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KANNAN TYPE ASYMPTOTIC CONTRACTIONS IN THE SENSE OF KIRK

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Abstract. Inspired by Kirk's pioneering work, we initiate the concept of Kannan type asymptotic contractions. Using two constructive lemmas, we establish a fixed point theorem for Kannan type asymptotic contractions. What's more, we present a fixed point result for asymptotic contractions concerning the generalization of weakly Ćirić type. This theorem generalizes and improves the results for Kirk's asymptotic contraction introduced by Suzuki and for Kannan type asymptotic contraction.

1. Introduction

In 1922, Banach [1] established the famous fixed point theorem known as the "Banach Contraction Principle". This theorem is one of the fundamental results of analysis and considered as the main source of metric fixed point theory. Since then, there have been numerous extensions of a milder form of Banach contraction principle, see e.g., [2, 3]. In 2003, Kirk [4] introduced a class of mappings called asymptotic contractions and obtained a fixed point theorem for such a mapping in metric spaces, which is a generalization of Banach contraction principle. Now we recall the results on asymptotic contraction as follows.

Definition 1.1 ([4]). *Let (X,d) be a metric space and $T : X \to X$ be a mapping. We say that T is an* asymptotic contraction *if for any $x, y \in X$ and $n \in \mathbb{N}$*

$$d(T^n x, T^n y) \leq \phi_n(d(x,y)), \tag{1.1}$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, $\phi(t) < t$ for all $t > 0$, and $\phi_n : \mathbb{R}^+ \to \mathbb{R}^+$ are such that $\phi_n \to \phi$ uniformly on the range of d.

Theorem 1.1 ([4]). *Let (X,d) be a complete metric space and $T : X \to X$ be an asymptotic contraction for which ϕ_n in (1.1) are also continuous. Assume also that some orbit of T is bounded. Then T has a unique fixed point $z \in X$ such that the Picard sequence $\{T^n x\}_{n=1}^{\infty}$ converges to z for all $x \in X$.*

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After this pioneering work, many authors investigated and extended asymptotic contractions in various directions, see e.g., [5, 6, 7, 8, 9]. It is worth mentioning that, as written in [4], "there is nothing in the formulation of the theorem to suggest that such a proof is necessary." In order to give a constructive proof, Jachymski and Jóźwik [10] modified the conditions of [4, Theorem 2.1] and established some results for asymptotic φ-contraction. Also, an oversight of Kirk that T should be continuous was point out in [10], where the counterexample and correction were given. We recall this result as below

Let (X, d) be a metric space. Following [10], mapping $T : X \to X$ is called an *asymptotic φ-contraction* if there exist a sequence $\{\varphi_n\}_{n=1}^{\infty}$, $\varphi_n : \mathbb{R}^+ \to \mathbb{R}^+$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$d(T^n x, T^n y) \leq \varphi_n(d(x, y)), \tag{1.2}$$

for all $x, y \in X$ and $n \in \mathbb{N}$, where $\varphi_n \to \varphi$ uniformly on $\mathbb{R}^+$ and $\varphi(t) < t$ $(t > 0)$.

Theorem 1.2 ([10])**.** *Assume that (X, d) is complete and T is a uniformly continuous asymptotic φ-contraction, where φ is upper semicontinuous and satisfies*

$$\lim_{t \to \infty} (t - \varphi(t)) = \infty. \tag{1.3}$$

Then T has a unique fixed point x^*. Moreover, x^* is both contractive and approximate fixed point.

In 2005, Aranđelović [11] presented another constructive proof of modified theorem from Kirk, but Jachymski [12] showed that the proof of Aranđelović's theorem contained an error, which was due to be fixed point free. Around the same time, Suzuki [13] introduced the concept of asymptotic contractions of Meir-keeler type and gave a corresponding theorem, which was a generalization of fixed point theorems of Meir-keeler and Kirk.

On the other hand, Kannan [14] proved a new theorem on complete metric space in 1969. We observe that the fixed point theorem of Kannan is different from the Banach contraction principle, see e.g., [15, 16, 17, 18, 19, 20]. After that, many authors investigated and improved the fixed point theorem of Kannan in different directions, see e.g., [21, 22, 23, 24]. In 1971, Reich [25] established a fixed point theorem on complete metric space. We note that the Reich type fixed point theorem yields the Banach contraction principle and Kannan type fixed point theorem. As far as we know, no one presents the fixed point theorem for asymptotic contractions of Kannan type and Reich type.

In this paper, we establish two fixed point theorems for asymptotic contractions of Kannan type and Reich type, where the second result generalizes the fixed point theorem for Kirk's asymptotic contractions by Suzuki [13, Theorem 4]. In particular, two useful lemmas we present can simplify the proofs of fixed point theorems for asymptotic contractions.

2. Main results

In the definition of asymptotic φ-contractions, the property concerning φ is only $\varphi(t) < t$. Note that upper semicontinuity of φ is necessary in Jachymski's result. Now, we introduce two lemmas for functions φ and φ_n.

Lemma 2.1. *Let function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be upper semicontinuous and satisfying $\varphi(t) < t$ for all $t > 0$. Then for any $b > a > 0$,*

$$\inf_{t\in[a,b]}(t - \varphi(t)) > 0.$$

Proof. From the condition of φ, we can see that

$$\eta(t) := t - \varphi(t)$$

is lower semicontinuous and $\eta(t) > 0$ for all $t > 0$. Assume that $\inf_{t\in[a,b]} \eta(t) = 0$ for some $a, b \in \mathbb{R}^+$ with $b > a > 0$. Then there exist a sequence $\{t_n\} \subset [a, b]$ and $t^* \in [a, b]$ such that $\eta(t_n) \to 0$ and $t_n \to t^*$ as $n \to \infty$, which implies that $\eta(t^*) \leq 0$ since η is lower semicontinuous. That is a contradiction for $\eta(t) > 0$ $(t > 0)$. □

Then, we give a useful lemma for function sequence φ_n in asymptotic φ-contraction.

Lemma 2.2. *Let function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be upper semicontinuous and satisfying $\varphi(t) < t$ for all $t > 0$, $\varphi_n : \mathbb{R}^+ \to \mathbb{R}^+$ converge to φ uniformly on $\mathbb{R}^+$, for all $n \in \mathbb{N}$. Then, for any $b > a > 0$, there exist $\delta > 0$ and $N \in \mathbb{N}$ such that*

$$\varphi_n(t) \leq t - \delta, \tag{2.1}$$

for all $t \in [a, b]$ and integer $n \geq N$.

Proof. Let $b > a > 0$ be arbitrarily given. By Lemma 2.1, we have

$$\delta := \frac{1}{2}\inf_{t\in[a,b]}(t - \varphi(t)) > 0.$$

Obviously, we have $\varphi(t) < t - 2\delta$ for all $t \in [a, b]$. Since $\varphi_n \to \varphi$ uniformly, we can see that there exists $N \in \mathbb{N}$ such that,

$$|\varphi_n(t) - \varphi(t)| < \delta,$$

for all $t \in [a, b]$ and integer $n \geq N$, leading to that $\varphi_n(t) \leq \varphi(t) + \delta \leq t - 2\delta + \delta = t - \delta$. □

The following theorem shows that the asymptotic contraction of Kannan type has a unique fixed point.

Theorem 2.3. *Let (X, d) be a complete metric space, and suppose that $T : X \to X$ is a continuous mapping. If there exists a sequence $\{\varphi_n\}_{n=1}^{\infty}$, $\varphi_n : \mathbb{R}^+ \to \mathbb{R}^+$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$d(T^n x, T^n y) \leq \frac{1}{2}\varphi_n(d(x, Tx)) + \frac{1}{2}\varphi_n(d(y, Ty)), \tag{2.2}$$

for all $x, y \in X$ and $n \in \mathbb{N}$, where $\varphi_n \to \varphi$ uniformly on $\mathbb{R}^+$, the function φ is upper semicontinuous and $\varphi(t) < t$ $(t > 0)$. Then, T has a unique fixed point $x^* \in X$.

Proof. Let $x_0 \in X$ be given. Denote $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$. Since $\varphi_n \to \varphi$ uniformly, we have $\varphi_n(d(x_0, x_1)) \to \varphi(d(x_0, x_1))$ and $\varphi_n(d(x_1, x_2)) \to \varphi(d(x_1, x_2))$ as $n \to \infty$. Thus the sequence $\{\varphi_n(d(x_0, x_1))\}$ and the sequence $\{\varphi_n(d(x_1, x_2))\}$ are bounded. Notice that for any $n \in \mathbb{N}$,

$$d(x_n, x_{n+1}) = d(T^n x_0, T^n x_1) \leq \frac{1}{2}\varphi_n(d(x_0, x_1)) + \frac{1}{2}\varphi_n(d(x_1, x_2)).$$

Then $\{d(x_n, x_{n+1})\}$ is bounded.

Now, we shall prove that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Assume not, then there exist $\varepsilon_0 > 0$ and a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that

$$d(x_{n_k}, x_{n_k+1}) \geq \varepsilon_0.$$

Note that $\{d(x_n, x_{n+1})\}$ is bounded. Without loss of generality, there exists a subsequence denoted by $\{n_k\}$, such that $d(x_{n_k}, x_{n_k+1}) \to a \geq \varepsilon_0$ as $k \to \infty$. From Lemma 2.2, for $\frac{3a}{2} > \frac{a}{2} > 0$, there exist $\delta > 0$ (now suppose $\delta \leq a$) and $N \in \mathbb{N}$ such that

$$\varphi_n(t) \leq t - \delta,$$

for all $t \in \left[\frac{a}{2}, \frac{3a}{2}\right]$ and integer $n \geq N$. Notice that there exists an integer k_0 such that $d(x_{n_k}, x_{n_k+1}) \in \left(a - \frac{\delta}{2}, a + \frac{\delta}{2}\right) \subset \left[\frac{a}{2}, \frac{3a}{2}\right]$ for all $k \geq k_0$. Thus, we can find a $k \geq k_0$ such that $n_k - n_{k_0} \geq N$. Then, we have

$$\begin{aligned} d(x_{n_k}, x_{n_k+1}) &= d(T^p x_{n_{k_0}}, T^p x_{n_{k_0}+1}) \\ &\leq \frac{1}{2}\varphi_p(d(x_{n_{k_0}}, x_{n_{k_0}+1})) + \frac{1}{2}\varphi_p(d(x_{n_{k_0}+1}, x_{n_{k_0}+2})) \\ &\leq \frac{1}{2}[d(x_{n_{k_0}}, x_{n_{k_0}+1}) - \delta] + \frac{1}{2}[d(x_{n_{k_0}+1}, x_{n_{k_0}+2}) - \delta] \\ &\leq a - \frac{\delta}{2}, \end{aligned}$$

where $p = n_k - n_{k_0}$. That is a contradiction. Therefore, we conclude that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Notice that

$$d(x_n, x_{n+k}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+k-1}, x_{n+k}).$$

This amounts to say that $d(x_n, x_{n+k}) \to 0$ as $n \to \infty$ for all integers $k \geq 1$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. If not, then there exist $\varepsilon > 0$ and two subsequences $\{x_{m_k}\}$ $\{x_{n_k}\}$ of $\{x_n\}$ such that $n_k > m_k > k$, $d(x_{m_k}, x_{n_k}) \geq \varepsilon$. Since $\varphi_n \to \varphi$ uniformly, there exists $N \in \mathbb{N}$ such that

$$|\varphi_n(t) - \varphi(t)| < \frac{\varepsilon}{2},$$

for all $t \geq 0$ and integer $n \geq N$. Then we have

$$\varphi_n(t) \leq \varphi(t) + \frac{\varepsilon}{2} \leq t + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

for $t \in [0, \frac{\varepsilon}{2})$ and integer $n \geq N$. Since $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$, then there exists $k_0 \geq 2N$ such that $d(x_{m_{k_0}-N}, x_{m_{k_0}-N+1}) < \frac{\varepsilon}{2}$ and $d(x_{n_{k_0}-N}, x_{n_{k_0}-N+1}) < \frac{\varepsilon}{2}$. We can see that

$$\begin{aligned} d(x_{m_{k_0}}, x_{n_{k_0}}) &= d(T^N x_{m_{k_0}-N}, T^N x_{n_{k_0}-N}) \\ &\leq \frac{1}{2}\varphi_N(d(x_{m_{k_0}-N}, x_{m_{k_0}-N+1})) + \frac{1}{2}\varphi_N(d(x_{n_{k_0}-N}, x_{n_{k_0}-N+1})) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon, \end{aligned}$$

which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Note that the mapping T is continuous. We exhibit the following

$$Tx^* = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*.$$

Thus, x^* is a fixed point for T. Finally, we show that the fixed point for T is unique. If not, then there exists $y^* \in X$ such that $y^* \neq x^*$ and $Ty^* = y^*$. It follows that

$$d(x^*, y^*) = d(T^n x^*, T^n y^*) \leq \frac{1}{2}\varphi_n(d(x^*, Tx^*)) + \frac{1}{2}\varphi_n(d(y^*, Ty^*)) = \varphi_n(0),$$

for all $n \in \mathbb{N}$. Since $\varphi_n \to \varphi$ and $\varphi(t) < t$ for all $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\varphi_{n_0}(0) < 0$, which implies that $d(x^*, y^*) < 0$. That is a contradiction. Therefore, T has the unique fixed point $x^* \in X$. □

Example 2.1. *Let* $X = \left\{\frac{1}{n}\right\}_{n=1}^{\infty} \cup \left\{-\frac{1}{n}\right\}_{n=1}^{\infty} \cup \{0\}$ *and* $d(x, y) = |x - y|$. *Define a mapping* $T : X \to X$ *by*

$$Tx = \begin{cases} -x, & x > 0; \\ -\dfrac{x}{x+1}, & x < 0; \\ 0, & x = 0. \end{cases}$$

Let $\varphi(t) = \dfrac{t}{2}$ $(t \geq 0)$ *and the sequence of function* $\{\varphi_n\}_{n=1}^{\infty}$ *be such that*

$$\varphi_n(t) = \varphi(t) + \frac{2}{n+1}.$$

Then the following hold:

(1) (X, d) *is a complete metric space and the sequence* $\{\varphi_n\}_{n=1}^{\infty}$ *satisfy* $\varphi_n \to \varphi$ *uniformly on* $\mathbb{R}^+$, *where the* φ *is upper semicontinuous and* $\varphi(t) < t$ $(t > 0)$*;*
(2) T *is continuous in* X *and satisfies* (2.2) *for all* $x, y \in X$*;*
(3) T *has the unique fixed point* $x^* = 0$.

Proof. It is clear that (X, d) is a metric space. Note that X is a closed set in $\mathbb{R}$, and so X with the normal metric is complete, leading to that (X, d) is a complete metric space. Also, we can see that

$$\max_{t \in \mathbb{R}^+} |\varphi_n(t) - \varphi(t)| = \frac{1}{n}.$$

Thus, the sequence $\{\varphi_n\}_{n=1}^{\infty}$ converges uniformly to φ on $\mathbb{R}^+$. Moreover, it is clear that $\frac{t}{2} < t$ $(t > 0)$ and φ is continuous, so it is upper semicontinuous. Thus, the conclusion (1) is proved. Clearly, (3) immediately follows from the definition of T. Now we proceed to prove conclusion (2).

Note that 0 is the only accumulation point in (X, d), and for any $\{x_n\} \subset X$ with $x_n \to 0$ $(n \to \infty)$, there is $d(Tx_n, 0) \leq d(x_n, 0)$, leading to that $Tx_n \to 0 = T(0)$ as $n \to \infty$. So we can see that T is continuous in X.

Next, we show that (2.2) holds for all $x, y \in X$. Let $x_0 = 1$ and $x_n = Tx_{n-1}$, then we have $\{x_n\}_{n=0}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty} \cup \left\{-\frac{1}{n}\right\}_{n=1}^{\infty}$. Let $n \in \mathbb{N}^*$ be arbitrary, then we show the following inequality hold:

$$d(T^n x, 0) \leq \frac{1}{2}\varphi_n(d(x, Tx)). \tag{2.3}$$

For any $x \in X$, since the case $x = 0$ is trivial, now we suppose that $x = x_p$ for some $p \in \mathbb{N}$. Let us consider the following two cases

Case 1. If $n = 2k$ for some $k \in \mathbb{N}^*$, then we have $T^n x_p = x_{n+p}$. Note that φ_n is increasing and $\varphi_n\left(\frac{4}{n+1}\right) = \frac{4}{n+1}$, and so

$$\frac{4}{n+1} \leq \varphi_n(t) \leq t \ \left(t \geq \frac{4}{n+1}\right) \text{ and } \frac{4}{n+1} \geq \varphi_n(t) \geq t \ \left(0 \leq t \leq \frac{4}{n+1}\right). \quad (2.4)$$

Since $d(x_p, 0)$ is decreasing with respect to p, we can see that

$$d(x_{n+p}, 0) \leq d(x_{2k}, 0) = \frac{1}{k+1} = \frac{2}{2k+2} < \frac{1}{2} \cdot \frac{4}{2k+1}.$$

If $d(x_p, Tx_p) \geq \frac{4}{n+1}$, then we get

$$d(x_{n+p}, 0) < \frac{1}{2} \cdot \frac{4}{2k+1} = \frac{1}{2} \cdot \frac{4}{n+1} \leq \frac{1}{2}\varphi_n(d(x_p, Tx_p));$$

If $d(x_p, Tx_p) < \frac{4}{n+1}$, then we have

$$d(x_{n+p}, 0) \leq d(x_{p+1}, 0) \leq \frac{1}{2}[d(x_p, 0)+d(x_{p+1}, 0)] = \frac{1}{2}d(x_p, Tx_p) \leq \frac{1}{2}\varphi_n(d(x_p, Tx_p)).$$

Thus, (2.3) holds in the case $n = 2k$ for some $k \in \mathbb{N}^*$.

Case 2. If $n = 2k - 1$ for some $k \in \mathbb{N}^*$. Since $d(x_p, 0)$ is decreasing with respect to p, from (2.4) we can see that

$$d(x_{n+p}, 0) \leq d(x_{2k-1}, 0) = \frac{1}{k} = \frac{2}{2k} = \frac{1}{2} \cdot \frac{4}{n+1}.$$

If $d(x_p, Tx_p) \geq \frac{4}{n+1}$, then we get

$$d(x_{n+p}, 0) \leq \frac{1}{2} \cdot \frac{4}{n+1} \leq \frac{1}{2}\varphi_n(d(x_p, Tx_p));$$

If $d(x_p, Tx_p) < \frac{4}{n+1}$, then we have

$$d(x_{n+p}, 0) \leq d(x_{p+1}, 0) \leq \frac{1}{2}[d(x_p, 0)+d(x_{p+1}, 0)] = \frac{1}{2}d(x_p, Tx_p) \leq \frac{1}{2}\varphi_n(d(x_p, Tx_p)).$$

Therefore, we obtain that (2.3) holds for all $x \in X$.

For any $x, y \in X$, if $y = 0$, by (2.3) we have

$$d(T^n x, T^n y) = d(T^n x, 0) \leq \frac{1}{2}\varphi_n(d(x, Tx)) \leq \frac{1}{2}\varphi_n(d(x, Tx)) + \frac{1}{2}\varphi_n(d(y, Ty)).$$

If $x \neq 0$ and $y \neq 0$, from (2.3) we can see that

$$d(T^n x, T^n y) \leq d(T^n x, 0) + d(T^n y, 0) \leq \frac{1}{2}\varphi_n(d(x, Tx)) + \frac{1}{2}\varphi_n(d(y, Ty)).$$

Therefore, we prove that (2.2) holds for all $x, y \in X$. $\square$

Remark. *The mapping T in Example 2.1 can not be considered as a Kannan contraction, as it is not even a contraction at certain points. Indeed, let $z_n = \frac{1}{n}$ for $n \in \mathbb{N}^*$ and $z_0 = 0$. Then we have $Tz_n = -z_n$ and $Tz_0 = z_0$. It follows that*

$$d(Tz_n, Tz_0) = d(-z_n, 0) = \frac{1}{n} = \frac{1}{2}[d(z_n, Tz_n) + d(z_0, Tz_0)].$$

Thus, the mapping T is not a Kannan contraction for arbitrary coefficient.

Next, we provide a counterexample to show that the continuity condition for mapping T cannot be omitted.

Example 2.2. *Let $X = [0,1]$ and $d(x,y) = |x-y|$ for all $x,y \in X$. The mapping $T : X \to X$ is defined by*

$$Tx = \begin{cases} \frac{1}{3}x, & x > 0; \\ 1, & x = 0. \end{cases}$$

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ and $\varphi_n : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$\varphi(t) = \frac{2}{3}t, \ \varphi_1(t) = 1 \text{ and } \varphi_n(t) = \varphi(t) \ (n \geq 2).$$

Then the following hold:

(1) *All conditions in Theorem 2.3 are satisfied except for the continuity of the mapping T.*
(2) *T is fixed point free in X.*

Proof. It is clear that (X,d) is a complete metric space, $\varphi_n \to \varphi$ uniformly as $n \to \infty$, φ satisfies continuity and $\varphi(t) < t$ $(t > 0)$, and T is fixed point free by the construction of T. Next, we show that (2.2) holds for all $n \in \mathbb{N}^*$ and $x,y \in X$.

Let $x,y \in X$ be any given. Since the case $x = y$ is trivial, without loss of generality, we always assume that $x < y$. Let us consider the following two cases.

Case 1. If $x > 0$ and $y > 0$, then we have $Tx = \frac{x}{3} > 0$ and $Ty = \frac{y}{3} > 0$. Since $d(u,v) \leq 1$ for all $u,v \in X$, we obtain that

$$d(Tx,Ty) \leq \frac{1}{3} < \frac{1}{2} = \frac{1}{2}\varphi_1(d(x,Tx)) \leq \frac{1}{2}[\varphi_1(d(x,Tx)) + \varphi_1(d(y,Ty))].$$

Note that $d(T^{n+1}x, T^{n+1}y) = \frac{1}{3}d(T^n x, T^n y) \leq (T^n x, T^n y)$. Then for any integer $n \geq 2$, we have

$$\begin{aligned} d(T^n x, T^n y) &\leq d(T^2 x, T^2 y) \leq d(0, T^2 y) = \frac{1}{9}y = \frac{1}{2}\frac{d(y,Ty)}{3} < \frac{1}{2}\varphi(d(y,Ty)) \\ &= \frac{1}{2}\varphi_n(d(y,Ty)) \leq \frac{1}{2}[\varphi_n(d(x,Tx)) + \varphi_n(d(y,Ty))]. \end{aligned}$$

Case 2. If $x = 0$ and $y > 0$, then we have $Tx = 1$ and $Ty = \frac{y}{3} > 0$. Following the similar discussion above, we can see that

$$d(Tx,Ty) \leq 1 = \frac{1}{2} \cdot 2 = \frac{1}{2}[\varphi_1(d(x,Tx)) + \varphi_1(d(y,Ty))],$$

Note that $Tx = 1 > Ty$, and so $T^n x > T^n y$. Then for any integer $n \geq 2$, we have

$$\begin{aligned} d(T^n x, T^n y) &\leq d(T^2 x, T^2 y) \leq d(0, T^2 x) = \frac{1}{3} = \frac{1}{2}\varphi(d(x,Tx)) \\ &= \frac{1}{2}\varphi_n(d(x,Tx)) \leq \frac{1}{2}[\varphi_n(d(x,Tx)) + \varphi_n(d(y,Ty))]. \end{aligned}$$

Therefore, we conclude that (2.2) holds for all $n \in \mathbb{N}^*$ and $x,y \in X$. □

According to the weakly Ćirić type fixed point theorem, we can obtain the following result.

Theorem 2.4. *Let (X, d) be a complete metric space, and suppose that $T : X \to X$ is a continuous mapping. If there exists a sequence $\{\varphi_n\}_{n=1}^{\infty}$, $\varphi_n : \mathbb{R}^+ \to \mathbb{R}^+$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$d(T^n x, T^n y) \leq \max\{\varphi_n(d(x,y)), \varphi_n(d(x,Tx)), \varphi_n(d(y,Ty))\}, \tag{2.5}$$

for all $x, y \in X$ and $n \in \mathbb{N}$, where $\varphi_n \to \varphi$ uniformly on $\mathbb{R}^+$, the function φ is upper semicontinuous and $\varphi(t) < t$ $(t > 0)$. Then, T has a unique fixed point $x^* \in X$.

Proof. Let $x_0 \in X$ be an arbitrary but fixed element. Define a sequence of iterates $\{x_n\}$ in X by $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$. Since $\varphi_n \to \varphi$ uniformly, we have $\varphi_n(d(x_0, x_1)) \to \varphi(d(x_0, x_1))$ and $\varphi_n(d(x_1, x_2)) \to \varphi(d(x_1, x_2))$ as $n \to \infty$. Thus the sequence $\{\varphi_n(d(x_0, x_1))\}$ and the sequence $\{\varphi_n(d(x_1, x_2))\}$ are bounded. By virtue of $d(x_n, x_{n+1}) = d(T^n x_0, T^n x_1) \leq \max\{\varphi_n(d(x_0, x_1)), \varphi_n(d(x_1, x_2))\}$ for all $n \in \mathbb{N}$, we have $\{d(x_n, x_{n+1})\}$ is bounded.

Now, we show that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Assume the contrary, then there exist $\varepsilon_0 > 0$ and a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that

$$d(x_{n_k}, x_{n_k+1}) \geq \varepsilon_0.$$

Note that $\{d(x_n, x_{n+1})\}$ is bounded. Without loss of generality, there exists a subsequence denoted by $\{n_k\}$, such that $d(x_{n_k}, x_{n_k+1}) \to a \geq \varepsilon_0$ as $k \to \infty$. Applying Lemma 2.2, for $\frac{3a}{2} > \frac{a}{2} > 0$, we can find $\delta_1 > 0$ (now suppose $\delta_1 \leq a$) and $N \in \mathbb{N}$ such that

$$\varphi_n(t) \leq t - \delta_1,$$

for all $t \in \left[\frac{a}{2}, \frac{3a}{2}\right]$ and integer $n \geq N$. Notice that there exists an integer k_0 such that $d(x_{n_k}, x_{n_k+1}) \in \left(a - \frac{\delta_1}{2}, a + \frac{\delta_1}{2}\right) \subset \left[\frac{a}{2}, \frac{3a}{2}\right]$ for all $k \geq k_0$. Then, we can find $k \geq k_0$ such that $n_k - n_{k_0} \geq N$. Hence, we have

$$\begin{aligned}
d(x_{n_k}, x_{n_k+1}) &= d(T^p x_{n_{k_0}}, T^p x_{n_{k_0}+1}) \\
&\leq \max\{\varphi_p(d(x_{n_{k_0}}, x_{n_{k_0}+1})), \varphi_p(d(x_{n_{k_0}+1}, x_{n_{k_0}+2}))\} \\
&\leq \max\{d(x_{n_{k_0}}, x_{n_{k_0}+1}) - \delta_1, d(x_{n_{k_0}+1}, x_{n_{k_0}+2}) - \delta_1\} \\
&\leq a + \frac{\delta_1}{2} - \delta_1 \\
&\leq a - \frac{\delta_1}{2},
\end{aligned}$$

where $p = n_k - n_{k_0}$. That is a contradiction. Therefore, we conclude that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Notice that

$$d(x_n, x_{n+k}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+k-1}, x_{n+k}).$$

Thus, we have $d(x_n, x_{n+k}) \to 0$ as $n \to \infty$ for all integers $k \geq 1$.

Next, we will prove that $\{x_n\}$ is a Cauchy sequence. If not, then there exist $\varepsilon > 0$ and two subsequences $\{x_{m_k}\}$ $\{x_{n_k}\}$ of $\{x_n\}$ such that $n_k > m_k > k$, $d(x_{m_k}, x_{n_k}) \geq \varepsilon$ and $d(x_{m_k}, x_{n_k-1}) < \varepsilon$. By Lemma 2.2, for $2\varepsilon > \frac{\varepsilon}{2} > 0$, we can find $\delta_2 > 0$ (now suppose $\delta_2 \leq \frac{\varepsilon}{2}$) and $N_1 \in \mathbb{N}$ such that

$$\varphi_{N_1}(t) \leq t - \delta_2, \tag{2.6}$$

for all $t \in \left[\frac{\varepsilon}{2}, 2\varepsilon\right]$. Since $\varphi_n \to \varphi$ uniformly, we can see that there exists $N_2 \in \mathbb{N}$ such that

$$|\varphi_n(t) - \varphi(t)| < \frac{\varepsilon}{4},$$

for all $t \geq 0$ and integer $n \geq N_2$. Then we have

$$\varphi_n(t) \leq \varphi(t) + \frac{\varepsilon}{4} \leq t + \frac{\varepsilon}{4} < \frac{\delta_2}{2} + \frac{\varepsilon}{4} \leq \frac{\varepsilon}{2},$$

for $t \in [0, \frac{\delta_2}{2})$ and integer $n \geq N = \max\{N_1, N_2\}$. Since $\lim_{n \to \infty} d(x_n, x_{n+k}) = 0$ for all $k \in \mathbb{N}$, then there exists $k_0 \geq N + 1$ such that $d(x_{m_{k_0}}, x_{m_{k_0}+1}) < \frac{\delta_2}{2}$

, $d(x_{m_{k_0}}, x_{m_{k_0}+N}) < \frac{\delta_2}{2}$, $d(x_{n_{k_0}-1}, x_{n_{k_0}}) < \frac{\delta_2}{2}$, $d(x_{n_{k_0}}, x_{n_{k_0}+N-1}) < \frac{\delta_2}{2}$ and $d(x_{n_{k_0}-1}, x_{n_{k_0}+N-1}) < \frac{\delta_2}{2}$. By the triangular inequality, we have

$$d(x_{m_{k_0}}, x_{n_{k_0}-1}) \geq d(x_{m_{k_0}}, x_{n_{k_0}}) - d(x_{n_{k_0}-1}, x_{n_{k_0}}) \geq \varepsilon - \frac{\delta_2}{2},$$

and

$$\begin{aligned}
\varepsilon - 2 \cdot \frac{\delta_2}{2} &\leq d(x_{m_{k_0}}, x_{n_{k_0}}) - d(x_{m_{k_0}}, x_{m_{k_0}+N}) - d(x_{n_{k_0}}, x_{n_{k_0}+N-1}) \\
&\leq d(x_{m_{k_0}+N}, x_{n_{k_0}+N-1}) \\
&\leq d(x_{m_{k_0}+N}, x_{m_{k_0}}) + d(x_{m_{k_0}}, x_{n_{k_0}-1}) + d(x_{n_{k_0}-1}, x_{n_{k_0}+N-1}) \\
&\leq \varepsilon + 2 \cdot \frac{\delta_2}{2},
\end{aligned}$$

Then, we can see that $d(x_{m_{k_0}}, x_{n_{k_0}-1}) \in [\varepsilon - \frac{\delta_2}{2}, \varepsilon) \subset [\frac{\varepsilon}{2}, 2\varepsilon]$ and $d(x_{m_{k_0}+N}, x_{n_{k_0}+N-1}) \in [\varepsilon - 2 \cdot \frac{\delta_2}{2}, \varepsilon + 2 \cdot \frac{\delta_2}{2}) \subset [\frac{\varepsilon}{2}, 2\varepsilon]$. By (2.5), we deduce that

$$\begin{aligned}
&d(x_{m_{k_0}+N}, x_{n_{k_0}+N-1}) \\
&= d(T^N x_{m_{k_0}}, T^N x_{n_{k_0}-1}) \\
&\leq \max\{\varphi_N(d(x_{m_{k_0}}, x_{n_{k_0}-1})), \varphi_N(d(x_{m_{k_0}}, x_{m_{k_0}+1})), \varphi_N(d(x_{n_{k_0}-1}, x_{n_{k_0}}))\} \\
&\leq \max\{d(x_{m_{k_0}}, x_{n_{k_0}-1}) - \delta_2, \varphi_N(d(x_{m_{k_0}}, x_{m_{k_0}+1})), \varphi_N(d(x_{n_{k_0}-1}, x_{n_{k_0}}))\} \\
&< \max\{\varepsilon - \delta_2, \frac{\varepsilon}{2}\} \\
&= \varepsilon - \delta_2.
\end{aligned}$$

That is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Note that T is continuous. It is simliar to the proof of Theorem 2.3, we can obtain x^* is a fixed point for T. Finally, we show that the fixed point for T is unique. Assume that, on the contrary, there exists $y^* \in X$ such that $y^* \neq x^*$ and $Ty^* = y^*$. It follows that for any $n \in \mathbb{N}$,

$$\begin{aligned}
d(x^*, y^*) &= d(T^n x^*, T^n y^*) \\
&\leq \max\{\varphi_n(d(x^*, y^*)), \varphi_n(d(x^*, Tx^*)), \varphi_n(d(y^*, Ty^*))\} \\
&= \max\{\varphi_n(d(x^*, y^*)), \varphi_n(0)\}.
\end{aligned}$$

Since $\varphi_n \to \varphi$ and $\varphi(t) < t$ for all $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\varphi_{n_0}(d(x^*, y^*)) < d(x^*, y^*)$ and $\varphi_{n_0}(0) < 0$, which implies that $d(x^*, y^*) < d(x^*, y^*)$. That is a contradiction. Therefore, T has the unique fixed point $x^* \in X$. □

Example 2.3. *Let* $X = \{x_n\}_{n=1}^{\infty} \cup \{y_n\}_{n=1}^{\infty} \cup \{z_n\}_{n=1}^{\infty} \cup \{\theta\}$ *(denote that* $A_1 := \{x_n\}$, $A_2 =: \{y_n\}$ *and* $A_3 =: \{z_n\}$*),* $X^* = X \setminus \{\theta\}$ *and metric* $d : X \times X \to \mathbb{R}^+$ *be defined* *by*

$$d(u,v)=d(v,u)=\begin{cases} 0, & u=\theta \text{ and } v=\theta; \\ \dfrac{1}{n}, & u=\theta, v=x_n \text{ or } v=y_n \text{ or } \\ & v=z_n \text{ for some } n\in\mathbb{N}^*; \\ \left|\dfrac{1}{m}-\dfrac{1}{n}\right|, & (u=x_m, v=x_n) \text{ or } (u=y_m, v=y_n) \\ & \text{or } (u=z_m, v=z_n) \text{ for some } m,n\in\mathbb{N}^*; \\ \max\left\{\dfrac{1}{m},\dfrac{1}{n}\right\}, & (u=x_m, v=y_n) \text{ or } (u=x_m, v=z_n) \\ & \text{or } (u=y_m, v=z_n) \text{ for some } m,n\in\mathbb{N}^*; \end{cases}$$

Define the mapping $T: X\to X$ such that:

$$Tx_n=y_n,\ Ty_n=z_n,\ Tz_n=x_{n+1} \text{ and } T\theta=\theta.$$

Let $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ and $\varphi_n:\mathbb{R}^+\to\mathbb{R}^+$ be defined by

$$\varphi(t)=\frac{t}{t+1},\ \varphi_1(t)=\varphi(t)+1,\ \varphi_2(t)=1 \text{ and } \varphi_n(t)=\varphi(t)\ (n\geq 3).$$

Then the following hold:

(1) *$d(x,y)\leq\max\{d(x,\theta),d(y,\theta)\}$ for all $x,y\in X$;*
(2) *$(A_i\cup\{\theta\},d)$ $(i=1,2,3)$ is a complete metric space;*
(3) *All conditions in Theorem 2.4 are satisfied and T has the unique fixed point $x=\theta$.*

Proof. **(1)** Let $x,y\in X$ be any given. If $x=\theta$ or $y=\theta$, then the conclusion is trivial. Now we suppose that $x,y\in X^*$. let us consider the following two cases.
Case 1. If $x,y\in A_i$ for some $i=1,2,3$. Without loss of generality, let $x=x_m$ and $y=x_n$ with $m\leq n$, then we have

$$d(x,y)=\frac{1}{m}-\frac{1}{n}\leq\frac{1}{m}=d(x,\theta)\leq\max\{d(x,\theta),d(y,\theta)\}.$$

Case 2. If $x\in A_i$ and $y\in A_j$ for some $i,j\in\{1,2,3\}$ with $i\neq j$. Without loss of generality, let $x=x_m$ and $y=y_n$ for some $m,n\in\mathbb{N}^*$, then we get

$$d(x,y)=\max\left\{\frac{1}{m},\frac{1}{n}\right\}=\max\left\{d(x_m,\theta),d(y_n,\theta)\right\}=\max\{d(x,\theta),d(y,\theta)\}.$$

Thus, we obtain that $d(x,y)\leq\max\{d(x,\theta),d(y,\theta)\}$ for all $x,y\in X$.

(2) Without loss of generality, we consider the case of $i=1$ to show the conclusion. Define a mapping $f: A_1\cup\{\theta\}\to\left\{\frac{1}{n}\right\}_{n=1}^{\infty}\cup\{0\}$ by

$$fx_n=\frac{1}{n} \text{ and } f\theta=0.$$

Then we can see that

$$d(x_m,x_n)=\left|\frac{1}{m}-\frac{1}{n}\right| \text{ and } d(x_n,\theta)=\frac{1}{n}=\left|\frac{1}{n}-0\right|.$$

It follows that f is an isometric mapping, and so $(A_1\cup\{\theta\},d)$ and $\left(\left\{\frac{1}{n}\right\}_{n=1}^{\infty}\cup\{0\},|\cdot|\right)$ are isometric. Thus, $(A_1\cup\{\theta\},d)$ is a complete metric space.

(3) First, we prove that (X,d) is a metric space. Indeed, it is sufficient to prove the triangle inequality holds in (X,d). Let $u,v,w\in X$ be any given. Due to the triviality of the proof when two of the three points coincide, we assume that the three points u,v,w are distinct. From (2) we can see $(A_i\cup\{\theta\},d)$ is a metric space for $i \in \{1,2,3\}$. Next, let us consider the following three cases where three points u, v, w are not in the same A_i.

Case a. Assume that $u = \theta$ or $v = \theta$ or $w = \theta$. Without loss of generality, let $u = \theta$. Since v, w are not in the same A_i, we suppose that $v \in A_1$ and $w \in A_2$. Then, we may assume $v = x_m$, $w = y_n$ for some $m, n \in \mathbb{N}^*$, leading to that

$$d(u,v) = \frac{1}{m} \le \max\left\{\frac{1}{m}, \frac{1}{n}\right\} = d(v,w) < d(v,w) + d(u,w),$$

$$d(u,w) = \frac{1}{n} \le \max\left\{\frac{1}{m}, \frac{1}{n}\right\} = d(v,w) < d(v,w) + d(u,v),$$

and

$$d(v,w) = \max\left\{\frac{1}{m}, \frac{1}{n}\right\} < \left|\frac{1}{m} + \frac{1}{n}\right| = d(u,v) + d(u,w).$$

Case b. Assume that $u, v, w \in X^*$ and two of these points belong to the same set A_i. Without loss of generality, let $u = x_m$, $v = x_n$ and $w = y_p$ for some $m, n, p \in \mathbb{N}^*$ with $m > n$. Then, we deduce that

$$d(u,w) = \max\left\{\frac{1}{m}, \frac{1}{p}\right\} \le \max\left\{\frac{1}{n}, \frac{1}{p}\right\} = d(v,w)$$

$$d(u,v) = \frac{1}{n} - \frac{1}{m} \le \frac{1}{n} \le \max\left\{\frac{1}{n}, \frac{1}{p}\right\} = d(v,w).$$

If $p \ge n$, then we have $d(v,w) = d(u,w)$. If $p < n$, then we obtain that

$$d(v,w) = \max\left\{\frac{1}{n}, \frac{1}{p}\right\} = \frac{1}{n} \le \frac{1}{n} - \frac{1}{m} + \max\left\{\frac{1}{m}, \frac{1}{p}\right\} = d(u,v) + d(u,w).$$

Case c. Assume that $u, v, w \in X^*$ with each point belonging to one of the three sets A_i $(i = 1,2,3)$. Without loss of generality, let $u = x_m$, $v = y_n$ and $w = z_p$ for some $m, n, p \in \mathbb{N}^*$ with $m \ge n \ge p$. Then we can see that $d(u,v) = \frac{1}{n}$, $d(u,w) = \frac{1}{p}$ and $d(v,w) = \frac{1}{p}$, leading to that

$$d(u,v) \le d(u,w) = d(v,w).$$

Therefore, the triangle inequality holds for all $u, v, w \in X$.

Next, we prove that (X,d) is complete. Denote that for each $i \in \{1,2,3\}$

$$\overline{A_i} = A_i \cup \{\theta\}.$$

Let $\{u_n\} \subset X$ be a Cauchy sequence. Note that the convergence of a Cauchy sequence is equivalent to the convergence of any of its subsequences. So it is sufficient to prove that there exists a subsequence of $\{u_n\}$ convergent in (X,d). Remarking that

$$X = A_1 \cup A_2 \cup A_3 \cup \{\theta\},$$

by the pigeonhole principle, there exits a subsequence $\{u_{n_k}\} \subset \{u_n\}$ that is contained within one of the four mentioned sets. If $\{u_{n_k}\} \subset \{\theta\}$, then we have $\{u_{n_k}\}$ converge to θ. If $\{u_{n_k}\} \subset A_i$ for some $i \in \{1,2,3\}$, then we get $\{u_{n_k}\} \subset \overline{A_i}$. Since $\{u_{n_k}\}$ is also a Cauchy sequence and $(\overline{A_i}, d)$ is a complete metric space (conclusion (2)), we obtain that $\{u_{n_k}\}$ converges in $\overline{A_i}$. Therefore, we conclude that $\{u_n\}$ converges in (X,d) and so (X,d) is complete.

By the construction of T, φ and φ_n, we can see that T is continuous, $\varphi_n \to \varphi$ as $n \to \infty$ uniformly and $x^* = \theta$ is the unique fixed point for T. Finally, we show that (2.5) holds for all $n \in \mathbb{N}^*$ and $x, y \in X$. From (1) we can see

$$d(T^n x, T^n y) \leq \max\{d(T^n x, \theta), d(T^n x, \theta)\} \leq 1 \leq \varphi_n(d(x, y)) \ (n = 1, 2).$$

Since the case of $x = y$ is trivial, it is sufficient to prove (2.5) for all $n \geq 3$ and $x, y \in X$ with $x \neq y$. We consider the following three cases.

Case I. Assume that $x = \theta$ or $y = \theta$. Without loss of generality, let $y = \theta$ and $x = x_p$ for $p \in \mathbb{N}^*$. By the construction of T, we have $T^3 x_p = x_{p+1}$ and $d(T^n x_p, \theta) \leq d(T^3 x_p, \theta)$ for $n \geq 3$. From (1) we obtain that

$$d(T^n x, T^n y) = d(T^n x_p, T^n \theta) \leq d(T^3 x_p, \theta) = \frac{1}{p+1} = \varphi(d(x_p, \theta)) = \varphi_n(d(x, y))$$

for all $n \geq 3$.

Case II. Assume that $x \in A_i$ and $y \in A_j$ for $i, j \in \{1, 2, 3\}$ and $i \neq j$. Without loss of generality, let $x = x_p$ and $y = y_q$ for $p, q \in \mathbb{N}^*$. In this case, we have $Tx \in A_{i+1}$ $(A_4 = A_1)$ and so $d(x, Tx) = d(x, \theta)$. Since $\varphi(t)$ is increasing, from (1) and Case I we obtain that for any $n \geq 3$,

$$\begin{aligned} d(T^n x, T^n y) &\leq \max\{d(T^n x, \theta), d(T^n y, \theta)\} \leq \max\{\varphi_n(d(x, \theta)), \varphi_n(d(y, \theta))\} \\ &= \max\{\varphi_n(d(x, Tx)), \varphi_n(d(y, Ty))\} \\ &\leq \max\{\varphi_n(d(x, y)), \varphi_n(d(x, Tx)), \varphi_n(d(y, Ty))\}. \end{aligned}$$

Case III. Assume that $x, y \in A_i$ for some $i \in \{1, 2, 3\}$. Without loss of generality, let $x = x_p$ and $y = x_q$ for $p, q \in \mathbb{N}^*$ and $p > q$. By the construction of T, we have $T^{3k+1} x_m = y_{m+k}$, $T^{3k+2} x_m = z_{m+k}$ and $T^{3k} x_m = x_{m+k}$. It follows that for any $n \geq 3$,

$$\begin{aligned} d(T^n x_p, T^n x_q) &\leq d(T^3 x_p, T^3 x_q) = \frac{1}{q} - \frac{1}{p} = \frac{p-q}{pq+p+q+1} \leq \frac{p-q}{pq+p-q} \\ &= \varphi(d(x_p, x_q)) = \varphi_n(d(x, y)) \\ &\leq \max\{\varphi_n(d(x, y)), \varphi_n(d(x, Tx)), \varphi_n(d(y, Ty))\}. \end{aligned}$$

Therefore, we conclude that (2.5) holds for all $n \geq 3$ and $x, y \in X$ with $x \neq y$. □

Remark. *We can see that Theorem 2.4 yields the fixed point theorem for Kirk's asymptotic contraction by Suzuki* [13, Theorem 4] *and the Theorem 2.3.*

3. Conclusion

In conclusion, this paper has established two fixed point theorems for asymptotic contractions of Kannan type and weakly Ćirić type. The second result extends the fixed point theorem for Kirk's asymptotic contractions, as originally presented by Suzuki. The paper introduces two useful lemmas that greatly simplify the proofs of fixed point theorems for asymptotic contractions. Additionally, three examples have been provided to validate our theorems and highlight the essential requirement of continuity for the mapping, as stated in the theorems. These findings contribute to the field of fixed point theory for asymptotic contractions and provide valuable insights for future research in this area.

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