligned} \tag{2.19}$$

Using (2.17),(2.18) and (2.19), we have

$$\tau + F\Big(\psi\big(\sigma(z,x)\big)\Big) \le F\Big(\psi\big(\sigma(z,x)\big)\Big) - \phi\big(\sigma(z,x)\big),$$

which is a contradiction in view of $F1, \psi$ hypothesis of and ϕ.
Then $\sigma(z,x) = 0$. This implies that $z = x$.
Case II. If x is not comparable to z, then there exists $y \in X$ comparable to x and z. The monotonicity of f implies that $f^n y$ is comparable to $f^n x = x$ and $f^n z = z$, for $n = 1,2,3,\cdots$
Moreover,

$$\begin{aligned} \tau + F\Big(\psi\big(\sigma(z,f^n y)\big)\Big) &= \tau + F\Big(\psi\big(\sigma(f^n z,f^n y)\big)\Big) \\ &\le F\Big(\psi\big(M(f^{n-1}z,f^{n-1}y)\big)\Big) - \phi\big(M(f^{n-1}z,f^{n-1}y)\big) \\ &\quad + L\big(N(f^{n-1}z,f^{n-1}y)\big). \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} N(f^{n-1}z,f^{n-1}y) &= \min\Big\{\sigma^s(f^{n-1}z,ff^{n-1}z),\sigma^s(f^{n-1}y,ff^{n-1}y),\sigma^s(f^{n-1}z,f^n y),\sigma^s(f^{n-1}y,f^n z)\Big\} \\ &= \min\Big\{\sigma^s(f^{n-1}z,f^n z),\sigma^s(f^{n-1}y,f^n y),\sigma^s(f^{n-1}z,f^n y),\sigma^s(f^{n-1}y,f^n z)\Big\} \\ &= \min\Big\{\sigma^s(z,z),\sigma^s(f^{n-1}y,f^n y),\sigma^s(z,f^n y),\sigma^s(f^{n-1}y,z)\Big\} \\ &= 0. \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} M(f^{n-1}z,f^{n-1}y) &= \max\Big\{\sigma(f^{n-1}z,f^{n-1}y),\sigma(f^{n-1}z,ff^{n-1}z),\sigma(f^{n-1}y,ff^{n-1}y), \\ &\qquad \frac{\sigma(f^{n-1}z,ff^{n-1}y)+\sigma(ff^{n-1}z,f^{n-1}y)}{2}\Big\} \\ &= \max\Big\{\sigma(f^{n-1}z,f^{n-1}y),\sigma(f^{n-1}z,f^n z),\sigma(f^{n-1}y,f^n y), \\ &\qquad \frac{\sigma(f^{n-1}z,f^n y)+\sigma(f^n z,f^{n-1}y)}{2}\Big\} \\ &= \max\Big\{\sigma(z,f^{n-1}y),\sigma(z,z),\sigma(f^{n-1}y,f^n y),\frac{\sigma(z,f^n y)+\sigma(z,f^{n-1}y)}{2}\Big\} \\ &\le \max\Big\{\sigma(z,f^{n-1}y),\sigma(z,f^n y)\Big\} \end{aligned} \tag{2.22}$$

for n sufficiently large, because $\sigma(f^{n-1}y,f^{n-1}y) \to 0$ and $\sigma(f^{n-1}y,f^n y) \to 0$ when $n \to \infty$. Similarly as in the proof of Theorem 2.1, it can be shown that $\sigma(z,f^n y) \le M(z,f^{n-1}y) \le \sigma(z,f^{n-1}y)$. It follows that the sequence $\{\sigma(z,f^n y)\}$ is non-negative decreasing and, consequently, there exists $\alpha \ge 0$ such that

$$\lim_{n\to\infty} \sigma(z,f^n y) = \lim_{n\to\infty} M(z,f^{n1}y) = \alpha.$$

We suppose that $\alpha > 0$. Then letting $n \to \infty$ in (2.20), we have

$$\tau + F\big(\psi(\alpha)\big) \le F\big(\psi(\alpha)\big) - \phi(\alpha),$$

which is a contradiction. Hence $\alpha = 0$. Similarly, it can be proved that

$$\lim_{n\to\infty} \alpha(x, f^n y) = 0.$$

Now, passing to the limit in $\sigma(x,z) \le \sigma(x, f^n y) + \sigma(f^n y, z)$, it follows that $\sigma(x,z) = 0$, so $x = z$, and the uniqueness of the fixed point is proved. □

We now discuss the following consequence of Theorem 2.1

As $\phi : [0,\infty] \to [0,\infty]$ and taking $L = 0$ in Theorem 2.1, we get the following corollary.

Corollary 2.4. *Let $(X, \sigma, \precsim)$ be a complete partially ordered metric like space and $f : X \to X$ be a continuous mapping. If there exist $\tau > 0, F \in \Delta_F$ and $\psi \in \Psi$ such that for all $x, y \in X$ with $x \neq y$,*

$$\begin{aligned} &\sigma(fx, fy) > 0 \Rightarrow \\ &\tau + F\Big(\psi\big(\sigma(fx, fy)\big)\Big) \le F\Big(\psi\big(M(x,y)\big)\Big) - \phi\big(M(x,y)\big) \end{aligned} \tag{2.23}$$

where

$$M(x,y) = \max\Big\{\sigma(x,y), \sigma(x, fx), \sigma_b(y, fy), \frac{\sigma(x, fy) + \sigma(fx, y)}{2}\Big\},$$

Then f has a unique fixed point.

Now, we present an example to support the useability of our results.

Example 2.1. *Let $X = [0,1]$ be equipped with partial order relation $\preceq$ defined by*

$$x \precsim y \Rightarrow x > y.$$

Define function $\sigma : X \times X \to [0,\infty)$ by

$$\sigma(x,y) = \begin{cases} 2x, & x = y \\ \max(x,y), & \textit{otherwise.} \end{cases}$$

Then (X, σ) is a complete metric-like space.
Let $fx = \frac{x}{\sqrt{x^3+2}}$. Clearly for all $x \in X, x \preceq fx$.
Taking $\psi = \log 3^x$, $\phi(t) = \frac{1}{100+t^3}$ and $F(\alpha) = \alpha + \log \alpha$.
Here $x \preceq y$ then without loss of generality it is assumed that $x > y$. Now we calculate values

$$\sigma(fx, fy) = \frac{x}{\sqrt{x^3+2}}, \quad \sigma(x,y) = x, \quad \sigma(x, fx) = x,$$

$$\sigma(y, fy) = y, \quad \sigma(x, fy) = x, \quad \sigma(fx, y) = fx \textit{ or } y, \quad \textit{and} \quad M(x,y) = x$$

and

$$\begin{aligned} \sigma^s(x, fx) &= |2\sigma(x, fx) - \sigma(x,x) - \sigma(fx, fy)| \\ &= |2x - 2x - 2fx| \\ &= 2\frac{x}{\sqrt{x^3+2}}, \end{aligned}$$

$\sigma^s(x, fy) = 2\frac{y}{\sqrt{y^3+2}}$,
when $fx > y$, $\quad \sigma^s(y, fx) = |2fx - 2y - 2fx| = 2y$,

when $fx < y$, $\quad \sigma^s(y, fx) = |2y - 2y - 2fx| = 2\frac{x}{\sqrt{x^3+2}}$,
$\sigma^s(y, fy) = |2y - 2y - 2fx| = 2\frac{y}{\sqrt{y^3+2}}$, *so*

$$N(x,y) = \min\{\sigma^s(x,fx), \sigma^s(x,fy), \sigma^s(y,fx), \sigma^s(y,fy)\} = \frac{2y}{\sqrt{y^3+2}}$$

then L.H.S. of (2.1) *becomes*

$$\tau + \log\left(\log 3^{\frac{x}{\sqrt{x^3+2}}}\right) + \log 3^{\frac{x}{\sqrt{x^3+2}}}$$

and R.H.S. comes out

$$\log(\log 3^x) + \log 3^x - \frac{1}{100 + x^3} + L\frac{2y}{\sqrt{y^3+2}}.$$

Following figures shows that R.H.S. function with colored figure dominates L.H.S. function with black and white checked function.

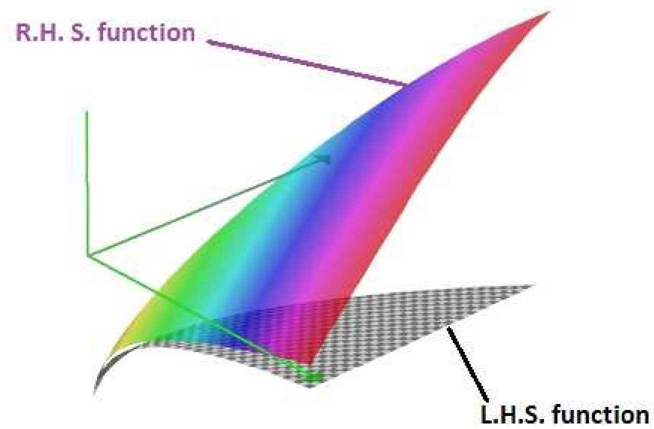

FIGURE 1. Domination of R.H.S. over L.H.S., exactly in [0, 1]

Thus we see that (2.1) *is satisfied with* $\tau \in (0, 0.13]$ *and* $L = [3, \infty)$.
Hence all the conditions of Theorem 2.1 are fulfilled then Tx *has a fixed point as* $x = 0$ *which is indeed unique in* $[0, 1]$, *which is demonstrated by following figure.*

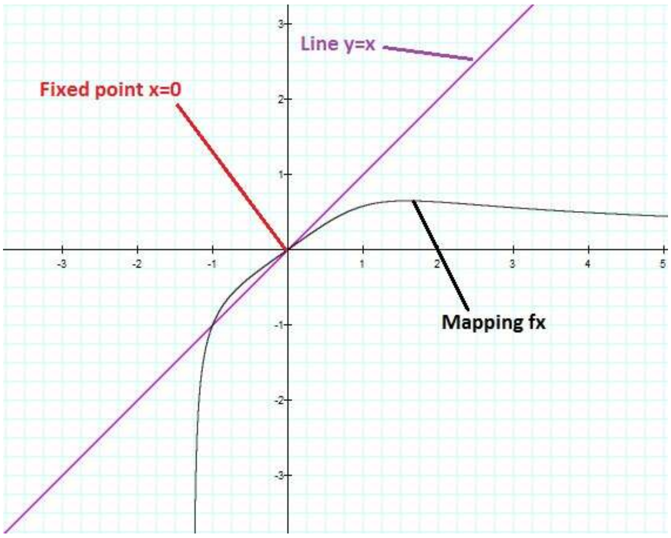

FIGURE 2. Fixed Point of f

3. Application to integral equation

Fixed point theory is critical in determining the solution of many nonlinear models in engineering and research. This section will use the previously stated fixed point findings to explain the existence of a solution to a class of integral equations 3.1. Consider the following integral equation

$$\theta(t) = P(t) + \int_a^b K(t,u)f(u,\theta(u)du, \forall t \in [a,b] \tag{3.1}$$

Consider the space $X = C([a,b],R)$ of continuous functions on $[a,b]$. For $\theta \in X$, we define

$$\|\theta_1 - \theta_2\| = \sup_{s\in[a,b]} \{|\theta_1(t) - \theta_2(t)|\}. \tag{3.2}$$

Define metric-like $\sigma : X \times X \to \mathbb{R}^+$ by

$$\sigma(x,y) = \|x-y\| + \|x\| + \|y\|, \quad for\ \ all\ \ x,y \in X, \tag{3.3}$$

where $\|x-y\|$ is defined by (3.2).
Equivalent metric to metric-like space is given by

$$d_\sigma(x,y) = 2\sigma(x,y) - \sigma(x,x) - \sigma(y,y) = 2\|x-y\|.$$

Clearly $d_\sigma(x,y)$ is complete and hence (X,σ) is also complete.
Consider the self map $A : X \to X$ defined by

$$A(\theta(t)) = \int_a^b K(t,u)f(u,\theta(u))du, \ \ t \in [a,b] \text{ and for all } \theta \in X.$$

It is evident that θ is a solution of equation (3.1) if and only if θ is fixed point of A. Now succeeding theorem is established for the guarantee of the existence of a fixed point of A.

Theorem 3.1. Consider the problem (3.1) and assume that

(I) $f : [a, b] \times R \to R$ is continuous;
(II) $P : [a, b] \to R$ is continuous;
(III) $K : [a, b] \times R \to [0, \infty)$ is continuous;
(IV) $\psi \in \Psi$ such that for all $\theta_1, \theta_2 \in R$, such that

$$|f(u, \theta_1) - f(u, \theta_2)| \leq e^{-\tau}|\theta_1 - \theta_2| \text{ for every } u \in [a, b];$$

(V)
there exists a continuous function $z : I \to \mathbb{R}^+$ such that

$$|f(u, \theta_1)| \leq e^{-\tau}|\theta_1| \text{ for every } u \in [a, b];$$

(VI) assume that $\sup\limits_{t\in[a,b]} = \int_a^b K(t, u)du \leq 1$.
Then the integral equation (3.1) has a solution in X.

Proof. For all $\theta_1, \theta_2 \in X$ such that $A\theta_1(t) \neq A\theta_2(t)$, we have

$$\begin{aligned}
|A\theta_1(t) - A\theta_2(t)| &\leq \left| \int_a^b k(t, u)\big(f(u, \theta_1(u)) - f(u, \theta_2(u))\big) \right| du \\
&\leq \int_a^b K(t, u)\, |f(u, \theta_1(u)) - f(u, \theta_2(u))|\, du \\
&\leq e^{-\tau} \int_a^b K(t, u)\Big|\theta_1(u) - \theta_2(u)\Big|du \\
&\leq e^{-\tau}\|\theta_1 - \theta_2\| \sup_{t\in[a,b]} \int_a^b K(t, u)du \\
&\leq e^{-\tau}\|\theta_1 - \theta_2\|.
\end{aligned}$$

Ultimately we have

$$\|A\theta_1(t) - A\theta_2(t)\| \leq e^{-\tau}\|\theta_1 - \theta_2\|. \tag{3.4}$$

Also, we have

$$\begin{aligned}
\Big|A\theta_1(t)\Big| &= \left| \int_a^b K(t, u)f(u, \theta(u)) \right| du \\
&\leq \int_a^b K(t, u)\Big|f(u, \theta(u))\Big|du \\
&\leq \int_a^b K(t, u)e^{-\tau}\Big|\theta_1\Big|du \\
&\leq e^{\tau}\|\theta_1\| \sup_{t\in[a,b]} \int_a^b K(t, u)du \\
&\leq e^{\tau}\|\theta_1\|.
\end{aligned}$$

Thus, one can get

$$\|A\theta_1(t)\| \leq \alpha_2 e^{-\tau}\|\theta_1\|. \tag{3.5}$$

With the similar treatment, we can get

$$\|A\theta_2(t)\| \leq e^{-\tau}\|\theta_2\|. \tag{3.6}$$

Now we know that from (3.3),

$$\sigma(A\theta_1, A\theta_2) = \|A\theta_1 - A\theta_2\| + \|A\theta_1\| + \|A\theta_2\|.$$

Utilizing (3.4), (3.5) and (3.6), we get

$$\begin{aligned}\sigma(A\theta_1, A\theta_2) &\leq e^{-\tau}\Big[\|\theta_1-\theta_2\| + \|\theta_1\| + \|\theta_2\|\Big]\\ &= e^{-\tau}\sigma(\theta_1,\theta_2)\\ &\leq e^{-\tau}M(\theta_1,\theta_2).\end{aligned}$$

Passing logarithm both sides, we get

$$\log(\sigma(A\theta_1, A\theta_2)) \leq log(M(\theta_1,\theta_2)) - \tau.$$

Here we note that the function $F : \mathbb{R}^+ \to \mathbb{R}$ defined by $F(\theta) = \log(\theta)$ for every $\theta \in C[a,b]$ and for $\tau > 0$ is in Δ_F. Consequently with $\psi = t$ all the conditions of Corollary 2.4 are satisfied. Subsequently A has a fixed point which is the solution of Integral equation (3.3). □

4. Conclusion

In this study, recognizing the concept of F-contraction, some fixed point theorems for (ψ,ϕ) Berinde-type F- contraction in partially ordered metric- like space are established. The applications and illustrative examples show the high degree of reliability to other authors to generalize and improve these results for future research.

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References

[1] M. A. Alghamdi, N. Hussain, P. Salimi, *Fixed point and coupled fixed point theorems on b-metric-like spaces*, J. Inequal. Appl. 2013,(2013):402.

[2] A. Amini-Harandi *Metric-like spaces, partial metric spaces and fixed points*, Fixed Point Theory and Applications, 2012,2012:204.

[3] M Asadi, E Karapinar, *Coincidence Point Theorem on Hilbert Spaces via Weak Ekeland Variational Principle and Application to Boundary Value Problem,* Thai Journal of Mathematics **19 (1)**, (2021) 1–7.

[4] M. Asadi, M. Gabeleh, C. Vetro, *A new approach to the generalization of darbos fixed point problem by using simulation functions with application to integral equations,* Results in Mathematics, **74**, 86 (2019).

[5] M. Asadi, *Discontinuity of Control Function in the -Contraction in Metric Spaces,* FILOMAT **31 (17)**, (2017) 5427–5433.

[6] V. Berinde, *Iterative approximation of fixed points*, Springer-Verlag, Berlin-Heidelberg, (2007),2007.

[7] V. Berinde, *Approximating fixed points of weak φ-contractions using the Picard iteration*, Fixed Point Theory, **4** (2003),131–142.

[8] C. F. Chen, J. Dong, C. X. Zhu, *Some fixed point theorems in b-metric-like spaces*, Fixed Point Theory Appl., 2015 (2015), 1–10.

[9] R. Heckmann, *Approximation of metric spaces by partial metric spaces*, Applied Categorical Structures, **7 (1-2)**, (1999)71–83.

[10] P. Hitzler, A. Seda, *Mathematical aspects of logic programming semantics*, CRC Studies in Informatic Series, Chapman and Hall and CRC Press, 2011.

[11] S.K. Malhotra, S. Radenović, S. Shukla, *Some fixed point results without monotone property in partially ordered metric-like spaces*, Journal of the Egyptian Mathematical Society, **22**, (2014)83–89.

[12] S.G. Mathews,*Partial metric topology, in Proceeding of the 8th summer conference on General Topology and Application*, Ann. New York Acad. Sci., vol. **728**, (1994) 183–197.

[13] S. J. ÓNeill, *Partial metrics, valuations, and domain theory*, in Proceedings of the 11th Summer Conference on General Topology and Applications, vol. **806**, (1996) 304–315, Annals of the New York Academy of Sciences, .

[14] D. Singh, V. Chauhan, P. Kumam, V. Joshi, *Some applications of fixed point results for generalized two classes of BoydWongs F-contraction in partial b-metric spaces,* Mathematical Sciences **12**, (2018) 111–127.

[15] D. Singh, V. Chauhan, I. Altun, *Common fixed point of a power graphic (F, psi)-contraction pair on partial b-metric spaces with application,* Nonlinear analysis: Modelling and Control **22 (5)**, (2017) 662–678 .

[16] D. Singh, V. Chauhan, R. Wangkeeree, *Geraghty type generalized F-contractions and related applications in partial b-metric spaces,* International Journal of Analysis, Article ID 8247925, 14 pages, (2017).

[17] D. Singh, V. Joshi, P. Kumam, N. Singh, *Fixed point results for generalized F-contractive and Roger Hardy type F-contractive mappings in G-metric spaces,* Revista de la Real Academia de Ciencias Exactas, Fsicas y Naturales.Serie A. Matemticas, 111, (2017) 473–487, .

[18] M. Younis, A. Sretenovic, S. Radenovic, *Some critical remarks on "Some new fixed point results in rectangular metric spaces with an application to fractional-order functional differential equations"*. Nonlinear Analysis: Modelling and Control, **27 (1)**,(2022) 163–178 .

[19] M. Younis, D. Bahuguna, *A unique approach to graph-based metric spaces with an application to rocket ascension*. Computational and Applied Mathematics, **42 (1)**, 44 (2023).

[20] M. Younis, D. Singh, L. Chen, M. Metwali, *A study on the solutions of notable engineering models*. Mathematical Modelling and Analysis, **27 (3)**, (2022) 492–509.

[21] M. Younis, H. Ahmad, L. Chen, M. Han, *Computation and convergence of fixed points in graphical spaces with an application to elastic beam deformations,* Journal of Geometry and Physics, (2023). https://doi.org/10.1016/j.geomphys.2023.104955.

[22] M Younis, D Singh, A A N Abdou, *A fixed point approach for tuning circuit problem in dislocated bmetric spaces,* Mathematical Methods in the Applied Sciences, **45 (4)**, (2022) 2234–2253.

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