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"ESS LIM INF" PROPERTY OF PLURISUPERHARMONIC FUNCTIONS

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Abstract. We present an analytic approach to plurisuperharmonic functions based on the Levi form in the sense of distributions in $\mathbb{C}^n$. We observe that the Sobolev space $W^{1,2}_{\mathrm{loc}}$ is a favourable function space to consider plurisuperharmonic functions. Using the distributional approach we derive several properties of plurisuperharmonic functions, mostly well-known in pluripotential theory but deduced applying different methods. In particular, we show that plurisuperharmonic functions satisfy so called "ess lim inf" property.

1. Introduction

The class of superharmonic functions consists of supersolutions to the Laplace equation, that is, lower semicontinuous functions satisfying $-\Delta u \geqslant 0$ in a weak sense. A natural function space to search weak supersolutions to the Laplace equation is the Sobolev space $W^{1,2}_{\mathrm{loc}}$, see [4]. It is compatible with the theory of p-superharmonic functions in the case $p = 2$, see [5]. In pluripotential theory (potential theory in several complex variables) a special subclass of superharmonic functions, plurisuperharmonic functions are parallel to superharmonic functions, originating to the work of Lelong [9, 10, 11]. The Levi form as well as the Monge–Ampère operator correspond to the Laplace operator, and the Levi form can be applied distributionally in a similar way to the Laplace operator.

From a measure theoretic point of view, Δu behaves well and it is a regular Borel measure whenever u is subharmonic. Contrary to this, the Monge–Ampère operator $(dd^c)^n$ cannot be well-defined as a regular Borel measure $(dd^c u)^n$ for arbitrary plurisubharmonic functions u in open subsets of $\mathbb{C}^n$, see e.g. [6]. Błocki studied this problem in the early 2000's. In the first paper [2] he proved the following result in the case $n = 2$: for a plurisubharmonic function u the Monge–Ampère measure $(dd^c u)^n$ can be well-defined if and only if u belongs to the Sobolev space $W^{1,2}_{\mathrm{loc}}$. In a subsequent paper [3] he generalized the result for each $n \geqslant 2$.

These observations inspire us to consider $W^{1,2}_{\mathrm{loc}}$-functions whose Levi form is negative semidefinite in the sense of distributions. Corresponding considerations

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of distributional derivatives are standard in the space of L^1_{loc}-functions, see e.g. [7, Theorem 2.9.11], but in our study we restrict the space of admissible solutions. Our proofs and justification of the results rely heavily on the corresponding results for superharmonic functions and Laplace operator.

2. Preliminaries

Let Ω be a nonempty open subset of $\mathbb{C}^n$, $n \geqslant 1$, and consider real-valued functions in Ω. The class of Lebesgue integrable (1-integrable) functions in Ω is denoted by $L^1(\Omega)$. The corresponding local space $L^1_{\text{loc}}(\Omega)$ is defined in the obvious manner: a function $u \in L^1_{\text{loc}}(\Omega)$ if and only if $u \in L^1(G)$ for each open set $G \Subset \Omega$ (that is, $\overline{G}$ is a compact subset of Ω). The *support* of a function $u\colon \Omega \to \mathrm{R}$, denoted by $\operatorname{spt} u$, is the smallest closed set such that u vanishes outside $\operatorname{spt} u$. The classes of compactly supported continuous and compactly supported infinitely smooth functions in Ω are denoted by $C_0(\Omega)$ and $C_0^\infty(\Omega)$, respectively.

Suppose that $u, g_j, h_k, f_{jk} \in L^1_{\text{loc}}(\Omega)$ for $j, k = 1, \ldots, n$. We say that $\partial_j u = \frac{\partial u}{\partial z_j} = g_j$ *weakly* (or *in the sense of distributions*) in Ω if

$$\int_\Omega u \,\frac{\partial \varphi}{\partial z_j}\, dm = -\int_\Omega g_j \varphi \, dm \tag{2.1}$$

for all $\varphi \in C_0^\infty(\Omega)$. Correspondingly, $\overline{\partial}_k u = \frac{\partial u}{\partial \bar{z}_k} = h_k$ weakly in Ω if

$$\int_\Omega u \,\frac{\partial \varphi}{\partial \bar{z}_k}\, dm = -\int_\Omega h_k \varphi \, dm \tag{2.2}$$

for all $\varphi \in C_0^\infty(\Omega)$. The functions g_j and h_k are called the *weak first order partial derivatives* of u in Ω. Further, $\partial_j \overline{\partial}_k u = \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} = f_{jk}$ weakly in Ω if

$$\int_\Omega u \,\frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}\, dm = \int_\Omega f_{jk} \varphi \, dm \tag{2.3}$$

for all $\varphi \in C_0^\infty(\Omega)$. The functions f_{jk} are called the *weak second order partial derivatives* of u in Ω. Note that the previous definitions of the weak derivatives are motivated by integration by parts, and this explains the signature in the formulas (2.1), (2.2) and (2.3). In addition, we define that $\partial u = g = (g_1, \ldots, g_n)$, $\overline{\partial} u = h = (h_1, \ldots, h_n)$ and

$$\partial\overline{\partial} u = f = \begin{bmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{bmatrix} \tag{2.4}$$

weakly in Ω if $\partial_j u = g_j$, $\overline{\partial}_k u = h_k$ and $\partial_j \overline{\partial}_k u = f_{jk}$ weakly in Ω for each $j, k = 1, \ldots, n$.

The class of Lebesgue 2-integrable functions in Ω is denoted by $L^2(\Omega)$. The *Sobolev space* $W^{1,2}(\Omega)$ is defined as the space of functions $u \in L^2(\Omega)$ whose weak first order partial derivatives $\partial_j u$ and $\overline{\partial}_k u$ exist and belong also to $L^2(\Omega)$. The corresponding local space $W^{1,2}_{\text{loc}}(\Omega)$ is defined again as follows: a function $u \in W^{1,2}_{\text{loc}}(\Omega)$ if and only if $u \in W^{1,2}(G)$ for each open set $G \Subset \Omega$. The space $W^{1,2}_0(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,2}(\Omega)$.

The space $L^2(\Omega)$ is equipped with a norm

$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega |u|^2 \, dm \right)^{\frac{1}{2}}, \tag{2.5}$$

and

$$\|u\|_{W^{1,2}(\Omega)} = \|u\|_{L^2(\Omega)} + \sum_{j=1}^{n} \left(\|\partial_j u\|_{L^2(\Omega)} + \|\overline{\partial}_j u\|_{L^2(\Omega)} \right) \tag{2.6}$$

is the *Sobolev norm* of a function $u \in W^{1,2}(\Omega)$. The Sobolev spaces $W^{1,2}(\Omega)$ and $W^{1,2}_{\mathrm{loc}}(\Omega)$ are Banach spaces under the norm $\|\cdot\|_{W^{1,2}(\Omega)}$. It is clear that these definitions of the Sobolev spaces in $\mathbb{C}^n$ are parallel to the definitions of the Sobolev spaces in $\mathbb{R}^{2n}$.

Recall next some terminology associated with the operators in the divergence form, especially the Laplace operator. Suppose that Ω is still an open subset of $\mathbb{C}^n$; here $\mathbb{C}^n$ is identified to $\mathbb{R}^{2n}$. If a C^1-function $u\colon \Omega \to \mathbb{R}$ is regarded as a C^1-function that takes the real variables $x_1, y_1, \ldots, x_n, y_n$ to the real variable u, then we use the standard notation

$$\partial_j u = \frac{\partial u}{\partial z_j} = \frac{1}{2}\left(\frac{\partial u}{\partial x_j} - i \frac{\partial u}{\partial y_j} \right) \quad \text{and} \quad \overline{\partial}_j u = \frac{\partial u}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial u}{\partial x_j} + i \frac{\partial u}{\partial y_j} \right),$$

where $j = 1, \ldots, n$. The *gradient* of u is the vector

$$\nabla u = (\partial_1 u, \partial_2 u, \ldots, \partial_{2n-1} u, \partial_{2n} u) = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial y_1}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial u}{\partial y_n} \right). \tag{2.7}$$

Suppose that $u \in L^1_{\mathrm{loc}}(\Omega)$. A function $v = (v_1, \ldots, v_{2n}) \in L^1_{\mathrm{loc}}(\Omega)$ is the *weak gradient* of u if

$$\int_\Omega u \, \partial_j \varphi \, dm = - \int_\Omega v_j \, \varphi \, dm \tag{2.8}$$

for all $\varphi \in C_0^\infty(\Omega)$ and $j = 1, 2, \ldots, 2n$. Then we denote $v = \nabla u$.

The *Laplace operator* $\Delta = \nabla \cdot \nabla = \operatorname{div} \nabla$ is of order 2 and in the divergence form. By Weyl's lemma every weak solution u of the Laplace equation $-\Delta u = 0$ is a smooth solution. Therefore it is justified to define the class of harmonic functions as a subclass of twice differentiable functions: A real-valued function $h \in C^2(\Omega)$ is said to be *harmonic* in Ω if it satisfies the *homogeneous Laplace equation*

$$\Delta h = \sum_{j=1}^{n} \left(\frac{\partial^2 h}{\partial x_j^2} + \frac{\partial^2 h}{\partial y_j^2} \right) = 4 \sum_{j=1}^{n} \frac{\partial^2 h}{\partial z_j \partial \bar{z}_j} = 0 \quad \text{in } \Omega. \tag{2.9}$$

A function $u\colon \Omega \to (-\infty, \infty]$ is called *superharmonic* in Ω if

(i) u is lower semicontinuous in Ω,
(ii) $u \not\equiv \infty$ in any component of Ω, and
(iii) for each open $G \Subset \Omega$ the comparison principle holds: if $h \in C(\overline{G})$ is harmonic in G and $u|_{\partial G} \geqslant h|_{\partial G}$, then $u \geqslant h$ in G.

A function v is called *subharmonic* in Ω if $-v$ is superharmonic in Ω. It is well-known that a function $u \in C^2(\Omega)$ is superharmonic if $-\Delta u \geqslant 0$ in Ω.

3. Elementary definitions and results for the distributional approach

A set Ω is any open nonempty subset of $\mathbb{C}^n$ as always in this paper.

Definition 3.1. *A function* $u\colon \Omega \to (-\infty, \infty]$ *is called* plurisuperharmonic *in* Ω *if*

(i) *u is lower semicontinuous in Ω,*
(ii) *u is not identically ∞ on any component of Ω, and*
(iii) *for each $z \in \Omega$ and $w \in \mathbb{C}^n$, the function $\lambda \longmapsto u(z + \lambda w)$ is superharmonic or identically ∞ on every component of the set $\{\lambda \in \mathbb{C}\colon z + \lambda w \in \Omega\}$.*

A function $v\colon \Omega \to [-\infty, \infty)$ is called *plurisubharmonic* in Ω if $-v$ is plurisuperharmonic in Ω. Moreover, a function $u \in C^2(\Omega)$ is said to be *pluriharmonic* in Ω if

$$\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} = 0 \quad \text{in } \Omega$$

for all $j, k = 1, \ldots, n$.

Recall that the *Levi form* of a function $u \in C^2(\Omega)$ at $z \in \Omega$ is the Hermitian form

$$\langle Lu(z)b, c\rangle = \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z) b_j \bar{c}_k, \tag{3.1}$$

where $b, c \in \mathbb{C}^n$. It is well-known that a function $u \in C^2(\Omega)$ is plurisuperharmonic in Ω if $-\langle Lu(z)b, b\rangle$ is positive semidefinite in Ω, i.e.,

$$-\langle Lu(z)b, b\rangle = -\sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z) b_j \bar{b}_k \geqslant 0 \tag{3.2}$$

for all $z \in \Omega$ and $b \in \mathbb{C}^n$. Note that $-\langle Lu(z)b, b\rangle \geqslant 0$ is equivalent to $\langle Lu(z)b, b\rangle \leqslant 0$ for $z \in \Omega$ and $b \in \mathbb{C}^n$, end hence we say that the Levi form of u is negative semidefinite in Ω.

Definition 3.2. *A function $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ satisfies*

$$-\langle Lu(z)b, c\rangle = -\sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z) b_j \bar{c}_k \geqslant 0 \tag{3.3}$$

weakly in Ω for $b, c \in \mathbb{C}^n$ if

$$\int_\Omega \sum_{j,k=1}^{n} \big(\overline{\partial}_k u \partial_j \varphi + \partial_j u \overline{\partial}_k \varphi\big) b_j \bar{c}_k \, dm \geqslant 0 \tag{3.4}$$

whenever $\varphi \in C_0^\infty(\Omega)$ is a nonnegative test function. If $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and $-\langle Lu(z)b, b\rangle \geqslant 0$ weakly in Ω for all $b \in \mathbb{C}^n$, then the Levi form of u is said to be negative semidefinite in the sense of distributions *in Ω.*

Remark. *Suppose that $u \in C^2(\Omega)$. If $\varphi \in C_0^\infty(\Omega)$ is nonnegative and $b \in \mathbb{C}^n$, then integration by parts gives*

$$\int_\Omega \sum_{j,k=1}^{n} \big(\overline{\partial}_k u(z) \partial_j \varphi(z) + \partial_j u(z) \overline{\partial}_k \varphi(z)\big)\, b_j \bar{b}_k \, dm(z) = -2 \int_\Omega \langle Lu(z)b, b\rangle \varphi(z) \, dm(z)$$

which is nonnegative for each $b \in \mathbb{C}^n$ if and only if $-\langle Lu(z)b, b\rangle \geqslant 0$ for each $z \in \Omega$ and $b \in \mathbb{C}^n$. This is a condition for a twice continuously differentiable function to be plurisuperharmonic, indeed.

A function $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ satisfies the equation

$$-\partial_j\overline{\partial}_k u - \overline{\partial}_k\partial_j u = 0 \tag{3.5}$$

weakly in Ω for $j, k = 1, \dots, n$ if

$$\int_\Omega \left(\overline{\partial}_k u \partial_j\varphi + \partial_j u\overline{\partial}_k\varphi\right) dm = 0 \tag{3.6}$$

whenever $\varphi \in C_0^\infty(\Omega)$. In addition, a function $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a (weak) supersolution of the equation (3.5) in Ω if

$$\int_\Omega \left(\overline{\partial}_k u \partial_j\varphi + \partial_j u\overline{\partial}_k\varphi\right) dm \geqslant 0 \tag{3.7}$$

whenever $\varphi \in C_0^\infty(\Omega)$ is nonnegative. Then we may write that

$$-\partial_j\overline{\partial}_k u - \overline{\partial}_k\partial_j u \geqslant 0 \tag{3.8}$$

weakly in Ω for $j, k = 1, \dots, n$.

Functions having the Levi form negative semidefinite in the sense of distributions are weak supersolutions of the equation (3.5):

Lemma 3.3. *If the Levi form of $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is negative semidefinite in the sense of distributions in Ω, then*

$$-\partial_j\overline{\partial}_k u - \overline{\partial}_k\partial_j u \geqslant 0$$

weakly in Ω for all $j, k = 1, \dots, n$.

Proof. Let $\varphi \in C_0^\infty(\Omega)$ be a nonnegative test function and fix $j, k \in \{1, \dots, n\}$. Choose $b \in \mathbb{C}^n$ such that $b_j = \bar{b}_k = 1$ and $b_l = 0$ for each $l \neq j, k$. Since $-\langle Lu(z)b, b\rangle \geqslant 0$ weakly for all $z \in \Omega$ and $b \in \mathbb{C}^n$, we have

$$\int_\Omega \left(\overline{\partial}_k u \partial_j\varphi + \partial_j u\overline{\partial}_k\varphi\right) dm = \int_\Omega \sum_{j,k=1}^n \left(\overline{\partial}_k u \partial_j\varphi + \partial_j u\overline{\partial}_k\varphi\right) b_j\bar{b}_k \, dm \geqslant 0. \qquad \square$$

Next theorem and its corollary state the relations between weak supersolutions of the Laplace equation and functions whose Levi form is negative semidefinite in the sense of distributions.

Theorem 3.4. *If the Levi form of $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is negative semidefinite in the sense of distributions in Ω, then u is a weak supersolution of the Laplace equation in Ω.*

Proof. Let $\varphi \in C_0^\infty(\Omega)$ be a nonnegative test function. Now

$$\overline{\partial}_j u \partial_j \varphi = \frac{1}{4}\left(\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_j} - i\left(\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial y_j} - \frac{\partial u}{\partial y_j}\frac{\partial \varphi}{\partial x_j}\right) + \frac{\partial u}{\partial y_j}\frac{\partial \varphi}{\partial y_j}\right)$$

and

$$\partial_j u \overline{\partial}_j \varphi = \frac{1}{4}\left(\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_j} + i\left(\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial y_j} - \frac{\partial u}{\partial y_j}\frac{\partial \varphi}{\partial x_j}\right) + \frac{\partial u}{\partial y_j}\frac{\partial \varphi}{\partial y_j}\right)$$

for each $j = 1, \dots, n$. This yields by Lemma 3.3 and Fubini's theorem that

$$\begin{aligned}\int_\Omega \nabla u \cdot \nabla\varphi \, dm &= \int_\Omega \sum_{j=1}^n \left(\frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_j} + \frac{\partial u}{\partial y_j} \frac{\partial \varphi}{\partial y_j} \right) dm = \int_\Omega \sum_{j=1}^n 2 \left(\overline{\partial}_j u \partial_j \varphi + \partial_j u \overline{\partial}_j \varphi \right) dm \\ &= 2 \sum_{j=1}^n \int_\Omega \left(\overline{\partial}_j u \partial_j \varphi + \partial_j u \overline{\partial}_j \varphi \right) dm \geqslant 0.\end{aligned}$$

□

Corollary 3.5. *If $u \in W^{1,2}_{\text{loc}}(\Omega)$ satisfies the equation (3.5) weakly in Ω for all $j, k = 1, \dots, n$, then u is a weak solution to the Laplace equation in Ω.*

In the end of this section we consider pointwise behavior of functions whose Levi form is negative semidefinite in the sense of distributions. Let $u\colon \Omega \to \mathbb{R} \cup \{\pm\infty\}$ be a function, and denote

$$A = \{\lambda \in \mathbb{R} \colon u(z) \geqslant \lambda \text{ for almost every } z \in \Omega\}.$$

If $A \neq \emptyset$, then u is said to be *essentially bounded below.* The *essential lower bound* of u in Ω is defined by

$$\operatorname{ess\,inf} u = \sup A. \tag{3.9}$$

If $A = \emptyset$, then $\operatorname{ess\,inf} u = -\infty$, and if A is not bounded above, then $\operatorname{ess\,inf} u = +\infty$. Hence the essential lower bound is always defined. Further, we set

$$\operatorname{ess} \liminf_{w \to z} u(w) = \lim_{r \to 0} \operatorname{essinf}_{w \in B(z,r)} u(w). \tag{3.10}$$

Then for a lower semicontinuous function

$$u(z) \leqslant \liminf_{w \to z} u(w) \leqslant \operatorname{ess} \liminf_{w \to z} u(w),$$

and if a function $u\colon \Omega \to \mathbb{R} \cup \{+\infty\}$ satisfies

$$u(z) = \operatorname{ess} \liminf_{w \to z} u(z)$$

for all $z \in \Omega$, then u is lower semicontinuous.

Theorem 3.6. *Suppose that the Levi form of $u \in W^{1,2}_{\text{loc}}(\Omega)$ is negative semidefinite in the sense of distributions in Ω. Then u is locally essentially bounded below and there is a lower semicontinuous representative of u such that*

$$u(z) = \operatorname{ess} \liminf_{w \to z} u(w) \tag{3.11}$$

for each $z \in \Omega$.

Proof. In view of Theorem 3.4, the result follows from the same result for supersolutions of the Laplace equation, see [5, Theorem 3.63]. □

Next theorem states an important convergence result.

Theorem 3.7. *Suppose that (u_i) is an increasing and locally bounded sequence of functions in $W^{1,2}_{\text{loc}}(\Omega)$. If the Levi form of every u_i is negative semidefinite in the sense of distributions in Ω, then the same is true for the limit function $u = \lim_{i\to\infty} u_i$.*

Proof. According to Theorem 3.6 we may suppose that every u_i is lower semicontinuous and satisfies the "ess lim inf" property (3.11). Fix an open set $D \Subset \Omega$. Then we know that $u \in W^{1,2}(D)$. Moreover, again in view of Theorem 3.4, $\overline{\partial}_k u_i \to \overline{\partial}_k u$ and $\partial_j u_i \to \partial_j u$ weakly in $L^2(D)$, see [5, Theorem 3.75]. Suppose now that $b \in \mathbb{C}^n$ and $\varphi \in C_0^\infty(\Omega)$ is a nonnegative test function such that $\operatorname{spt}\varphi \subset D$. Then

$$0 \leqslant \int_\Omega \sum_{j,k=1}^n \big(\overline{\partial}_k u_i \partial_j \varphi + \partial_j u_i \overline{\partial}_k \varphi\big) b_j \overline{b}_k \, dm = \sum_{j,k=1}^n \int_D \big(\overline{\partial}_k u_i \partial_j \varphi + \partial_j u_i \overline{\partial}_k \varphi\big) b_j \overline{b}_k \, dm$$
$$\to \sum_{j,k=1}^n \int_D \big(\overline{\partial}_k u \partial_j \varphi + \partial_j u \overline{\partial}_k \varphi\big) b_j \overline{b}_k \, dm = \int_\Omega \sum_{j,k=1}^n \big(\overline{\partial}_k u \partial_j \varphi + \partial_j u \overline{\partial}_k \varphi\big) b_j \overline{b}_k \, dm.$$

Hence the Levi form of u is negative semidefinite in the sense of distributions in Ω. □

Remark. *Regarding Theorem 3.7, the same conclusion holds if we assume that the limit function u belongs to $W^{1,2}_{\text{loc}}(\Omega)$ instead of the sequence (u_i) being locally bounded.*

4. Plurisuperharmonic functions and Levi form in the sense of distributions

In this section we study relationship between the Levi form in the sense of distributions and plurisuperharmonic functions. It will turn out that plurisuperharmonicity can be characterized in terms of negative semidefinite Levi form in the sense of distributions.

Theorem 4.1. *Suppose that the Levi form of $u \in W^{1,2}_{\text{loc}}(\Omega)$ is negative semidefinite in the sense of distributions in Ω. If*

$$u(z) = \operatorname*{ess\,lim\,inf}_{w \to z} u(w) \tag{4.1}$$

for each $z \in \Omega$, then u is plurisuperharmonic in Ω.

Proof. Since u is locally essentially bounded below by Theorem 3.6, $u > -\infty$. The lower semicontinuity of u follows from (4.1), and $u \not\equiv \infty$ on every component of Ω because it belongs to $W^{1,2}_{\text{loc}}(\Omega)$.

Let $\varepsilon > 0$ and choose a usual ε-mollifier η_ε in $\mathbb{C}^n$. Let the convolution

$$u_\varepsilon(z) = \eta_\varepsilon * u(z) = \int_{\mathbb{C}^n} \eta_\varepsilon(z-w) u(w) \, dm(z)$$

be the usual ε-mollification of u defined in

$$\Omega_\varepsilon = \{z \in \Omega \colon \operatorname{dist}(z, \partial\Omega) > \varepsilon\},$$

see [12, Section 1.6]. Then $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$. Because the Levi form of u is negative semidefinite in the sense of distributions in Ω, Fubini's theorem gives

$$-\int_\Omega \langle Lu_\varepsilon(z)b, b\rangle \varphi(z) \, dm(z) \geqslant 0$$

for all $b \in \mathbb{C}^n$ and nonnegative $\varphi \in C_0^\infty(\Omega_\varepsilon)$. This implies that $-\langle Lu_\varepsilon(z)b, b\rangle \geqslant 0$ for each $b \in \mathbb{C}^n$ and $z \in \Omega_\varepsilon$, and hence u_ε is plurisuperharmonic in Ω_ε.

We know that u is a supersolution of the Laplace equation in Ω (Theorem 3.4), and thus by [5, Theorem 7.16], u is superharmonic in Ω. Hence u_ε decreases with decreasing ε and $\lim_{\varepsilon\to 0} u_\varepsilon = u$ pointwise in Ω. Thus the limit function u is plurisuperharmonic in Ω. □

Next result follows now from Theorem 3.6 and Theorem 4.1.

Corollary 4.2. *If the Levi form of $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is negative semidefinite in the sense of distributions in Ω, then there is a plurisuperharmonic function v in Ω such that $v = u$ almost everywhere in Ω.*

Theorem 4.3. *If u is plurisuperharmonic in Ω and locally bounded above, then $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and the Levi form of u is negative semidefinite in the sense of distributions in Ω.*

Proof. Since plurisuperhamonic functions are superharmonic, we have $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ by [5, Corollary 7.20]. Moreover, Theorem 3.7 implies that the Levi form of u is negative semidefinite in the sense of distributions in Ω. □

It is essential to assume in the previous theorem that u is locally bounded above. This is because there exist plurisuperharmonic functions which do not belong to $W^{1,2}_{\mathrm{loc}}(\Omega)$, see [6].

Corollary 4.4. *If a plurisuperharmonic function u belongs to $W^{1,2}_{\mathrm{loc}}(\Omega)$, then the Levi form of u is negative semidefinite in the sense of distributions in Ω.*

Proof. By Remark 3 the result is obtained if we apply Corollary 4.3 to the plurisuperharmonic functions $u_i = \min(u, i)$, $i = 1, 2, \ldots$. □

Theorem 4.5. *If u is a plurisuperharmonic function in Ω, then*

$$u(z) = \operatorname{ess}\liminf_{w\to z} u(w)$$

for each $z \in \Omega$.

Proof. Since plurisuperharmonic functions are superharmonic, the result follows from [5, Theorem 7.22]. □

Corollary 4.6. *If the Levi form of $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is negative semidefinite in the sense of distributions in Ω, then a plurisuperharmonic representative of u is unique.*

Proof. The result follows from Corollary 4.2 and Theorem 4.5. □

Corollary 4.7. *Suppose that u and v are plurisuperharmonic functions in Ω. If $u = v$ almost everywhere in Ω, then $u(z) = v(z)$ for all $z \in \Omega$.*

Proof. The result follows from Theorem 4.5. □

As a final comment, the most properties of plurisuperharmonic functions presented in this section belong to the standard literature in pluripotential theory, but the methods to deduce the results are different than usually.

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