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ON VECTOR VARIATIONAL INEQUALITIES AND NONSMOOTH VECTOR OPTIMIZATION PROBLEMS WITH GENERALIZED APPROXIMATE INVEXITY

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Abstract. In this paper, we consider two types of vector variational inequalities namely, Minty vector variational inequalities(MVVI) and Stampacchia Vector variational inequalities(SVVI) for a nonsmooth vector optimization problem involving locally Lipschitz generalized approximate invex functions. We formulate approximate (MVVI) and (SVVI) involving Clarke's generalized Jacobians and exploit them to characterize an approximate efficient solutions of the nonsmooth vector optimization problems to approximate (MVVI) and (SVVI) of different types. We also give an example to show the validity of main results. Our newly proved results generalize some well-known results in the literature.

1. Introduction

In 1980, Giannessi [6] introduced vector variational inequality(VVI) in a finite-dimensional Euclidean space and gave some of its applications. Chen and Cheng [3] studied the (VVI) in infinite-dimensional space and applied it to vector optimization problems. Giannessi [7] introduced (VVI) of Minty type as a generalization of the variational inequalities and established the necessary and sufficient conditions for a point to be an efficient solution of a vector optimization problem involving differentiable and convex functions. In multiobjective optimization, the notion of efficiency is a widely used solution concept. The concept of approximate efficiency may be given more flexibility by making the error depending on the decision variables. This led to the development of the idea of quasi efficiency. Dutta and Vetrivel [5] defined the notion of weak quasi efficiency and obtained necessary and sufficient optimality conditions for nonsmooth multiobjective optimization problems.

Approximation methods are crucial in optimization theory because finding an exact solution is sometimes unattainable or computationally very expensive. As a result, approximate efficient solutions help in overcoming the difficulties posed

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by computational imperfections and modeling restrictions. Mishra and Laha [15] gave the idea of approximate efficient solutions for a vector optimization problem using locally Lipschitz approximately convex functions and characterize these approximate efficient solutions by using approximate vector variational inequalities of Minty and Stampacchia type in terms of the Clarke subdifferentials. Later, Wang [22] proved that under higher order generalized invexity assumptions, the solutions of generalized vector variational-like inequalities in terms of the generalized Jacobian are the generalized quasi efficient solutions of nonsmooth multiobjective programming problems. Yang and Zheng [23] obtained the necessary and sufficient optimality conditions for a point to be the approximate solution of a vector variational inequality problem.

In the last three decades, several definitions extending the concept of convex function have been purposed by different researchers. The significant generalization of a convex function is an invex function that preserves many properties of the convex function. The notions of generalized invex function for differentiable functions were introduced by Osuna-Gomez et al. [19] in a finite-dimensional space. Using generalized Jacobian the generalized invex function has been extended to Lipschitz functions. Further development on generalized invex function and their application can be found in [18, 19, 21].

Recently, the class of invex functions has received a lot of interest, since it allows us to relax the smoothness and convex function assumptions for practical applications. In 2013, Bhatia et al. [2] introduced four new classes of generalized approximate convex functions and established sufficient optimality conditions for quasi efficient solutions of a vector optimization problem involving these functions. Mishra and Upadhyay [16] study the effort of [2, 8, 17] and consider a class of nonsmooth vector optimization problems and two vector variational inequality problems. Certain relations between vector variational inequality problems and nonsmooth vector optimization problems are recognized by using the quasi efficiency and generalized approximate convexity hypotheses. Gupta and Mishra [10] gave the idea of generalized approximate convex functions and established some relationship between vector variational inequalities and vector optimization problems in terms of Clarke's subdifferentials. Jennane et al. [12] formulate necessary and sufficient optimality conditions based on Stampacchia and Minty types of vector variational inequalities involving Clarke's generalized Jacobians, and established the relationship between local quasi weak efficient solutions and vector critical points. Joshi [13] considers vector optimization problem involving locally Lipschitz generalized approximately convex functions and formulate approximate (VVI) of Minty and Stampacchia type.

Motivated and inspired by the work of Gupta and Mishra [10], Jennane et al. [12], Joshi [13], Mishra and Laha [15], we introduce a class of generalized approximate invex function and establish some relationship between nonsmooth vector variational inequality problems and nonsmooth vector optimization problems.

The rest sections of this paper are organized as follows:

In section 2, we review some notions and definitions that will be used in this paper. In section 3, we establish the relationship between vector variational inequalities in the sense of Minty and Stampacchia and approximate efficient solutions of the nonsmooth vector optimization problems by using generalized approximate invex assumptions. Numerical examples to justify the main results have been shown in section 4.

2. Preliminaries

In this section, we recall some notions of nonsmooth analysis. For more detail, see [4]. Suppose $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{R}^n_+$ be its nonnegative orthant. In the sequel, let X be a nonempty subset of $\mathbb{R}^n$.
The following stipulation for equalities and inequalities will be used throughout this paper. If $x, y \in \mathbb{R}^n$, then
$x \leq y \;\Leftrightarrow\; x_i \leq y_i,\ i = 1, 2, 3, ..., n$ with strict inequality holding for at least one i;
$x \leq y \;\Leftrightarrow\; x_i \leq y_i,\ i = 1, 2, 3, ..., n;$
$x = y \;\Leftrightarrow\; x_i = y_i,\ i = 1, 2, 3, ..., n;$
$x < y \;\Leftrightarrow\; x_i < y_i,\ i = 1, 2, 3, ..., n.$
Suppose $X \subseteq \mathbb{R}^n$ be a nonempty set, $b : X \times X \to \mathbb{R}_+$, $\phi : \mathbb{R} \to \mathbb{R}$ be two functions and $\eta : X \times X \to \mathbb{R}^n$ be a continuous map. First of all, we recall some definitions.

Definition 2.1. *A function $f : X \to \mathbb{R}$ is said to be Lipschitz near $x \in X$ if ,*

$$\|f(y) - f(z)\| \leq k \|y - z\|,$$

for some $k > 0$ and for all y, z within a neighborhood of x.

We say that $f : X \to \mathbb{R}$ is Locally Lipschitz on X if it is Lipschitz near any point of X.

Definition 2.2. *Suppose $f : X \to \mathbb{R}$ is Lipschitz at $x \in X$, the generalized derivative (in the sense of Clarke) of f at $x \in X$ in the direction $v \in \mathbb{R}^n$, is denoted by $f^0(x, v)$ and is defined as*

$$f^0(x, v) = \limsup_{y \to x, \lambda \to 0} \frac{f(y + \lambda v) - f(y)}{\lambda}.$$

Definition 2.3. *The Clarke's subdifferential of f at $x \in X$ is denoted by $\partial f(x)$, and is defined as follows:*

$$\partial f(x) = \{\xi \in \mathbb{R}^n : f^0(x, v) \geq \langle \xi, v \rangle,\ \textit{for all } v \in \mathbb{R}^n\}.$$

It follows that, for any $v \in \mathbb{R}^n$

$$f^0(x, v) = max\{\langle \xi, v \rangle : \xi \in \partial f(x)\}.$$

These definitions and properties can be extended to a locally Lipschitz vector-valued function $f : X \to \mathbb{R}^p$. Denote by $f_i,\ i = 1, 2, 3, ..., p$ the components of f. The Clarke generalized gradient of f at $x \in X$ is the set $\partial f(x) = \partial f_1(x) \times \partial f_2(x) \times \partial f_3(x) \times ... \times \partial f_p(x)$.

Definition 2.4. [13] *Suppose $f : X \to \mathbb{R}^p$ be locally Lipschitz function on X, then f is said to be approximate invex function at $x_0 \in X$, if for all $\epsilon > 0$, there exist $\delta > 0$ such that*

$$f(x) - f(y) \geq \langle \xi, \eta(x, y) \rangle - e\|\eta(x, y)\|,\ \forall\ \xi \in \partial f(y),\ x, y \in B(x_0, \delta),$$

where $e = (\epsilon, \epsilon, ..., \epsilon) \in int\mathbb{R}^p_+$

The function f is said to be approximate invex on X, if the above condition is satisfied for all $x_0 \in X$.

Definition 2.5. *Suppose $f : X \to \mathbb{R}^p$ be locally Lipschitz function on X, then f is said to be approximate pseudo-invex function of type I at $x_0 \in X$, if for all $\epsilon > 0$, there exist $\delta > 0$ such that, whenever $x, y \in B(x_0, \delta)$ and if*

$$\langle \xi, \eta(x, y)\rangle \geq 0, \text{ for some } \xi \in \partial f(y),$$

then

$$f(x) - f(y) \geq -e\|\eta(x, y)\|.$$

Definition 2.6. *Suppose $f : X \to \mathbb{R}^p$ be locally Lipschitz function on X, then f is said to be approximate pseudo-invex function of type II (strictly approximate pseudo-invex function of type II) at $x_0 \in X$, if for all $\epsilon > 0$, there exist $\delta > 0$ such that, whenever $x, y \in B(x_0, \delta)$ and if*

$$\langle \xi, \eta(x, y)\rangle + e\|\eta(x, y)\| \geq 0, \text{ for some } \xi \in \partial f(y),$$

then

$$f(x) - f(y) \geq (>) 0.$$

Definition 2.7. *Suppose $f : X \to \mathbb{R}^p$ be locally Lipschitz function on X, then f is said to be approximate quasi-invex function of type I at $x_0 \in X$, if for all $\epsilon > 0$, there exist $\delta > 0$ such that, whenever $x, y \in B(x_0, \delta)$ and if*

$$f(x) - f(y) \leq 0,$$

then

$$\langle \xi, \eta(x, y)\rangle - e\|\eta(x, y)\| \leq 0, \ \forall\ \xi \in \partial f(y).$$

Definition 2.8. *Suppose $f : X \to \mathbb{R}^p$ be locally Lipschitz function on X, then f is said to be approximate quasi-invex function of type II at $x_0 \in X$, if for all $\epsilon > 0$, there exist $\delta > 0$ such that, whenever $x, y \in B(x_0, \delta)$ and if*

$$f(x) - f(y) \leq +e\|\eta(x, y)\|,$$

then

$$\langle \xi, \eta(x, y)\rangle \leq 0, \ \forall\ \xi \in \partial f(y).$$

3. Approximate Minty and Stampacchia Vector Variational Inequalities

In this section, using generalized approximate invexity, we establish some relationships between Minty and Stampacchia vector variational inequalities, and nonsmooth vector optimization problems.

We consider the following nonsmooth vector optimization problem (for short, VOP)

$$\min \ \{f(x) = (f_1(x), f_2(x), ..., f_p(x))\} \text{ such that } x \in X,$$

where $f_i : X \to \mathbb{R},\ i = 1, 2, 3, ..., p$ are non-differentiable locally Lipschitz functions on X.

Definition 3.1. [16] *Let $f : X \to \mathbb{R}^p$ be a function. Then a point $x_0 \in X$ is said to be:*

(i) *efficient solution to the nonsmooth VOP, if there exists no $y \in X$ such that $f(y) \leq f(x_0)$,*

(ii) *local weak efficient solution to the nonsmooth VOP, if there exists a $\delta > 0$ such that the following inequality does not hold*

$$f(y) < f(x_0),\ \forall\ y \in X \cap B(x_0, \delta).$$

Theorem 3.2. *([14]P.71, Theorem 5.1.3) Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be a locally Lipschitz at $x_0 \in X$. If f attains its local minimum at x_0, then $0 \in \partial f(x_0)$.*

Theorem 3.3. *Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be a locally Lipschitz and approximate pseudo-invex of type II at $x_0 \in X$. f attains its local minimum at x_0 iff $0 \in \partial f(x_0)$.*

Proof. Suppose f attains its local minimum at x_0, then by Theorem 3.2, we have $0 \in \partial f(x_0)$.

On the other hand, if $0 \in \partial f(x_0)$, then for every $\epsilon > 0$ and $x \in X$, we have

$$\langle \xi, \eta(x, x_0)\rangle + e\|\eta(x, x_0)\| = \langle 0, \eta(x, x_0)\rangle + e\|\eta(x, x_0)\| \geq 0,$$

where $e = \underbrace{(\epsilon, \epsilon, \ldots, \epsilon)}_{p} \in \text{int}\mathbb{R}^p_+$.

Using the definition of approximate pseudo-invex of type II, there exists a $\delta > 0$ such that

$$f(x) - f(x_0) \geq 0,\ \forall\ x \in X \cap B(x_0, \delta).$$

□

Joshi [13] introduced the following concepts of approximate efficient solutions, which are beneficial when the existence of an efficient solution fails.

Definition 3.4. *Let $f : X \to \mathbb{R}^p$ be a function. A vector $y \in X$ is said to be:*

(i) *approximate efficient solution of type I of the nonsmooth VOP, denoted by $(AES)_1$, iff for any $e = \underbrace{(\epsilon, \epsilon, \ldots, \epsilon)}_{p}$, $\epsilon > 0$ sufficiently small, there does not exist $\delta > 0$ such that*

$$f(x) - f(y) \leq e\|\eta(x, y)\|,\ \forall\ x \in B(y, \delta) \setminus \{y\},$$

(ii) *approximate efficient solution of type II of the nonsmooth VOP, denoted by $(AES)_2$, iff for any $\epsilon > 0$ sufficiently small, there exist $\delta > 0$ such that*

$$f(x) - f(y) \nleq e\|\eta(x, y)\|,\ \forall\ x \in B(y, \delta),$$

(iii) *approximate efficient solution of type III of the nonsmooth VOP, denoted by $(AES)_3$, iff for any $\epsilon > 0$ sufficiently small, there does not exist $\delta > 0$ such that*

$$f(x) - f(y) \nleq -e\|\eta(x, y)\|,\ \forall\ x \in B(y, \delta).$$

The following vector variational inequality problems of Minty type involving Clarke's subdifferential introduced by Joshi [13], which will be used in the sequel to characterize an approximate efficient solution of the nonsmooth VOP.

$(AMVVI)_1$: Find $x_0 \in X$ such that, for any $\epsilon > 0$ sufficiently small, there does not exist $\delta > 0$ such that

$$\langle \xi, \eta(x, x_0) \rangle \leq e\|\eta(x, x_0)\|, \ \forall \ x \in B(x_0, \delta) \setminus \{x_0\}, \ \xi \in \partial f(x).$$

$(AMVVI)_2$: Find $x_0 \in X$ such that, for any $\epsilon > 0$ sufficiently small, there exist $\delta > 0$ such that

$$\langle \xi, \eta(x, x_0) \rangle \nleq e\|\eta(x, x_0)\|, \ \forall \ x \in B(x_0, \delta), \ \xi \in \partial f(x).$$

$(AMVVI)_3$: Find $x_0 \in X$ such that, for any $\epsilon > 0$ sufficiently small, there does not exist $\delta > 0$ such that

$$\langle \xi, \eta(x, x_0) \rangle \nleq -e\|\eta(x, x_0)\|, \ \forall \ x \in B(x_0, \delta), \ \xi \in \partial f(x).$$

The following theorem gives the conditions under which an (AES) of the VOP is a solution of $(AMVVI)$.

Theorem 3.5. *Let $f : X \to \mathbb{R}^p$ be a locally Lipschitz function and $\eta : X \times X \to \mathbb{R}^p$ be such that $\eta(x, y) = -\eta(y, x)$. Then*

(i) *If f is an approximate pseudo-invex of type II at $x_0 \in X$ and x_0 is an $(AES)_1$ of the VOP, then x_0 is also a solution of the $(AMVVI)_1$.*
(ii) *If f is an approximate pseudo-invex of type II at $x_0 \in X$ and x_0 is an $(AES)_2$ of the VOP, then x_0 is also a solution of the $(AMVVI)_2$.*
(iii) *If f is strictly approximate pseudo-invex of type II at $x_0 \in X$ and x_0 is an $(AES)_3$ of the VOP, then x_0 is also a solution of the $(AMVVI)_3$.*

Proof. (i) Suppose on the contrary that x_0 is not a solution of the $(AMVVI)_1$. Then, for some $\epsilon > 0$ sufficiently small, there exist $\tilde{\delta} > 0$ such that

$$\langle \xi, \eta(x, x_0) \rangle \leq e\|\eta(x, x_0)\|, \ \text{for all } x \in B(x_0, \tilde{\delta}) \text{ and } \xi \in \partial f(x),$$

where $e = \underbrace{(\epsilon, \epsilon, \ldots, \epsilon)}_{p} \in \text{int}\mathbb{R}^p_+$. We can write it as follows :

$$\langle \xi, \eta(x_0, x) \rangle + e\|\eta(x_0, x)\| \geq 0. \tag{3.1}$$

Since f is approximate pseudo-invex function of type II at $x_0 \in X$, it follows that for every $\epsilon > 0$, there exists $\check{\delta} > 0$ such that, whenever $x, x_0 \in B(x_0, \check{\delta})$ and if

$$\langle \xi, \eta(x_0, x) \rangle + e\|\eta(x_0, x)\| \geq 0, \ \text{for some } \xi \in \partial f(x),$$

then

$$f(x) - f(x_0) \leq 0.$$

Using (3.1) and the definition of approximate pseudo-invex function of type II, and by setting $\hat{\delta} = \min\{\tilde{\delta}, \check{\delta}\}$, we have

$$f(x) - f(x_0) \leq e\|\eta(x, x_0)\|, \ \text{for all } x \in B(x_0, \hat{\delta}) \text{ and } \xi \in \partial f(x),$$

which is a contradiction that x_0 is an $(AES)_1$ of VOP.

(ii) Suppose on the contrary that x_0 is not a solution of the $(AMVVI)_2$. Then, for some $\epsilon > 0$ sufficiently small and for all $\tilde{\delta} > 0$, there exist $x \in B(x_0, \tilde{\delta})$ and $\xi \in \partial f(x)$ such that

$$\langle \xi, \eta(x, x_0) \rangle \leq e\|\eta(x, x_0)\|,$$

where $e = \underbrace{(\epsilon, \epsilon, \ldots, \epsilon)}_{p} \in \text{int}\mathbb{R}^p_+$. We can write it as follows :

$$\langle \xi, \eta(x_0, x) \rangle + e\|\eta(x_0, x)\| \geq 0. \tag{3.2}$$

Since f is approximate pseudo-invex function of type II at $x_0 \in X$, it follows that for all $\epsilon > 0$, there exists $\check{\delta} > 0$ such that, whenever $x, x_0 \in B(x_0, \check{\delta})$ and if

$$\langle \xi, \eta(x_0, x) \rangle + e\|\eta(x_0, x)\| \geq 0, \text{ for some } \xi \in \partial f(x),$$

then

$$f(x) - f(x_0) \leq 0.$$

Using (3.2) and the definition of approximate pseudo-invex function of type II, and by setting $\hat{\delta} = \min\{\tilde{\delta}, \check{\delta}\}$, we have

$$f(x) - f(x_0) \leq e\|\eta(x, x_0)\|, \text{ for some } x \in B(x_0, \hat{\delta}) \text{ and } \xi \in \partial f(x),$$

which is a contradiction that x_0 is an $(AES)_2$ of VOP.

(iii) Suppose on the contrary that x_0 is not a solution of the $(AMVVI)_3$. Then, for some $\epsilon > 0$ and for all $\tilde{\delta} > 0$, we have

$$\langle \xi, \eta(x, x_0) \rangle \leq -e\|\eta(x, x_0)\| < e\|\eta(x, x_0)\|, \text{ for all } x \in B(x_0, \tilde{\delta}) \text{ and } \xi \in \partial f(x),$$

where $e = \underbrace{(\epsilon, \epsilon, \ldots, \epsilon)}_{p} \in \text{int}\mathbb{R}^p_+$. We can write it as follows :

$$\langle \xi, \eta(x_0, x) \rangle + e\|\eta(x_0, x)\| \geq 0. \tag{3.3}$$

Since f is strictly approximate pseudo-invex function of type II at $x_0 \in X$, it follows that for every $\epsilon > 0$, there exist $\check{\delta} > 0$ such that, whenever $x, x_0 \in B(x_0, \check{\delta})$ and if

$$\langle \xi, \eta(x_0, x) \rangle + e\|\eta(x_0, x)\| \geq 0, \text{ for some } \xi \in \partial f(x),$$

then

$$f(x) - f(x_0) < 0.$$

Using (3.3) and the definition of strictly approximate pseudo-invex function of type II, and by setting $\hat{\delta} = \min\{\tilde{\delta}, \check{\delta}\}$, we have

$$f(x) - f(x_0) < 0, \text{ for all } x \in B(x_0, \hat{\delta}) \text{ and } \xi \in \partial f(x).$$

This implies there exists $\epsilon > 0$ sufficiently small such that

$$f(x) - f(x_0) \leq -e\|\eta(x, x_0)\|,$$

which is a contradiction that x_0 is $(AES)_3$ of VOP. □

Here we consider the approximate Stampacchia vector variational inequality problems involving Clarke's subdifferential introduced by Joshi [13] as follows:

$(ASVVI)_1$: Find $x_0 \in X$ such that, for any $\epsilon > 0$ sufficiently small, there exist $x \in X \setminus \{x_0\}$ and $\xi_0 \in \partial f(x_0)$ such that

$$\langle \xi_0, \eta(x, x_0) \rangle \nleq e\|\eta(x, x_0)\|.$$

$(ASVVI)_2$: Find $x_0 \in X$ such that, for any $\epsilon > 0$ sufficiently small, for every $x \in X$ and $\xi_0 \in \partial f(x_0)$ such that

$$\langle \xi_0, \eta(x, x_0) \rangle \nleq e\|\eta(x, x_0)\|.$$

$(ASVVI)_3$: Find $x_0 \in X$ such that, for any $\epsilon > 0$, there exist $\delta > 0$ such that

$$\langle \xi_0, \eta(x, x_0) \rangle \nleq -e\|\eta(x, x_0)\| \text{ for all } x \in B(x_0, \delta),\ \xi_0 \in \partial f(x_0).$$

The following theorem gives the conditions under which a solution of $(ASVVI)$ is (AES) of the VOP.

Theorem 3.6. *Let $f : X \to \mathbb{R}^p$ be a locally Lipschitz function. Then*

(i) *If f is an approximate quasi-invex of type II at $x_0 \in X$ and x_0 is a solution of $(ASVVI)_1$, then x_0 is also an $(AES)_1$ of the VOP.*
(ii) *If f is an approximate quasi-invex of type II at $x_0 \in X$ and x_0 is a solution of $(ASVVI)_2$, then x_0 is also an $(AES)_2$ of the VOP.*
(iii) *If f is an approximate pseudo-invex of type II at $x_0 \in X$ and x_0 is a solution of $(ASVVI)_3$, then x_0 is also an $(AES)_3$ of the VOP.*

Proof. (i) Suppose on the contrary that x_0 is not an $(AES)_1$ of the VOP. Then, for some $\epsilon > 0$ sufficiently small, there exists $\tilde{\delta} > 0$ such that

$$f(x) - f(x_0) \leq e\|\eta(x, x_0)\|, \text{ for all } x \in B(x_0, \tilde{\delta}),\ x \neq x_0, \tag{3.4}$$

where $e = \underbrace{(\epsilon, \epsilon, \ldots, \epsilon)}_{p} \in \text{int } \mathbb{R}^p_+$.

Since f is approximate quasi-invex function of type II at $x_0 \in X$, it follows that for all $\epsilon > 0$, there exist $\check{\delta} > 0$ such that, whenever $x, x_0 \in B(x_0, \check{\delta})$ and if

$$f(x) - f(x_0) \leq e\|\eta(x, x_0)\|,$$

then

$$\langle \xi_0, \eta(x, x_0) \rangle \leq 0.$$

Using (3.4) and the definition of approximate quasi-invex function of type II, and by setting $\hat{\delta} = \min\{\tilde{\delta}, \check{\delta}\}$, we have

$$\langle \xi_0, \eta(x, x_0) \rangle \leq 0, \text{ for all } x \in B(x_0, \hat{\delta}),\ x \neq x_0 \text{ and } \xi_0 \in \partial f(x_0)$$

This implies, for $\epsilon > 0$,

$$\langle \xi_0, \eta(x, x_0) \rangle \leq e\|\eta(x, x_0)\|,$$

for all $x \in B(x_0, \hat{\delta}),\ x \neq x_0$ and hence for all $x \in X \setminus \{x_0\}$, which is a contradiction that x_0 is a solution of $(ASV)_1$.

(ii) Suppose that x_0 is a solution of the $(ASV)_2$. Then, for any $\epsilon > 0$ sufficiently small, for every $x \in X$ and $\xi_0 \in \partial f(x_0)$, we have

$$\langle \xi_0, \eta(x, x_0) \rangle \nleq e\|\eta(x, x_0)\|,$$

which implies

$$\langle \xi_0, \eta(x, x_0) \rangle \nleq 0. \tag{3.5}$$

Since f is approximate quasi-invex function of type II at $x_0 \in X$, it follows that for all $\epsilon > 0$, there exist $\check{\delta} > 0$ such that, whenever $x, x_0 \in B(x_0, \check{\delta})$ and if

$$f(x) - f(x_0) \leq e\|\eta(x, x_0)\|,$$

then

$$\langle \xi_0, \eta(x, x_0) \rangle \leq 0.$$

Using (3.5) and the definition of approximate quasi-invex function of type II, it follows that for x sufficiently close to x_0, we have

$$f(x) - f(x_0) \nleq e\|\eta(x, x_0)\|, \text{ for all } x \in B(x_0, \check{\delta}) \text{ and } x \neq x_0.$$

Hence x_0 is an $(AES)_2$ of the VOP.

(iii) Suppose on the contrary that x_0 is not an $(AES)_3$ of the VOP. Then, for some $\epsilon > 0$, and for all $\tilde{\delta} > 0$, there exists $x \in B(x_0, \tilde{\delta})$ such that

$$f(x) - f(x_0) \leq -e\|\eta(x, x_0)\| < 0, \tag{3.6}$$

where $e = \underbrace{(\epsilon, \epsilon, \ldots, \epsilon)}_{p} \in \text{int } \mathbb{R}^p_+$.

Since f is approximate pseudo-invex function of type II at $x_0 \in X$, it follows that for all $\epsilon > 0$, there exists $\check{\delta} > 0$ such that, whenever $x, x_0 \in B(x_0, \check{\delta})$ and if

$$\langle \xi_0, \eta(x, x_0) \rangle + e\|\eta(x, x_0)\| \geq 0, \text{ for some } \xi_0 \in \partial f(x_0),$$

then

$$f(x) - f(x_0) \leq 0.$$

Using (3.6) and the definition of approximate pseudo-invexity of type II, and by setting $\hat{\delta} = \min\{\tilde{\delta}, \check{\delta}\}$, we have

$$\langle \xi_0, \eta(x, x_0) \rangle < -e\|\eta(x, x_0)\|, \text{ for some } x \in B(x_0, \hat{\delta}) \text{ and for all } \xi_0 \in \partial f(x_0),$$

which is a contradiction that x_0 is a solution of $(ASV)_3$.

□

Remark. *Under some suitable conditions, the implications of Theorem* 3.5 *and Theorem* 3.6 *state that the solution of $ASVVI$ is also a solution of $AMVVI$.*

(i) *If f is approximate quasi-invex of type II and approximate pseudo-invex of type II at $x_0 \in X$. Then, $x_0 \in X$ is a solution of $(ASVVI)_1$ implies x_0 is a solution of $(AMVVI)_1$.*
(ii) *If f is approximate quasi-invex of type II and approximate pseudo-invex of type II at $x_0 \in X$. Then, $x_0 \in X$ is a solution of $(ASVVI)_2$ implies x_0 is a solution of $(AMVVI)_2$.*
(iii) *If f is strictly approximate pseudo-invex of type II at $x_0 \in X$. Then, $x_0 \in X$ is a solution of $(ASVVI)_3$ implies x_0 is a solution of $(AMVVI)_3$.*

4. Numerical Example

The authenticity of the main results have been shown in the following example:

Example 4.1. *Consider the VOP as follows:*

$$\textit{min } f(x) = (f_1(x), f_2(x)), \textit{ subject to } x \in \mathbb{R},$$

where

$$f_1(x) = \begin{cases} x + x^2, & \textit{if } x \geq 0, \\ 2x, & \textit{if } x < 0, \end{cases}$$

and

$$f_2(x) = \begin{cases} 2x - x^2, & \text{if } x \geq 0, \\ 3x, & \text{if } x < 0. \end{cases}$$

The Clarke subdifferential of f_1 and f_2 at x are defined as follows:

$$\partial f_1(x) = \begin{cases} 1 + 2x, & \text{if } x > 0, \\ [1, 2], & \text{if } x = 0, \\ 2, & \text{if } x < 0, \end{cases}$$

and

$$\partial f_2(x) = \begin{cases} 2 - 2x, & \text{if } x > 0, \\ [2, 3], & \text{if } x = 0, \\ 3, & \text{if } x < 0. \end{cases}$$

Let $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a bifunction defined by $\eta(x, y) = x - y$.
For any $0 < e < 1$, there exist $\delta > 0$ such that for every $x, y \in B(x_0, \delta)$, $x_0 = 0$, $\xi_1 \in \partial f_1(x)$, $\xi_1 \in \partial f_2(x)$, we have

$$\langle \xi_1, \eta(x,y) \rangle + e\|\eta(x,y)\| = \begin{cases} (1+2x)(x-y) + e\|x-y\| > 0, & \text{if } x > 0, y > 0, x - y > 0, \\ (1+2x)(x-y) + e\|x-y\| < 0, & \text{if } x > 0, y > 0, x - y < 0, \\ (1+2x)(x-y) + e\|x-y\| < 0, & \text{if } x < 0, y > 0, \\ 2(x-y) + e\|x-y\| > 0, & \text{if } x > 0, y < 0, \\ 2(x-y) + e\|x-y\| > 0, & \text{if } x < 0, y < 0, x - y > 0, \\ 2(x-y) + e\|x-y\| < 0, & \text{if } x < 0, y < 0, x - y < 0, \\ r_1(x-y) + e\|x-y\| > 0, & \text{if } y = 0, x > 0, r_1 \in [1,2], \\ r_1(x-y) + e\|x-y\| < 0, & \text{if } y = 0, x < 0, r_1 \in [1,2] \end{cases}$$

and

$$\langle \xi_2, \eta(x,y) \rangle + e\|\eta(x,y)\| = \begin{cases} (2-2x)(x-y) + e\|x-y\| > 0, & \text{if } x > 0, y > 0, x - y > 0, \\ (2-2x)(x-y) + e\|x-y\| < 0, & \text{if } x > 0, y > 0, x - y < 0, \\ (2-2x)(x-y) + e\|x-y\| < 0, & \text{if } x < 0, y > 0, \\ 3(x-y) + e\|x-y\| > 0, & \text{if } x > 0, y < 0, \\ 3(x-y) + e\|x-y\| > 0, & \text{if } x < 0, y < 0, x - y > 0, \\ 3(x-y) + e\|x-y\| < 0, & \text{if } x < 0, y < 0, x - y < 0, \\ r_2(x-y) + e\|x-y\| > 0, & \text{if } y = 0, x > 0, r_2 \in [2,3], \\ r_2(x-y) + e\|x-y\| < 0, & \text{if } y = 0, x < 0, r_2 \in [2,3]. \end{cases}$$

Also

$$\begin{aligned} f_1(x) - f_1(y) &= \begin{cases} (x-y)(y+x+1), & \text{if } x > 0, y > 0, x - y > 0, \\ x^2 + x - 2y, & \text{if } y < 0, x \geq 0, \\ 2(x-y), & \text{if } x < 0, y < 0, x - y > 0, \\ x^2 + x, & \text{if } y = 0, x > 0, \end{cases} \\ &> 0, \end{aligned}$$

and

$$\begin{aligned} f_2(x) - f_2(y) &= \begin{cases} (x-y)(2-x-y), & \text{if } x > 0, y > 0, x - y > 0, \\ 2x - x^2 - 3y, & \text{if } y < 0, x \geq 0, \\ 3(x-y), & \text{if } x < 0, y < 0, x - y > 0, \\ 2x - x^2, & \text{if } y = 0, x > 0. \end{cases} \\ &\geq 0. \end{aligned}$$

Hence $f = (f_1, f_2)$ is approximate pseudo-invex of type II at $x_0 = 0$.

Since, for every $x \in B(x_0, \delta)$, if $x > 0$, then

$$\langle \xi_{0_1} \eta(x, x_0)\rangle + e\|\eta(x, x_0)\| = r_1 x + e\|x\| > 0,\ \ r_1 \in [1, 2]\,,$$

and

$$\langle \xi_{0_1} \eta(x, x_0)\rangle + e\|\eta(x, x_0)\| = r_2 x + e\|x\| > 0,\ \ r_2 \in [2, 3]\,.$$

That is, $\langle \xi_0, \eta(x, x_0)\rangle + e\|\eta(x, x_0)\| \nleq 0$.
Hence $x_0 = 0$ is a solution of $(ASVVI)_3$.

Since, for every $\epsilon > 0$, there exist $\delta > 0$ such that for every $x \in B(x_0, \delta)$ and $x > 0$, we have

$$f_1(x) - f_1(x_0) + e\|\eta(x, x_0)\| = x + x^2 + e\|x\| > 0,$$

and

$$f_2(x) - f_2(x_0) + e\|\eta(x, x_0)\| = 2x - x^2 + e\|x\| > 0.$$

That is, $f(x) - f(x_0) + e\|\eta(x, x_0)\| \nleq 0$.
Hence $x_0 = 0$ is an $(AES)_3$ of the VOP.

Thus, Theorem 3.6 is verified.

Since, for every $x \in B(x_0, \delta)$, if $x > 0$,

$$\langle \xi_1, \eta(x, x_0)\rangle + e\|\eta(x, x_0)\| = x + x^2 + e\|x\| > 0,$$

and

$$\langle \xi_2, \eta(x_0, x)\rangle + e\|\eta(x, x_0)\| = 2x - 2x^2 + e\|x\| > 0.$$

That is, $\langle \xi, \eta(x, x_0)\rangle + e\|\eta(x, x_0)\| \nleq 0$.

Hence $x_0 = 0$ is a solution of $(AMVVI)_3$. Thus, Theorem 3.5 is verified.

5. Conclusion

In this paper, using the concept of approximate pseudo-invex function of type II and approximate quasi-invex function of type II, we established the relationship between the solution of $AMVVI$ and $ASVVI$ to the approximate efficient solution of the nonsmooth VOP by utilizing our results, we can establish some more relationships between the problems to approximate efficient solutions of the nonsmooth VOP, which guarantees the novelty of our results.

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