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A VISCOSITY IMPLICIT MIDPOINT ITERATIVE METHOD FOR NONEXPANSIVE MAPPING IN CAT(0) SPACES

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Abstract. In this paper, we suggest and analyze a viscosity implicit midpoint iterative method for a nonexpansive mapping in the framework of CAT(0) space. Further, under the some conditions, strong convergence theorem is proved by the sequence generated by the proposed iterative method, which, also solves the variational inequality problem. Further, the results presented in this paper may be treated as an extension and generalization of some corresponding ones in the literature.

1. Introduction

A metric space (X, d) is termed a CAT(0) space if it possesses geodetic connectedness and if every geodesic triangle within X is "thin" to at least the extent of its comparison triangle in the Euclidean plane. It is recognized that any complete, simply connected Riemannian manifold with non-positive sectional curvature qualifies as a CAT(0) space. Additional examples of CAT(0) spaces encompass pre-Hilbert spaces, R-trees [1], Euclidean buildings [2], among numerous others. A complete CAT(0) space earns the designation of a Hadamard space. In a CAT(0) space X, a subset K is deemed convex if, for any $x, y \in K$, the interval $[x, y] \subset K$, where $[x, y]$ signifies the uniquely geodesic path linking x and y. For an in-depth exploration of CAT(0) spaces and their pertinent properties that wield a significant role in geometry, we direct interested readers to [1] and the references therein.

Recall that, a mapping $S : X \to X$ on a metric space (X, d) is called contraction, if there exists a constant $\alpha \in (0, 1)$ such that

$$d(Sx, Sy) \leq d(x, y), \forall x, y \in X.$$

If $\alpha = 1$, S is called nonexpansive mapping.

A study of nonpositive curvature geodesic metric spaces was initiated and studied in the first decades of the twentieth century with an introduction of hyperbolic spaces by Hadamard and Cartan. Later on Gromov restated some features of global

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Riemannian geometry solely based on the so-called CAT(0) inequality; the letters C, A and T stand for Cartan, Alexandrov and Toponogov, respectively. For more details of these spaces and their properties in different branches of mathematics, we refer to Bridson and Haefliger [1] and references cited therein.

On the other hand, some iterative methods have been developed and studied to approximate the fixed points of nonexpansive mappings defined on suitable domains. These iterative algorithms have received much attention due to its applications in a variety of mathematical problems such as inverse problems, solution approximation of partial differential equations, image recovery, and signal processing see for example, [3–11] and references cited therein. The geometric properties of structure of a domain of nonexpansive mappings play a significant role for finding the solution of its fixed point equation. In this regard, Banach spaces and Hilbert spaces are natural choices to study the existence and approximation of fixed points of certain mappings. The problem of switching from linear structures to nonlinear structures has attracted the attention of several mathematicians. CAT(0) space is a typical example of a domain possessing a nonlinear structure. Berg and Nikolaev [31] introduced an inner product-like notion which is called quasilinearization in CAT(0) spaces to deal the problems of a nonlinear structure of an underlying domain.

Fixed-point theory in CAT(0) spaces was first studied by Kirk [13,14]. He showed that every nonexpansive mapping defined on a bounded, closed, convex subset of a complete CAT(0) space has always a fixed point. Since then, the fixed point theory for single-valued and multivalued mappings in CAT(0) spaces has been studied extensively.

The viscosity approximation method was introduced and studied by Moudafi [15] to approximate a fixed point of a nonexpansive mapping in the framework of a Hilbert space, which generates the sequence $\{x_n\}$ by the following iterative algorithm:

$$x_{n+1} = \alpha_n Q(x_n) + (1-\alpha_n)Sx_n,\ n \geq 0, \tag{1.1}$$

where $\{\alpha_n\} \subset [0,1]$ and Q is a contraction mapping on Hilbert space H. Further he proved that the sequences generated by (1.1) converge strongly to the unique solution $q \in \mathrm{Fix}(S)$ which also solves of variational inequality problem.

$$\langle (I-Q)q, x-q\rangle \geq 0,\ \forall x \in \mathrm{Fix}(S). \tag{1.2}$$

The implicit midpoint iterative method is one of the powerful numerical methods for solving ordinary differential equations and differential algebraic equations. For related works, we refer to [16–23] and the references cited therein.

In 2014, Alghamdi *et al.* [24] has been extended implicit midpoint iterative method to nonexpansive mappings, which generates a sequence $\{x_n\}$ by the following implicit iterative algorithm:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)S\left(\frac{x_n + x_{n+1}}{2}\right),\ n \geq 0. \tag{1.3}$$

Recently, Rizvi [25] extended and generalized the results of Alghamdi *et al.* [24] and presented the following general viscosity implicit midpoint iterative method for nonexpansive mapping, which generates a sequence $\{x_n\}$ by the following implicit iterative algorithm:

$$x_{n+1} = \alpha_n \gamma Q(x_n) + (1 - \alpha_n B) S \left(\frac{x_n + x_{n+1}}{2} \right), \; n \geq 0, \tag{1.4}$$

where $\{\alpha_n\} \subset [0, 1]$ and S is a nonexpansive mapping and B is a strongly positive self-adjoint bounded linear operator on H with constant $\bar{\gamma} > 0$ and $\gamma \in (0, \frac{\bar{\gamma}}{\alpha})$. He proved that under some mild conditions, the sequence generated by (1.4) converge in norm to fixed point of nonexpansive mapping, which, in addition, solves the variational inequality (1.2). For related work see [26].

In 2012, Shi and Chen [27] studied and extends the convergence result of Moudafi's viscosity approximation method for a nonexpansive mapping in the setting of CAT(0) spaces, which generates the sequence $\{x_n\}$ by the following iterative algorithm:

$$x_{n+1} = \alpha_n Q(x_n) \oplus (1 - \alpha_n) S x_n, \; n \geq 0, \tag{1.5}$$

where $\{\alpha_n\} \subset [0, 1]$ and Q is a contraction mapping and S is a nonexpansive mapping on CAT(0) space X. They proved that under some mild conditions, the sequence generated by (1.5) converge in norm to fixed point of nonexpansive mapping.

In 2017, Zhao *et al.* [28] introduced and studied viscosity approximation method for implicit midpoint iterative method of a nonexpansive mapping in CAT(0) spaces, which generates the sequence $\{x_n\}$ by the following iterative algorithm:

$$x_{n+1} = \alpha_n Q(x_n) \oplus (1 - \alpha_n) S \left(\frac{x_n \oplus x_{n+1}}{2} \right), \; n \geq 0, \tag{1.6}$$

where $\{\alpha_n\} \subset [0, 1]$. Further, they proved that under some mild conditions, the sequence generated by (1.6) converge in norm to fixed point of nonexpansive mapping, which, in addition, solves the variational inequality (1.2).

Motivated by the works of Moudafi [15], Rizvi [25], Shi and Chen [27], Alghamdi et al. [24], and Zhao et al. [28], as well as the ongoing research in this direction, we propose and analyze a viscosity implicit midpoint iterative method for a nonexpansive mapping within the framework of CAT(0) space. Furthermore, under certain conditions, we establish a strong convergence theorem for the sequence generated by the proposed iterative method. This sequence, moreover, constitutes the unique solution to the variational inequality problem. These results and methods presented herein further extend and generalize the corresponding findings and approaches outlined in [15, 24, 25, 27, 28].

2. Preliminaries

In this section, we recall some concepts and results which are needed in sequel.

Let X be a CAT(0) space, then for any $x, y, z \in X$ and $t \in [0, 1]$, we write $(1 - t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that

$$d(z, x) = \mathrm{t}d(x, y), \quad \text{and} \quad d(z, y) = (1 - t)d(x, y).$$

The following lemmas play an important role in for the subsequent sections.

Lemma 2.1. [29] *Let* X *be a* CAT(0) *space,* $x, y, z \in \mathrm{X}$ *and* $\mathrm{t} \in [0, 1]$. *Then*

(i) $d(tx \oplus (1-t)y, z) \leq td(x,z) + (1-t)d(y,z)$;
(ii) $d^2(tx \oplus (1-t)y, z) \leq td^2(x,z) + (1-t)d^2(y,z) - t(1-t)d^2(x,y)$.

Lemma 2.2. [30] *Let X be a $CAT(0)$ space,* $\mathrm{p,q,r,s} \in \mathrm{X}$ *and* $\lambda \in [0,1]$. *Then*

$$d(\lambda p \oplus (1-\lambda)q, \lambda r \oplus (1-\lambda)s) \leq \lambda d(p,r) + (1-\lambda)d(q,s).$$

In 2008, the concept of quasilinearization were introduced and studied by Berg and Nikolaev [31], which is defined as follows as follows. Let us denote a pair $(a,b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then, quasilinearization is defined as a map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times x) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\left(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)\right), \quad (a,b,c,d \in X).$$

It is easy to see that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ for all $a,b,c,d \in X$. We say that X satisfies the Cauchy-Schwartz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a,b)d(c,d)$$

for all $a,b,c,d \in X$. It is well-known [31] that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwartz inequality.

Let K be a nonempty closed convex subset of a complete CAT(0) space X. The metric projection $P_K : X \to K$ is defined by

$$u = P_K(x) \Leftrightarrow d(u,x) = \inf\{d(y,x) : y \in K\}, \quad \forall x \in X.$$

Lemma 2.3. [32] *Let K be a nonempty convex subset of a complete $CAT(0)$ space X, for $x \in X$ and $u \in K$. Then $u = P_K x$ if and only if u is a solution of the following variational inequality*

$$\langle \overrightarrow{yu}, \overrightarrow{ux} \rangle \geqslant 0, \quad \forall y \in K,$$

i.e., u satisfies the following inequality:

$$d^2(x,y) - d^2(y,u) - d^2(u,x) \geq 0, \forall y \in K.$$

Lemma 2.4. [33]. *Every bounded sequence in a complete $CAT(0)$ space always has a Δ-convergent subsequence.*

Lemma 2.5. [28]. *Let X be a complete $CAT(0)$ space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ Δ-converges to x if and only if $\limsup_{n\to\infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle \leq 0$, for all $y \in X$.*

Lemma 2.6. [34]. *Let X be a complete $CAT(0)$ space. Then for all $u,x,y \in X$, the following inequality holds*

$$d^2(x,u) \leq d^2(y,u) + 2\langle \overrightarrow{xy}, \vec{u} \rangle.$$

Lemma 2.7. [35]. *Let X be a complete $CAT(0)$ space. For any $t \in [0,1]$ and $u,v \in X$, let $u_t = tu \oplus (1-t)v$. Then, for all $x,y \in X$,*

Lemma 2.8. [36]. *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\beta_n)a_n + \delta_n, \ n \geq 0,$$

where $\{\beta_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \beta_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\beta_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Viscosity Implicit Midpoint Iterative Method

In this section, we prove a strong convergence theorem based on viscosity implicit midpoint iterative method for fixed point of nonexpansive mapping.

Theorem 3.1. *Let K be a nonempty, closed and convex subset of a complete $CAT(0)$ space X and $S : K \to K$ be a nonexpansive mapping such that* $\mathrm{Fix}(S) \neq \emptyset$. *Let $Q : K \to K$ be a contraction mapping with constant $\alpha \in (0,1)$, let the iterative sequence $\{x_n\}$ be generated by the following viscosity implicit midpoint iterative algorithm:*

$$x_{n+1} = \alpha_n Q(x_n) \oplus \beta_n x_n \oplus \gamma_n S\left(\frac{x_n \oplus x_{n+1}}{2}\right), \ n \geq 0, \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $(0,1)$ and satisfying the following conditions

(i) $\alpha_n + \beta_n + \gamma_n = 1$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$;
(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iv) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$.

Then the sequence $\{x_n\}$ converge strongly to $z \in \mathrm{Fix}(S)$, where $z = P_{\mathrm{Fix}(S)}Q(z)$. In other words, which is also unique solution of variational inequality (1.2).

Proof. It is easy to observe that, if for $\mu \in C$, one define a mapping.

$$x \mapsto S_\mu x := \alpha Q(\mu) \oplus \beta_\mu \oplus \gamma S\left(\frac{\mu \oplus x}{2}\right).$$

Therefore, we compute

$$\begin{aligned} d(S_\mu x, S_\mu y) &= \gamma d\left(S\left(\frac{\mu \oplus x}{2}\right), S\left(\frac{\mu \oplus y}{2}\right)\right) \\ &\leq \gamma d\left(\frac{\mu \oplus x}{2}, \frac{\mu \oplus y}{2}\right) \\ &\leq \frac{\gamma}{2} d(x,y). \end{aligned}$$

Since $\gamma \in (0,1)$. Hence $S_\mu x$ is a contraction mapping. Hence (3.1) is well defined.

Let $p \in \mathrm{Fix}(S)$, we compute

$$\begin{aligned}
d(x_{n+1},p) &= d\Big(\alpha_n Q(x_n) \oplus \beta_n x_n \oplus \gamma_n S\left(\frac{x_n \oplus x_{n+1}}{2}\right), p\Big) \\
&\leq \alpha_n d(Q(x_n),p) + \beta_n d(x_n,p) + \gamma_n d\left(S\left(\frac{x_n \oplus x_{n+1}}{2}\right), p\right) \\
&\leq \alpha_n [d(Q(x_n),Q(p)) + d(Q(p),p)] + \beta_n d(x_n,p) + \gamma_n d\left(\left(\frac{x_n \oplus x_{n+1}}{2}\right), p\right) \\
&\leq \alpha_n k d(x_n,p) + \alpha_n d(Q(p),p) + \beta_n d(x_n,p) + \gamma_n d\left(\left(\frac{x_n \oplus x_{n+1}}{2}\right), p\right) \\
&\leq \alpha_n k d(x_n,p) + \alpha_n d(Q(p),p) + \beta_n d(x_n,p) + \frac{\gamma_n}{2}\left(d(x_n,p) + d(x_{n+1},p)\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
\left(1 - \frac{\gamma_n}{2}\right) d(x_{n+1},p) &\leq \left[\alpha_n k + \beta_n + \frac{\gamma_n}{2}\right] d(x_n,p) + \alpha_n d(Q(p),p) \\
d(x_{n+1},p) &\leq \left[\frac{1 + \beta_n + (2k-1)\alpha_n}{1 + \alpha_n + \beta_n}\right] d(x_n,p) + \frac{2\alpha_n}{1+\alpha_n+\beta_n} d(Q(p),p) \\
&\leq \left[1 - \frac{2(1-k)\alpha_n}{1+\alpha_n+\beta_n}\right] d(x_n,p) + \frac{2\alpha_n}{(1+\alpha_n+\beta_n)} d(Q(p),p) \\
&\leq \left[1 - \frac{2(1-k)\alpha_n}{1+\alpha_n+\beta_n}\right] d(x_n,p) + \frac{2\alpha_n}{(1+\alpha_n+\beta_n)} \frac{1}{(1-k)} d(Q(p),p).
\end{aligned}$$

Consequently, we get

$$d(x_{n+1},p) \leq \max\left\{d(x_n,p), \frac{d(Q(p),p)}{1-k}\right\}.$$

Therefore by using induction, we obtain

$$d(x_{n+1},p) \leq \max\left\{d(x_0,p), \frac{d(Q(p),p)}{1-k}\right\}. \tag{3.2}$$

Hence the sequence $\{x_n\}$ is bounded.

Next, we show that the sequence $\{x_n\}$ is asymptotically regular, i.e., $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. It follows from (3.1) that

$$\begin{aligned}
& d(x_{n+1}, x_n) = \\
& d\left(\alpha_n Q(x_n) \oplus \beta_n x_n \oplus \gamma_n S\left(\frac{x_n \oplus x_{n+1}}{2}\right), \alpha_{n-1} Q(x_{n-1}) \oplus \beta_{n-1} x_{n-1} \oplus \gamma_{n-1} S\left(\frac{x_n \oplus x_{n-1}}{2}\right)\right) \\
\leq\ & d\Big(\alpha_n Q(x_n) \oplus \beta_n x_n \oplus (1-\alpha_n-\beta_n) S\Big(\frac{x_n \oplus x_{n+1}}{2}\Big), \alpha_n Q(x_n) \oplus \beta_n x_{n-1} \\
& \oplus (1-\alpha_n-\beta_n) S\Big(\frac{x_n \oplus x_{n-1}}{2}\Big)\Big) \\
& + d\Big(\alpha_n Q(x_n) \oplus \beta_n x_{n-1} \oplus (1-\alpha_n-\beta_n) S\Big(\frac{x_n \oplus x_{n-1}}{2}\Big)\Big), \alpha_n Q(x_{n-1}) \oplus \beta_n x_{n-1} \\
& \oplus (1-\alpha_n-\beta_n) S\Big(\frac{x_n \oplus x_{n-1}}{2}\Big)\Big) \\
& + d\Big(\alpha_n Q(x_{n-1}) \oplus \beta_n x_{n-1} \oplus (1-\alpha_{n-1}-\beta_{n-1}) S\Big(\frac{x_n \oplus x_{n-1}}{2}\Big)\Big), \alpha_{n-1} Q(x_{n-1}) \\
& \oplus \beta_{n-1} x_{n-1} \oplus (1-\alpha_{n-1}-\beta_{n-1}) S\Big(\frac{x_n \oplus x_{n-1}}{2}\Big)\Big) \\
\leq\ & \alpha_n d(Q(x_n) - Q(x_{n-1})) + \beta_n d(x_n - x_{n-1}) \\
& + (1-\alpha_{n-1}-\beta_{n-1}) d\Big(S\Big(\frac{x_{n+1} \oplus x_n}{2}\Big), S\Big(\frac{x_n \oplus x_{n-1}}{2}\Big)\Big) \\
& + |\alpha_n - \alpha_{n-1}| d\Big(Q(x_{n-1}), S\Big(\frac{x_n \oplus x_{n-1}}{2}\Big)\Big) + |\beta_n - \beta_{n-1}| d\Big(x_{n-1}, S\Big(\frac{x_n \oplus x_{n-1}}{2}\Big)\Big) \\
\leq\ & \alpha_n k d(x_n - x_{n-1}) + \beta_n d(x_n - x_{n-1}) + |\alpha_n - \alpha_{n-1}| M_1 + |\beta_n - \beta_{n-1}| M_2 \\
& + (1-\alpha_n-\beta_n) d\Big(\frac{x_{n+1} \oplus x_n}{2}, \frac{x_n \oplus x_{n-1}}{2}\Big) \\
\leq\ & \alpha_n k d(x_n - x_{n-1}) + \beta_n d(x_n - x_{n-1}) + |\alpha_n - \alpha_{n-1}| M_1 + |\beta_n - \beta_{n-1}| M_2 \\
& + \frac{(1-\alpha_n-\beta_n)}{2} \Big[d(x_{n+1}, x_n) + d(x_n, x_{n-1})\Big],
\end{aligned}$$

where $M_1 := \sup\Big\{d\Big(Q(x_{n-1}), S\Big(\frac{x_n \oplus x_{n-1}}{2}\Big)\Big) : n \in \mathbb{N}\Big\}$ and $M_2 := \sup\Big\{d\Big(x_{n-1}, S\Big(\frac{x_n \oplus x_{n-1}}{2}\Big)\Big) : n \in \mathbb{N}\Big\}$. It follows that

$$\begin{aligned}
\Big[1 - \Big(\frac{1-\alpha_n-\beta_n}{2}\Big)\Big] d(x_{n+1}, x_n) &\leq \Big[\alpha_n k + \beta_n + \frac{1-\alpha_n-\beta_n}{2}\Big] d(x_n, x_{n-1}) + M_1 |\alpha_n - \alpha_{n-1}| \\
& \quad + M_2 |\beta_n - \beta_{n-1}| \\
d(x_{n+1}, x_n) &\leq \Big[1 - \frac{2(1-k)\alpha_n}{1+\alpha_n+\beta_n}\Big] d(x_n, x_{n-1}) + \frac{2M_1}{1+\alpha_n+\beta_n} |\alpha_n - \alpha_{n-1}| \\
& \quad + \frac{2M_2}{1+\alpha_n+\beta_n} |\beta_n - \beta_{n-1}|.
\end{aligned}$$

Since $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $(0, 1)$, then

$1 + \alpha_n + \beta_n < 3$ and $\dfrac{1}{1+\alpha_n+\beta_n} > \dfrac{1}{3}$

Therefore, we obtain

$1-\dfrac{2(1-k)\alpha_n}{1+\alpha_n+\beta_n} < (1-(1-k)\alpha_n)$, it follows that

$$d(x_{n+1},x_n) \le (1-(1-k)\alpha_n)d(x_n,x_{n-1}) + \frac{2M_1}{1+\alpha_n+\beta_n}|\alpha_n-\alpha_{n-1}| + \frac{2M_2}{1+\alpha_n+\beta_n}|\beta_n-\beta_{n-1}|.$$

By using the conditions (i)-(ii) of Lemma 2.8, we obtain

$$\lim_{n\to\infty} d(x_{n+1},x_n) = 0. \tag{3.3}$$

Next, we show that

$$\lim_{n\to\infty} d(x_n,Sx_n) = 0.$$

We, write

$$\begin{aligned}
d(x_n,Sx_n) &\le d(x_n,x_{n+1}) + d\Big(x_{n+1}, S\Big(\frac{x_n\oplus x_{n+1}}{2}\Big)\Big) + d\Big(S\Big(\frac{x_n\oplus x_{n+1}}{2}\Big), Sx_n\Big)\\
&\le d(x_n,x_{n+1}) + \alpha_n d\Big(Q(x_n), S\Big(\frac{x_n\oplus x_{n+1}}{2}\Big)\Big) + \beta_n d\Big(x_n, S\Big(\frac{x_n\oplus x_{n+1}}{2}\Big)\Big)\\
&\quad +(1-\alpha_n-\beta_n) d\Big(S\Big(\frac{x_n\oplus x_{n+1}}{2}\Big), Sx_n\Big)\\
&\le d(x_n,x_{n+1}) + \alpha_n M_1 + \beta_n M_2 + \frac{(1-\alpha_n-\beta_n)}{2} d(x_{n+1},x_n)\\
&\le \Big[1+\frac{1-\alpha_n-\beta_n}{2}\Big] d(x_{n+1},x_n) + \alpha_n M_1 + \beta_n M_2.
\end{aligned}$$

It follows from condition (i) and (3.3), we obtain

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ Δ-converges to $\hat{x}$ and

$$\limsup_{n\to\infty} (\overrightarrow{Q(\hat{x})\hat{x}}, \overrightarrow{x_n\hat{x}}) = \limsup_{k\to\infty} (\overrightarrow{Q(\hat{x})\hat{x}}, \overrightarrow{x_{n_k}\hat{x}}). \tag{3.4}$$

Since $\{x_{n_k}\}$ Δ-converges to $\hat{x}$. Therefore, it follows from (3.4) that

$$\limsup_{k\to\infty} (\overrightarrow{Q(\hat{x})\hat{x}}, \overrightarrow{x_{n_k}\hat{x}}) \le 0. \tag{3.5}$$

This together with (3.4), we have

$$\limsup_{n\to\infty} (\overrightarrow{Q(\hat{x})\hat{x}}, \overrightarrow{x_n\hat{x}}) = 0. \tag{3.6}$$

Finally, we show that $x_n \to z$ as $n\to\infty$. On Setting $u_n := \alpha_n\hat{x} \oplus (1-\alpha_n-\beta_n)y_n$ and $y_n := \dfrac{\beta_n}{1-\alpha_n-\beta_n}x_n \oplus \dfrac{\gamma_n}{1-\alpha_n-\beta_n} S\Big(\dfrac{x_n\oplus x_{n+1}}{2}\Big)$. It follows from Lemma 2.6 that

$$\begin{aligned}
d^2(x_{n+1},\hat{x}) &\leq d^2(u_n,\hat{x}) + 2\left\langle \overrightarrow{x_{n+1}u_n}, \overrightarrow{x_{n+1}\hat{x}} \right\rangle \\
&= d^2(\alpha_n\hat{x} \oplus (1-\alpha_n-\beta_n)y_n, \hat{x}) \\
&\quad +2\Big[\alpha_n \left\langle \overrightarrow{f(x_n)u_n}, \overrightarrow{x_{n+1}\hat{x}} \right\rangle + (1-\alpha_n-\beta_n)\left\langle \overrightarrow{y_n)u_n}, \overrightarrow{x_{n+1}\hat{x}} \right\rangle\Big] \\
&\leq d^2(\alpha_n\hat{x} \oplus (1-\alpha_n-\beta_n)y_n, \hat{x}) + 2\Big[\alpha_n^2 \left\langle \overrightarrow{f(x_n)\hat{x}}, \overrightarrow{x_{n+1}\hat{x}} \right\rangle \\
&\quad +\alpha_n(1-\alpha_n-\beta_n)\left\langle \overrightarrow{f(x_n)y_n}, \overrightarrow{x_{n+1}\hat{x}} \right\rangle + \alpha_n(1-\alpha_n-\beta_n)\left\langle \overrightarrow{y_n\hat{x}}, \overrightarrow{x_{n+1}\hat{x}} \right\rangle \\
&\quad +(1-\alpha_n-\beta_n)^2\left\langle \overrightarrow{y_n y_n}, \overrightarrow{x_{n+1}\hat{x}} \right\rangle\Big] \\
&\leq d^2(\alpha_n\hat{x} \oplus (1-\alpha_n-\beta_n)y_n, \hat{x}) + 2\Big[\alpha_n^2 \left\langle \overrightarrow{f(x_n)\hat{x}}, \overrightarrow{x_{n+1}\hat{x}} \right\rangle
\end{aligned}$$

$$\begin{aligned}
&+\alpha_n\beta_n\left\langle \overrightarrow{f(x_n)x_n}, \overrightarrow{x_{n+1}\hat{x}} \right\rangle + \alpha_n\gamma_n\left\langle \overrightarrow{f(x_n)S\left(\frac{x_{n+1}\oplus x_n}{2}\right)\hat{x}}, \overrightarrow{x_{n+1}\hat{x}} \right\rangle \\
&+\alpha_n\beta_n\left\langle \overrightarrow{x_n\hat{x}}, \overrightarrow{x_{n+1}\hat{x}} \right\rangle + \alpha_n\gamma_n\left\langle \overrightarrow{S\left(\frac{x_{n+1}\oplus x_n}{2}\right)\hat{x}}, \overrightarrow{x_{n+1}\hat{x}} \right\rangle\Big] \\
\leq\ & d^2(\alpha_n\hat{x} \oplus (1-\alpha_n-\beta_n)y_n, \hat{x}) + 2\Big[\alpha_n^2 \left\langle \overrightarrow{f(x_n)\hat{x}}, \overrightarrow{x_{n+1}\hat{x}} \right\rangle
\end{aligned}$$

$$\begin{aligned}
&+\alpha_n\beta_n\left\langle \overrightarrow{f(x_n)\hat{x}}, \overrightarrow{x_{n+1}\hat{x}} \right\rangle + \alpha_n\gamma_n\left\langle \overrightarrow{f(x_n)\hat{x}}, \overrightarrow{x_{n+1}\hat{x}} \right\rangle\Big] \\
\leq\ & d^2(\alpha_n\hat{x} \oplus (1-\alpha_n-\beta_n)y_n, \hat{x}) + 2\Big[\alpha_n(\alpha_n+\beta_n+\gamma_n)\left\langle \overrightarrow{f(x_n)\hat{x}}, \overrightarrow{x_{n+1}\hat{x}} \right\rangle\Big] \\
\leq\ & d^2(\alpha_n\hat{x} \oplus (1-\alpha_n-\beta_n)y_n, \hat{x}) + 2\alpha_n k d(x_n,\hat{x})d(x_{n+1},x_n) + 2\alpha_n\left\langle \overrightarrow{f(x_n)\hat{x}}, \overrightarrow{x_{n+1}\hat{x}} \right\rangle \\
\leq\ & d^2(\alpha_n\hat{x} \oplus (1-\alpha_n-\beta_n)y_n, \hat{x}) + \alpha_n k\Big[d^2(x_n,\hat{x}) + d^2(x_{n+1},\hat{x})\Big] + 2\alpha_n\left\langle \overrightarrow{f(x_n)\hat{x}}, \overrightarrow{x_{n+1}\hat{x}} \right\rangle \\
\leq\ & (1-\alpha_n-\beta_n)d^2(y_n,\hat{x}) + \alpha_n k\Big[d^2(x_n,\hat{x}) + d^2(x_{n+1},\hat{x})\Big] + 2\alpha_n\left\langle \overrightarrow{f(x_n)\hat{x}}, \overrightarrow{x_{n+1}\hat{x}} \right\rangle \\
\leq\ & (1-\alpha_n-\beta_n)d^2\left(\frac{\beta_n}{1-\alpha_n-\beta_n}x_n \oplus \frac{\gamma_n}{1-\alpha_n-\beta_n}S\left(\frac{x_{n+1}\oplus x_n}{2}\right), \hat{x}\right) \\
&+\alpha_n k\Big[d^2(x_n,\hat{x}) + d^2(x_{n+1},\hat{x})\Big] + 2\alpha_n\left\langle \overrightarrow{f(x_n)\hat{x}}, \overrightarrow{x_{n+1}\hat{x}} \right\rangle \\
=\ & \beta_n d^2(x_n,\hat{x}) + \frac{\gamma_n}{2}\Big[d^2(x_{n+1},\hat{x}) + d^2(x_n,\hat{x})\Big] + \alpha_n k\Big[d^2(x_n,\hat{x}) + d^2(x_{n+1},\hat{x})\Big] \\
&+2\alpha_n\left\langle \overrightarrow{f(x_n)\hat{x}}, \overrightarrow{x_{n+1}\hat{x}} \right\rangle.
\end{aligned}$$

This implies that

$$\left(1-\alpha_n k-\frac{\gamma_n}{2}\right)d^2(x_{n+1},\hat{x}) \leq \left(\alpha_n k+\beta_n+\frac{\gamma_n}{2}\right) d^2(x_n,\hat{x})+2\alpha_n \left\langle \overrightarrow{f(x_n)\hat{x}}, \overrightarrow{x_{n+1}\hat{x}}\right\rangle$$

$$\begin{aligned} d^2(x_{n+1},\hat{x}) &\leq \left(\frac{2\alpha_n k+\beta_n+\gamma_n}{2(1-\alpha_n k)-\gamma_n}\right) d^2(x_n,\hat{x})+\frac{2\alpha_n}{2(1-\alpha_n k)-\gamma_n}\left\langle \overrightarrow{f(x_n)\hat{x}}, \overrightarrow{x_{n+1}\hat{x}}\right\rangle \\ &= \frac{1-\alpha_n(1-2k)}{1-\alpha_n(2k-1)+\beta_n} d^2(x_n,\hat{x})+\frac{2\alpha_n}{2(1-\alpha_n k)-\gamma_n}\left\langle \overrightarrow{f(x_n)\hat{x}}, \overrightarrow{x_{n+1}\hat{x}}\right\rangle \\ &= \left[1-\frac{2\alpha_n(1-2k)-\beta_n}{1-\alpha_n(2k-1)+\beta_n}\right] d^2(x_n,\hat{x})+\frac{2\alpha_n}{2(1-\alpha_n k)-\gamma_n}\left\langle \overrightarrow{f(x_n)\hat{x}}, \overrightarrow{x_{n+1}\hat{x}}\right\rangle \\ &= (1-\delta_n)\|x_n-z\|^2+\delta_n\sigma_n, \end{aligned} \tag{3.7}$$

where $\delta_n = \dfrac{2\alpha_n(1-2k)-\beta_n}{1-\alpha_n(2k-1)+\beta_n}$ and $\sigma_n = \dfrac{2\alpha_n}{2(1-\alpha_n k)-\gamma_n}\left\langle \overrightarrow{f(x_n)\hat{x}}, \overrightarrow{x_{n+1}\hat{x}}\right\rangle$.

Since $\lim\limits_{n\to\infty}\alpha_n = 0$ and $\sum\limits_{n=0}^{\infty}\alpha_n=\infty$, it is easy to see that $\lim\limits_{n\to\infty}\delta_n=0$, $\sum\limits_{n=0}^{\infty}\delta_n=\infty$ and $\limsup\limits_{n\to\infty}\sigma_n\leq 0$. Hence from (3.4), (3.7) and Lemma 2.8, we deduce that $x_n\to z$. This completes the proof.

□

As a direct consequences of Theorem 3.1, we obtain the following result due to Zhao *et al.* [28] for fixed point of nonexpansive mapping. Take $\beta_n := 0$ and in Theorem 3.1 then the following Corollary is obtained.

Corollary 3.2. [28] *Let K be a nonempty, closed and convex subset of a complete CAT(0) and $S: K\to K$ be a nonexpansive mapping such that* $\mathrm{Fix}(S)\neq\emptyset$. *Let $Q: K\to K$ be a contraction mapping with constant $\alpha\in(0,1)$, let the iterative sequence $\{x_n\}$ be generated by the following viscosity implicit midpoint iterative algorithms:*

$$x_{n+1}=\alpha_n Q(x_n)\oplus(1-\alpha_n)S\left(\frac{x_n\oplus x_{n+1}}{2}\right),\ n\geq 0, \tag{3.8}$$

where $\{\alpha_n\}$ is the sequence in $(0,1)$ and satisfying the conditions (i)-(iv) of Theorem 3.1. *Then the sequence $\{x_n\}$ converge strongly to $z\in$* $\mathrm{Fix}(S)$, *which, in addition also solves variational inequality* (1.2).

Remark. *Since every Hilbert space is a complete CAT*(0) *space, therefore Theorem* 3.1 *is any extension, and generalization of the results of Xu et al.* [26] *and Alghamdi et al.* [24] *to a general viscosity implicit midpoint rule for a nonexpansive mappings.*

If we, take $\beta_n := 0$ and CAT(0) space X as an Hilbert space H in Theorem 3.1 then the following Corollary is obtained.

Corollary 3.3. [26] *Let H be a real Hilbert space and $Q: H\to H$ be a contraction mapping with constant $\alpha\in(0,1)$. Let $S: H\to H$ be a nonexpansive mapping such that* $\mathrm{Fix}(S)\neq\emptyset$. *Let the iterative sequence $\{x_n\}$ be generated by the following viscosity implicit midpoint iterative algorithms:*

$$x_{n+1}=\alpha_n Q(x_n)+(1-\alpha_n)S\left(\frac{x_n+x_{n+1}}{2}\right),\ n\geq 0, \tag{3.9}$$

where $\{\alpha_n\}$ *is the sequence in* $(0,1)$ *and satisfying the conditions (i)-(iv) of Theorem* 3.1*. Then the sequence* $\{x_n\}$ *converge strongly to* $z \in \mathrm{Fix}(S)$*, which, in addition also solves variational inequality* (1.2)*.*

Take $\beta_n := 0$ and $Q := I$ and CAT(0) space X as an Hilbert space H in Theorem 3.1 then the following Corollary is obtained.

Corollary 3.4. [24] *Let* H *be a real Hilbert space and* $Q : H \to H$ *be a contraction mapping with constant* $\alpha \in (0,1)$*. Let* $S : H \to H$ *be a nonexpansive mapping such that* $\mathrm{Fix}(S) \neq \emptyset$*. Let the iterative sequence* $\{x_n\}$ *be generated by the following viscosity implicit midpoint iterative algorithms:*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) S\left(\frac{x_n + x_{n+1}}{2}\right), \ n \geq 0, \tag{3.10}$$

where $\{\alpha_n\}$ *is the sequence in* $(0,1)$ *and satisfying the conditions (i)-(iii) of Theorem* 3.1*. Then the sequence* $\{x_n\}$ *converge strongly to* $z \in \mathrm{Fix}(S)$*.*

4. Numerical Example

Example 4.1. *Let* $X = \mathbb{R}$*, the set of all real numbers, with the usual metric* $d(x,y) = |x-y|$*,* $\forall x, y \in \mathbb{R}$*. Let* $K = [0,\infty)$*; let* $Q : K \to K$ *be define by* $Q(x) = \frac{1}{2}x$*,* $\forall x \in K$*,* $S : K \to K$ *be define by* $S(x) = \frac{1}{4}x$ $\forall x \in K$*. Let the sequences* $\{x_n\}$ *be generated by the iterative algorithms*

$$x_{n+1} = \frac{1}{(n+1)} Q(x_n) + \left(\frac{1}{5} + \frac{1}{(n+1)}\right) x_n + \left(\frac{1}{2} - \frac{1}{(n+1)}\right) S\left(\frac{x_n + x_{n+1}}{2}\right), \tag{4.1}$$

where $\alpha_n = \frac{1}{n+1}$*,* $\beta_n = \left(\frac{1}{5} + \frac{1}{(n+1)}\right)$*,* $\gamma_n = \left(\frac{1}{2} - \frac{1}{(n+1)}\right)$*. Then* $\{x_n\}$ *converges strongly to* $0 \in Fix(S)$*.*

Proof. It is easy to prove Q is contraction mapping with constant $\alpha = \frac{1}{2}$ and S is a nonexpansive mapping with constant $\frac{1}{4}$. Furthermore, it is easy to observe that $\mathrm{Fix}(S) = \{0\}$. After simplification, schemes (4.1) reduce to

$$x_{n+1} = \frac{x_n}{2(n+1)} + \left(\frac{1}{5} + \frac{1}{(n+1)}\right) x_n + \frac{1}{4}\left(\frac{1}{2} - \frac{1}{(n+1)}\right)\left(\frac{x_n + x_{n+1}}{2}\right), \tag{4.2}$$

Following the steps of proof of Theorem 3.1, we obtain that $\{x_n\}$ converge strongly to $q = 0 \in Fix(S)$ as $n \to \infty$. The proof is completed. □

Now, by using the software Matlab 7.0, we study the convergence behavior of $\{x_n\}$, which shows that $\{x_n\}$ converges strongly to 0.

Conclusion: The main aim of present work is to study the viscosity type implicit midpoint iterative method for nonexpansive mapping in the setting of CAT(0) space and proved the strong convergence theorem for solving fixed point for a nonexpansive mapping. Theorem 3.1 extends and generalize the viscosity implicit midpoint formula for Zhao *et al.* and Shi *et al.* [27], which also includes the results of [24, 26] as special cases.

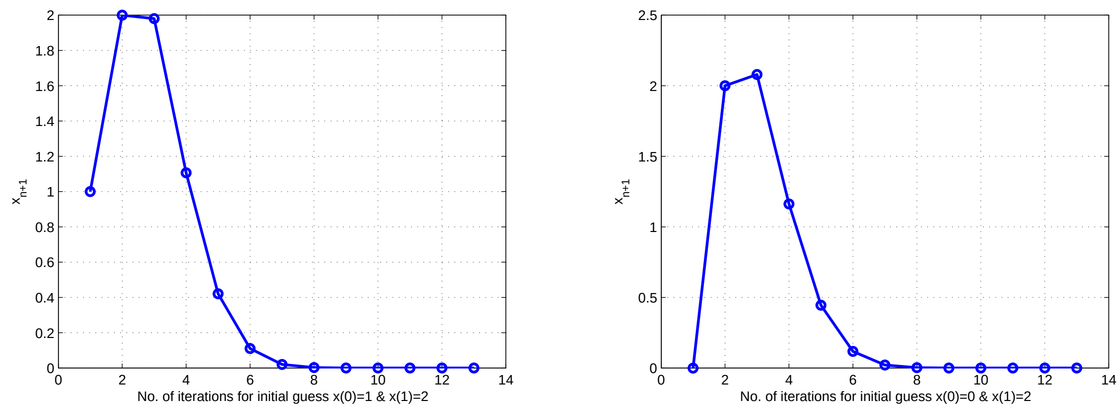

FIGURE 1. Convergence analysis for the sequence $\{x_n\}$

Abbreviations:
Fix(S): Fixed point set for a nonexpansive mapping S.

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