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NEW NOVEL ITERATIVE SCHEMES FOR SOLVING GENERAL ABSOLUTE VALUE EQUATIONS

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Abstract. In this paper, we consider the general absolute value equations, which can be used to study the odd-order and nonsymmetric boundary value problems. It is shown that Lax-Milgram lemma and absolute values equations can be obtained as special cases. We use the auxiliary principle technique to prove the existence of a solution to the general absolute value equations. This technique is also used to suggest some new iterative methods for solving the general absolute value equations. The convergence analysis of the proposed methods is analyzed under some mild conditions. Ideas and techniques of this paper may stimulate further research.

1. Introduction

Recently much attention has been given to solve the systems of absolute value equations, which were introduced and studied by Mangasarian and Meyer [15]. The system of absolute value equations is closely related to the complementarity problems. Consequently, a system of absolute value equations can be viewed as a special case of variational inequalities and related optimization problems. This interplay between these two different fields enables us to use various techniques, which have been developed for variational inequalities for solving the system of absolute value problems. For recent numerical methods for solving the absolute values equations, see [8, 9, 11, 22, 13, 14, 15, 16, 17, 19, 20, 23, 24, 26, 27, 28, 29, 30, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43] and the references therein. We would like to point out that the system of absolute value equations is equivalent to the optimization problem, if the operator A is positive and symmetric. It is known that only the even order and self-adjoint boundary value problems can be studied by the classical variational inequalities. In many problem, the involved operator A may not be positive and symmetric. In such cases, the operator may be made positive and symmetric with respect to an arbitrary map. For more details, see Fillopov [5] and Tonti [37] and the references therein. Motivated by this fact, Noor [21, 22] introduced the general variational inequalities, which are used to study the odd-order and nonsymmetric

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boundary value problems. In this paper, we introduce and study a new system of general absolute values equations with respect to an arbitrary operator. This system of generalized absolute value equations can be viewed as a weak formulation of the non-positive and nonsymmetric boundary value problems. It is shown that the system of absolute value equations [15] and Lax-Milgram [1, 4, 10] can be obtained as special cases. We use the auxiliary principle technique, which is mainly due to Lions and Stampacchia [11] and Glowinski et al. [7], to discuss the existence of a solution for the system of generalized absolute value equations. The auxiliary principle technique is used to suggest some iterative methods for solving the system of generalized absolute value equations. The convergence analysis of these methods is investigated under suitable pseudomonotonicity, which is a weaker condition.

In Section 2, we introduce new general absolute value equations and discuss their applications. It is shown that the third order boundary value problems can be studied in the general framework of general absolute value equations. In section 3 and section 4, we use the auxiliary principle technique to discuss the existence of a solution as well as to suggest some iterative methods for solving the general absolute value equations. The convergence analysis of the proposed methods is considered under some mild conditions. Several new iterative methods for solving the absolute values equations are obtained as novel applications of the results. The ideas and techniques of this paper may the starting point for further research.

2. Formulations and basic facts

Let H be a Hilbert space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively.

For given operators $L, g : H \longrightarrow H$, a continuous linear functional f and a constant λ, we consider the problem of finding $u \in H$ such that

$$\langle Lu + \lambda|u|, g(v) - g(u)\rangle = \langle f, g(v) - g(u)\rangle, \quad \forall v \in H, \tag{2.1}$$

which is called the system of general absolute value equations. Here $|u|$ denotes the component-wise absolute value of $u \in H$. A wide class of problems arising in pure and applied sciences can be studied via the absolute valued equations (2.1).
If the operator g is subjective, then the problem (2.1) is equivalent to finding $u \in H$ such that

$$\langle Lu + \lambda|u|, g(v)\rangle = \langle f, g(v)\rangle, \quad \forall v \in H. \tag{2.2}$$

If $\lambda = 0$, then problem (2.1) collapses to finding $u \in H$, such that

$$\langle Lu, g(v)\rangle = \langle f, g(v)\rangle, \quad \forall v \in H, \tag{2.3}$$

which can be viewed as a general equations, which was introduced and analyzed by Noor et al. [25]. This result can be used to discuss the unique existence to a solution of the odd-order and nonsymmetric boundary value problems. This result plays a significant role in the study of function spaces and partial differential equations. For the applications and generalizations of the Lax-Milgram Lemma, see [1, 2, 5, 9, 10, 17, 18, 19, 20, 24, 25, 26] and the references.

For $L = I$, the identity operator, problem 2.3 reduces to finding $u \in H$, such that

$$\langle u, g(v)\rangle = \langle f, g(v)\rangle, \quad \forall v \in H, \tag{2.4}$$

which is called the general Riesz-Frechet representation theorem. This general Riesz-Frechet theorem be used to study the existence results for nonsymmetric, nonpositive and odd order problems boundary value problems. If $= I$, then the problem is called the celebrated Riesz-Frechet theorem, which played a significant role in the development of functional analysis, differential equations and optimization problems. See [18, 19, 23, 24] for more details.

If $g = I$, the identity operator, then problem (2.2) reduces to finding $u \in H$ such that

$$\langle Lu + \lambda|u|, v\rangle = \langle f, v\rangle, \quad \forall v \in H,$$

which is equivalent to finding $u \in H$ such that

$$Lu + \lambda|u| = f, \tag{2.5}$$

which is known as the absolute value equation. Problems of type (2.5) have been discussed in a series of papers recently, see [7, 8, 11, 12, 13, 14, 15, 21, 22, 23, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37].

Remark. *For suitable and appropriate choice of the operators L, g and constant λ, one can obtain various classes of new and previous known classes of problems. This shows that the system of general absolute value equations is a unified one.*

It is known [5, 37] that, if the operator is not symmetric and non-positive, then it can be made symmetric and positive with respect to an arbitrary operator.

Definition 2.1. [5, 37] *An operator $L : H \longrightarrow H$ is said to be :*
(a) *g-symmetric , if and only if,*

$$\langle Lu, g(v)\rangle = \langle g(u), Lv\rangle, \quad \forall u, v \in H.$$

(b) *g-positive, if and only if,*

$$\langle Lu, g(u)\rangle \geq 0, \quad \forall u \in H.$$

(c) *g-coercive (g-elliptic), if there exists a constant $\alpha > 0$ such that*

$$\langle Lu, g(u)\rangle \geq \alpha\|g(u)\|^2, \quad \forall u \in H.$$

Note that g-coercivity implies g-positivity, but the converse is not true. It is also worth mentioning that there are operators which are not g-symmetric but g-positive. On the other hand, there are g-positive, but not g-symmetric operators. Furthermore, it is well-known [5, 37] that, if for a linear operator L, there exists an inverse operator L^{-1} on $R(L)$, the range of L, with $\overline{R(L)} = H$, then one can find an infinite set of auxiliary operators g such that the operator T is both g-symmetric and g-positive.

If the operator L is linear, g-positive, g-symmetric and the operator g is linear, then the problem (2.1) is equivalent to finding a minimum of the function $I[v]$ on H, where

$$I[v] = \langle Lv + \lambda|v|, g(v)\rangle - 2\langle f, g(v)\rangle, \quad \forall v \in H, \tag{2.6}$$

which is a nonlinear programming problem and can be solved using the optimization techniques.

We now consider the problem of finding the minimum of the functional $I[v]$, defined by (2.6) and this is the main motivation of our next result.

Theorem 2.1. *Let the operator $L : H \longrightarrow H$ be linear, g-symmetric and let $(L + \lambda|.|)$ be g-positive. If the operator $g : H \longrightarrow H$ is linear, then the function $u \in H$ minimizes the functional $I[v]$, defined by (2.6), if and only if,*

$$\langle Lu + \lambda|u|, g(v) - g(u)\rangle = \langle f, g(v) - g(u)\rangle, \quad \forall v \in H. \tag{2.7}$$

Proof. Let $u \in H$ satisfy (2.7). Then, using the g-positivity of $(L + \lambda|.|)$, we have

$$\langle Lv + \lambda|v|, g(v) - g(u)\rangle \geq \langle f, g(v) - g(u)\rangle. \tag{2.8}$$

$\forall u, v \in H, \quad \epsilon \geq 0$, let $v_\epsilon = u + \epsilon(v - u) \in H$. Taking $v = v_\epsilon$ in (2.8) and using the fact that g is linear, we have

$$\langle Lv_\epsilon + \lambda|v_\epsilon|, g(v_\epsilon) - g(u)\rangle \geq \langle f, g(v_\epsilon) - g(u)\rangle. \tag{2.9}$$

We now define the function

$$\begin{aligned} h(\epsilon) \quad &= \quad \epsilon\langle Lu + \lambda|u|, g(v) - g(u)\rangle + \frac{\epsilon^2}{2}\langle L(v - u) \\ &\quad\quad +\lambda|v - u|, g(v) - g(u)\rangle - \epsilon\langle f, g(v) - g(u)\rangle, \end{aligned} \tag{2.10}$$

such that

$$\begin{aligned} h'(\epsilon) \quad &= \quad \langle Lu + \lambda|u|, g(v) - g(u)\rangle + \epsilon\langle L(v - u) + \lambda|v - u|, g(v) - g(u)\rangle \\ &\quad\quad -\langle f, g(v) - g(u)\rangle \\ &\geq \quad 0, \quad \text{by (2.9)} . \end{aligned}$$

Using the g symmetry of $L + \lambda|.|$, we see that $h(\epsilon)$ is an increasing function on $[0, 1]$ and so $h(0) \leq h(1)$ gives us

$$\langle Lu + \lambda|u|, g(u)\rangle - 2\langle f, g(u) \leq \langle Lv + \lambda|v|, g(v)\rangle - 2\langle f, g(v)\rangle,$$

that is,

$$I[u] \leq I[v], \quad \forall v \in H,$$

which shows that $u \in H$ minimizes the functional $I[v]$, defined by (2.6).
Conversely, assume that $u \in H$ is the minimum of $I[v]$, then

$$I[u] \leq I[v], \quad \forall v \in H. \tag{2.11}$$

Taking $v = v_\epsilon \equiv u + \epsilon(v - u) \in H, \forall u, v \in H$ in (2.11), we have

$$I[u] \leq I[v_\epsilon].$$

Using (2.6), g-positivity and the linearity of L, we obtain

$$\begin{aligned} \langle Lu + \lambda|u|, g(v) - g(u)\rangle \quad &+ \quad \frac{\epsilon}{2}\langle L(g(v) - g(u)) + \lambda|g(v) - g(u)|, g(v) - g(u)\rangle \\ &\geq \langle f, g(v) - g(u)\rangle, \end{aligned}$$

from which, as $\epsilon \longrightarrow 0$, we have

$$\langle Lu + \lambda|u|, g(v) - g(u)\rangle \geq \langle f, g(v) - g(u)\rangle, \quad \forall v \in H. \tag{2.12}$$

Replacing $g(v) - g(u)$ by $(g(u) - g(v))$ in inequality (2.12), we have

$$\langle Lu + \lambda|u|, g(v) - g(u)\rangle \leq \langle f, g(v) - g(u)\rangle, \quad \forall v \in H. \tag{2.13}$$

From (2.12) and (2.13), it follows that $u \in H$ satisfies

$$\langle Lu + \lambda|u|, g(v) - g(u)\rangle = \langle f, g(v) - g(u)\rangle, \quad \forall v \in H, \tag{2.14}$$

the required result (2.7). □

We now show that the third order boundary value problems can be studied via problem (2.1).

Example 2.1. *Consider the third order absolute boundary value problem of finding u such that*

$$-\frac{D^3u}{dx^3} + \lambda|u| = f(x), \quad \forall x \in [a,b], \tag{2.15}$$

with boundary conditions

$$u(a) = 0, \quad u'(a) = 0, \quad u'(b) = 0, \tag{2.16}$$

where $f(x)$ is a continuous function. This problem can be studied in the general framework of the problem (2.1). To do so, let

$$H = \{u \in H_0^2[a,b] : u(a) = 0, \quad u'(a) = 0, \quad u'(b) = 0\}$$

be a Hilbert space, see [7]. One can easily show that the energy functional associated with (2.1) is:

$$\begin{aligned} I[v] &= -\int_a^b \frac{d^3v}{dx^3} v dx + \int_a^b \lambda|v|\frac{dv}{dx}dx - 2\int_a^b f\frac{dv}{dx}dx, \quad \forall \frac{dv}{dx} \in H_0^2[a,b] \\ &= \int_a^b (\frac{d^2v}{dx^2})^2 + \int_a^b \lambda|v|\frac{dv}{dx}dx - 2\int_a^b f\frac{dv}{dx}dx \\ &= \langle Lv, g(v)\rangle + \langle \lambda|v|, g(v)\rangle - 2\langle f, g(v)\rangle, \end{aligned}$$

where

$$\langle Lu, g(v)\rangle = -\int_a^b \frac{d^3u}{dx^3}\frac{dv}{dx}dx = \int_a^b (\frac{d^2u}{dx^2})(\frac{d^2v}{dx^2})dx, \tag{2.17}$$

and

$$\begin{aligned} \langle |u|, g(v)\rangle &= \int_a^b |u|\frac{dv}{dx}dx, \\ \langle f, g(v)\rangle &= \int_a^b f\frac{dv}{dx}dx, \end{aligned}$$

where $g = \frac{d}{dx}$ is linear operator. It is clear that the operator L defined by (2.17) is linear, g-symmetric, g-positive and g is a linear operator.
Thus the minimum of the functional $I[v]$ defined on the Hilbert space H can be characterized by equation (2.2). This shows that the third order absolute boundary value problems can be studied in the framework of (2.2).

Definition 2.2. *An operator $L : H \longrightarrow H$ is said to be;*
(i) strongly monotone, if there exists a constant $\alpha > 0$, such that

$$\langle Lu - Lv, u - v\rangle \geq \alpha\|u - v\|^2, \quad \forall u, v \in H.$$

(ii) Lipschitz continuous, if there exists a constant $\beta > 0$, such that

$$\|Lu - Lv\| \leq \beta\|u - v\|, \quad \forall u, v \in H.$$

(iii) monotone, if

$$\langle Lu - Lv, u - v\rangle \geq 0, \quad \forall u, v \in H.$$

(iv) *firmly strongly monotone, if*

$$\langle Lu - Lv, u - v\rangle \geq \|u - v\|^2, \quad \forall u, v \in H.$$

We remark that, if the operator L is both strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, respectively, then from (i) and (ii), it follows that $\alpha \leq \beta$.

3. Main results

In this section, we use the auxiliary principle technique, the origin of which can be traced back to Lions and Stampacchia [11] and Glowinski et al. [7], as developed by Noor [21] and Noor et al. [30, 32]. The main idea of this technique is to consider an auxiliary problem related to the original problem. This way, one defines a mapping connecting the solutions of both problems. To prove the existence of solution of the original problem, it is enough to show that this connecting mapping is a contraction mapping and consequently has a unique solution of the original problem. Another novel feature of this approach is that this technique enables us to suggest some iterative methods for solving the general absolute value equations.

Theorem 3.1. *Let the operator L be a strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, respectively. Let the operator g be firmly strongly monotone and Lipschitz continuous with constant β_1. If there exists a constant $\rho > 0$ such that*

$$\left|\rho - \frac{\alpha + \nu - 1}{\beta^2 - \lambda^2}\right| < \frac{\sqrt{(\alpha + \nu - 1)^2 - (\beta^2 - \lambda^2)\nu(2 - \nu)}}{\beta^2 - \lambda^2}, \quad \nu < 1, \tag{3.1}$$

$$\alpha > 1 - \nu + \sqrt{(\beta^2 - \lambda^2)\nu(2 - \nu)}, \quad \rho\lambda < 1 - \nu, \tag{3.2}$$

where

$$\nu = \sqrt{\beta_1^2 - 1}, \tag{3.3}$$

then the problem (2.1) *has a solution.*

Proof. We use the auxiliary principle technique to prove the existence of a solution of (2.1). For a given $u \in H$, consider the problem of finding $w \in H$ such that,

$$\begin{aligned}\langle \rho(Lu + \lambda|u|), g(v) - g(w)\rangle &+ \langle g(w) - g(u), g(v) - g(w)\rangle \\ &= \langle \rho f, g(v) - g(w)\rangle, \quad \forall v \in H,\end{aligned} \tag{3.4}$$

which is called the auxiliary problem, where $\rho > 0$ is a constant. It is clear that (3.4) defines a mapping w connecting the both problems (2.1) and (3.4). To prove the existence of a solution of (2.1), it is enough to show that the mapping w defined by (3.4) is a contraction mapping.

Let $w_1 \neq w_2 \in H$ (corresponding to $u_1 \neq u_2$) satisfy the auxiliary problem (3.4). Then

$$\begin{aligned}\langle \rho(Lu_1 + \lambda|u_1|), g(v) - g(w_1)\rangle &+ \langle g(w_1) - g(u_1), g(v) - g(w_1)\rangle \\ &= \langle \rho f, g(v) - g(w_1)\rangle, \quad \forall v \in H,\end{aligned} \tag{3.5}$$

$$\begin{aligned}\langle \rho(Lu_2 + \lambda|u_2|), g(v) - g(w_2)\rangle &+ \langle g(w_2) - g(u_2), g(v) - g(w_2)\rangle \\ &= \langle \rho f, g(v) - g(w_2)\rangle, \quad \forall v \in H.\end{aligned} \tag{3.6}$$

Taking $v = w_2$ in (3.5) and $v = w_1$ in (3.6) and adding the resultant, we have

$$\begin{aligned}\|g(w_1) - g(w_2)\|^2 &= \langle g(w_1) - g(w_2), g(w_1) - g(w_2)\rangle \\ &= \langle g(u_1) - g(u_2) - \rho(Lu_1 - Lu_2) \\ &\quad +\rho\lambda(|u_1| - |u_2|), g(w_1) - g(w_2)\rangle. \qquad (3.7)\end{aligned}$$

From (3.7), we have

$$\|g(w_1) - g(w_2)\|^2 \leq \|g(u_1) - g(u_2) - \rho(Lu_1 - Lu_2) + \rho\lambda(|u_1| - |u_2|)\|\|g(w_1) - g(w_2)\|$$

from which, using the firmly strongly monotonicity of g and positivity of λ, it follows that

$$\begin{aligned}\|w_1 - w_2| &\leq \|g(w_1) - g(w_2)\| \\ &\leq \|g(u_1) - g(u_2) - \rho(Lu_1 - Lu_2)\| + \rho\lambda\||u_1| - |u_2|\| \\ &\leq \|g(u_1) - g(u_2) - \rho(Lu_1 - Lu_2)\| + \rho\lambda\|u_1 - u_2\| \\ &\leq \|u_1 - u_2 - g(u_1) - g(u_2)\| + \|u_1 - u_2 - \rho(Lu_1 - Lu_2)\| \\ &\quad +\rho\lambda\|u_1 - u_2\|. \qquad (3.8)\end{aligned}$$

Using the strongly monotonicity and Lipschitz continuity of the operator L with constants $\alpha > 0$ and $\beta > 0$, we have

$$\begin{aligned}\|u_1 - u_2 - \rho(Lu_1 - Lu_2)\|^2 &= \langle u_1 - u_2 - \rho(Lu_1 - Lu_2), u_1 - u_2 - \rho(Lu_1 - Lu_2)\rangle \\ &= \langle u_1 - u_2, u_1 - u_2\rangle - 2\rho\langle Lu_1 - Lu_2, u_1 - u_2\rangle \\ &\quad +\rho^2\langle Lu_1 - Lu_2, Lu_1 - Lu_2\rangle \\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2)\|u_1 - u_2\|^2. \qquad (3.9)\end{aligned}$$

Similarly, using the strongly firmly monotonicity and Lipschitz continuity of the operator g with constant β_1, we have

$$\|u_1 - u_2 - (g(u_1) - g(u_2))\|^2 \leq \{\sqrt{\beta_1^2 - 1}\}\|u_1 - u_2\|^2. \qquad (3.10)$$

Combining (3.8),(3.9) and (3.10), we have

$$\begin{aligned}\|w_1 - w_2\| &\leq \big(\sqrt{\beta_1^2 - 1} + \rho\lambda + \sqrt{(1 - 2\rho\alpha + \rho^2\beta^2)}\big)\|u_1 - u_2\| \\ &= \theta\|u_1 - u_2\|, \qquad (3.11)\end{aligned}$$

where

$$\begin{aligned}\theta &= \big(\sqrt{\beta_1^2 - 1} + \rho\lambda + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\big) \\ &= \nu + \rho\lambda + \sqrt{1 - 2\rho\alpha + \beta^2\rho^2},\end{aligned}$$

and

$$\nu = \sqrt{\beta_1^2 - 1}.$$

From (3.1) and (3.2), it follows that $\theta < 1$, so the mapping w is a contraction mapping and consequently, it has a fixed point $w(u) = u \in H$ satisfying the problem (2.1). $\square$

Remark. *We point out that the solution of the auxiliary problem* (3.4) *is equivalent to finding the minimum of the functional* $I[w]$, *where*

$$I[w] = \frac{1}{2}\langle g(w) - g(u), g(w) - g(u)\rangle - \rho\langle Lu + \lambda|u| - f, g(w) - g(u)\rangle,$$

which is a differentiable convex functional associated with the inequality (3.4)*, if the operator* g *is differentiable. This alternative formulation can be used to suggest iterative methods for solving the general absolute value equations. This auxiliary functional can be used to find a kind of gap function, whose stationary points solves the problem* (2.2)*, see* [6, 21, 30].

It is clear that, if $w = u$, then w is a solution of (2.1). This observation shows that the auxiliary principle technique can be used to suggest the following iterative method for solving the generalized absolute value equations (2.1).

Algorithm 3.2. *For a given initial value* u_0, *compute the approximate solution* x_{n+1} *by the iterative scheme*

$$\langle Lu_n + \lambda|u_n| + g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1})\rangle = \langle f, g(v) - g(u_{n+1})\rangle, \forall v \in H.$$

From Algorithm 3.2, one can easily obtain the Picard type iterative method for solving the absolute value equation (2.5) and appears to be a new one.

Algorithm 3.3. *For a given initial value* u_0, *compute the approximate solution* x_{n+1} *by the iterative scheme*

$$g(u_{n+1}) = g(u_n) - \rho(Lu_n + \lambda|u_n| - f), \quad n = 0, 1, 2, 3...$$

We again use the auxiliary principle technique to suggest an implicit method for solving the problem (2.1).
For a given $u \in H$, consider the problem of finding $w \in H$ such that,

$$\begin{aligned}\langle \rho(Lw + \lambda|w|), g(v) - g(w)\rangle \quad + \quad & \langle g(w) - g(u), g(v) - g(w)\rangle \\ & = \rho\langle f, g(v) - g(w)\rangle, \quad \forall v \in H, \end{aligned} \tag{3.12}$$

which is called the auxiliary problem. We note that the auxiliary problems (3.4) and (3.12) are quite different.
Clearly $w = u \in H$ is a solution of (2.1). This observation allows us to suggest the following iterative method for solving the problem (2.1).

Algorithm 3.4. *For a given initial value* u_0, *compute the approximate solution* x_{n+1} *by the iterative scheme*

$$\begin{aligned}\langle \rho Lu_{n+1} + \lambda\rho|u_{n+1}| \quad + \quad & g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1})\rangle \\ & = \langle \rho f, g(v) - g(u_{n+1})\rangle, \forall v \in H, \end{aligned} \tag{3.13}$$

which is an implicit method.
From this implicit method, we can obtain the following iterative method for solving (2.5)

Algorithm 3.5. *For a given initial value* u_0, *compute the approximate solution* x_{n+1} *by the iterative scheme*

$$g(u_{n+1}) = g(u_n) - \rho(Lu_{n+1} + \lambda|u_{n+1}| - f), \quad n = 0, 1, 2, 3...$$

This is a new implicit method for solving the absolute value equations (2.5).

To implement the implicit method (3.4), one uses the explicit method as a predictor and implicit method as a predictor. Consequently, we obtain the two-step method for solving the problem (2.1).

Algorithm 3.6. *For a given initial value u_0, compute the approximate solution x_{n+1} by the iterative schemes*

$$\begin{aligned}\langle \rho L u_n + \lambda\rho|u_n| &+ g(y_n) - g(u_n), g(v) - g(u_{n+1})\rangle \\ &= \langle \rho f, g(v) - g(y_n)\rangle, \quad \forall v \in H, \\ \langle \rho L y_n + \lambda\rho|y_n| &+ g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1})\rangle \\ &= \langle \rho f, g(v) - g(u_{n+1})\rangle, \quad \forall v \in H,\end{aligned}$$

which is known as two-step iterative method for solving problem (2.1).

Based on the above arguments, we can suggest a new two-step (predictor-corrector) method for solving the absolute value equations (2.5).

Algorithm 3.7. *For a given initial value u_0, compute the approximate solution x_{n+1} by the iterative schemes*

$$\begin{aligned} g(y_n) &= g(u_n) - \rho(Lu_n + \lambda|u_n| - f) \\ g(u_{n+1}) &= g(u_n) - \rho(Ly_n + \lambda|y_n| - f), \quad n = 0, 1, 2... \end{aligned}$$

For the convergence analysis of the iterative methods, we need the following concept.

Definition 3.1. *The operator L is said to be pseudo g-monotone with respect to $\lambda|.|$, if*

$$\begin{aligned} &\langle Lu + \lambda|u|, g(v) - g(u)\rangle = \langle f, g(v) - g(u)\rangle, \quad \forall v \in H, \\ &\Rightarrow \\ &\langle Lv + \lambda|v|, g(v) - g(u)\rangle \geq \langle f, g(v) - g(u)\rangle, \quad \forall v \in H. \end{aligned}$$

We now consider the convergence analysis of Algorithm 3.4 and this is the main motivation of our next result.

Theorem 3.8. *Let $u \in H$ be a solution of problem* (2.1) *and let u_{n+1} be the approximate solution obtained from Algorithm 3.4. If L is a g-monotone operator with respect to $\lambda|.|$, then*

$$\|g(u_{n+1}) - g(u)\|^2 \leq \|g(u_n) - g(u)\|^2 - \|g(u_{n+1}) - g(u_n)\|^2. \tag{3.14}$$

Proof. Let $u \in H : g(u) \in H$ be a solution of (2.1). Then

$$\langle Lu + \lambda|u|, g(v) - g(u)\rangle = \langle f, g(v) - g(u)\rangle, \quad \forall v \in H,$$

which implies that

$$\langle Lv + \lambda|v|, g(v) - g(u)\rangle \geq \langle f, g(v) - g(u)\rangle, \quad \forall v \in H, \tag{3.15}$$

since the operator L is a monotone operator with respect to $\lambda|.|$.
Taking $v = u_{n+1}$ in (3.15) and $v = u$ in (3.13), we have

$$\langle Lu_{n+1} + \lambda|u_{n+1}|, g(u_{n+1}) - g(u)\rangle \geq \langle f, g(u_{n+1}) - g(u)\rangle, \quad \forall v \in H, \tag{3.16}$$

and

$$\begin{aligned}&\langle \rho L u_{n+1} + \rho\lambda|u_{n+1}| + g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1})\rangle \\ &= \langle \rho f, g(u) - g(u_{n+1})\rangle, \forall v \in H. \end{aligned} \tag{3.17}$$

From (3.17), we have

$$\begin{aligned}\langle g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1})\rangle \quad &\geq \quad \rho\langle (Lu_{n+1} + \lambda|u_{n+1}|), g(u_{n+1}) - g(u)\rangle \\ &\quad -\rho\langle f, g(u_{n+1}) - g(u)\rangle \\ &\geq 0, \end{aligned} \tag{3.18}$$

where we have used (3.16).
Using the relation $2\langle a, b\rangle = \|a+b\|^2 - \|a\|^2 - \|b\|^2, \quad \forall a, b \in H$, the Cauchy inequality and from (3.18), we have

$$\|g(u) - g(u_{n+1})\|^2 \leq \|g(u) - g(u_n)\|^2 - \|g(u_n) - g(u_{n+1})\|^2,$$

which is the required (3.14).

□

Theorem 3.9. *Let $\bar{u} \in H$ be a solution of* (2.1) *and let u_{n+1} be the approximate solution obtained from Algorithm 3.4. Let L be a monotone operator with respect to $\lambda|.|$, and g^{-1} exist. If g is linear, then*

$$\lim_{n\to\infty} u_{n+1} = \bar{u}. \tag{3.19}$$

Proof. Let $\bar{u} \in H$ be a solution of (2.1). From (3.14), it follows that the sequence $\{\|g(\bar{u}) - g(u_n)\|\}$ is noncreasing and consequently the sequence $\{g(u_n)\}$ is bounded. Also, from (3.14), we have

$$\sum_{n=0}^{\infty} \|g(u_{n+1}) - g(u_n)\|^2 \leq \|g(u_0) - g(\bar{u})\|^2,$$

which implies that

$$\lim_{n\to\infty} \|g(u_{n+1}) - g(u_n)\| = 0 \Longrightarrow \quad \lim_{n\to\infty} \|u_{n+1} - u_n\| = 0, \tag{3.20}$$

since g is linear and g^{-1} exits.
Let $\hat{u}$ be a cluster point of $\{u_n\}$ and the subsequences $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converges to $\bar{u} \in H$. Replacing u_n by u_{n_j} in (3.13), taking the limit as $n_j \to \infty$ and using (3.20), we have

$$\langle L\hat{u} + \lambda|\hat{u}|, g(v) - g(\hat{u})\rangle = \langle f, g(v) - g(\hat{u})\rangle, \quad \forall v \in H,$$

which shows that $\hat{u} \in H$ satisfies (2.1) and

$$\|u_{n+1} - u_n\|^2 \leq \|u_n - \hat{u}\|^2.$$

From the above inequality, it follows that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and $\lim_{n\to\infty} u_n = \hat{u}$. □

We now use the auxiliary principle technique involving the Bregman function to suggest and analyze the proximal method for solving general absolute value equations (2.1). For the sake of completeness and to convey the main ideas of the Bregman distance functions, we recall the basic concepts and applications.
For the sake of completeness and conveyance of the readers, we recall the following concepts [2, 21].

Definition 3.2. *A set K is said to be a general convex set, if*

$$g(u)+t(g(v)-g(u))\in K, \quad \forall u,v\in K, \quad t\in[0,1].$$

Definition 3.3. [2] *A function E on the general convex set K is said to be a strongly general convex function, if there exists a constant $\nu>0$, such that*

$$\begin{aligned} E((u)+t(g(v)-g(u))) &\leq (1-t)E(g(u))+tE(g(v)) \\ &\quad -t(1-t)\nu\|g(v)-g(u)\|^2, \quad \forall u,v\in K, \quad t\in[0,1]. \end{aligned}$$

Lemma 3.10. *Let E be a differentiable general convex function on the general convex set K. Then E is a strongly general convex function, if and only if, there exists a constant $\nu>0$, such that*

$$E(g(v))-E(g(u))\geq\langle E'(g(u)),g(v)-g(u)\rangle+\nu\|g(v)-g(u)\|^2, \quad \forall u,v\in K.$$

We now consider the Bregman distance function, which is defined as

$$\begin{aligned} B(u,w) &= E(g(u))-E(g(w))-\langle E'(g(w)),g(u)-g(w)\rangle \\ &\geq \nu\|g(u)-g(w)\|^2, \quad \forall u,w\in K, \end{aligned} \tag{3.21}$$

where we have used the strongly general convexity of the function E with modulus ν. The function $B(u,w)$ is called the general Bregman distance function associated with strongly general convex functions.
For $g=I$, we obtain the original Bregman distance function associated with strongly convex functions

$$B(u,w)=E(u)-E(w))-\langle E'(w),u-w)\rangle\geq\nu\|u-w\|^2.$$

For the applications of Bregman functions, see [22, 30, 44] and the references

It is important to emphasize that various types of convex function E gives different Bregman distance. We give the following important examples of some practical important types of function E and their corresponding Bregman distance functions.

Example 3.2.

(1) If $E(v)=\|v\|^2$, then $B(v,u)=\|v-u\|^2$, which is the squared Euclidean distance (SE).

(2) If $E(v)=\sum_{i=1}^n a_i\log\Big(g(v_i)\Big)$, which is known as Shannon entropy, then its corresponding Bregman distance is given as

$$B(v,u)=\sum_{i=1}^n\Big(v_i\log\big(\frac{g(v_i)}{g(u_i)}\big)+g(u_i)-g(v_i)\Big),$$

This distance is called general KullbackLeibler distance (KL) and as become a very important tool in several areas of applied mathematics such as machine learning.

(3) If $E(v)=-\sum_{i=1}^n\log\big(g(v_i)\big)$, which is called Burg entropy, then its corresponding Bregman distance function is given as

$$B(v,u)=\sum_{i=1}^n\Big(\log\big(\frac{g(v_i)}{g(u_i)}\big)+\frac{g(v_i)}{g(u_i)}-1\Big).$$

This is called ItakuraSaito distance (IS),which is very important in information theory.
It is a challenging problem to explore the applications of Bregman distance functions for other types of nonconvex functions as preinvex,
k-convex functions and harmonic functions.

For a given $u \in H$, find a $w \in H$ satisfying the auxiliary general absolute value equation

$$\begin{aligned}\langle \rho(Lu + \lambda|u|) \quad &+ \quad E'(g(w)) - E'(g(u)), g(v) - g(w)\rangle \\ &= \quad \langle \rho f, g(v) - g(w)\rangle, \quad \forall v \in H, \end{aligned} \tag{3.22}$$

where $E'(u)$ is the differential of a strongly general convex function E.
Note that, if $w = u$, then w is a solution of (2.1). Thus, we can suggest the following iterative method for solving (2.1).

Algorithm 3.11. *For a given $u_0 \in H$, calculate the approximate solution by the iterative scheme*

$$\begin{aligned}\langle \rho(Lu_n + \lambda|u_n|) \quad &+ \quad E'(g(u_{n+1}) - E'(g(u_n)), g(v) - g(u_{n+1})\rangle \\ &= \quad \langle \rho f, g(v) - g(u_{n+1})\rangle, \quad \forall v \in H, \end{aligned} \tag{3.23}$$

which is known as the proximal point method.

We again use the auxiliary principle technique involving the Bregman function to suggest and analyze the proximal implicit method for solving general absolute value equation (2.1).

For a given $u \in H$, find $w \in H$ satisfying the auxiliary general absolute value equations

$$\begin{aligned}\langle \rho(Lw + \lambda|w|) \quad &+ \quad E'(g(w)) - E'(g(u)), g(v) - g(w)\rangle \\ &= \quad \langle \rho f, g(v) - g(w)\rangle, \quad \forall v \in H, \end{aligned} \tag{3.24}$$

where $E'(u)$ is the differential of a strongly general convex function E.

It is clear that, if $w = u$, then w is a solution of (2.1). Thus, we can suggest the following iterative method for solving (2.1).

Algorithm 3.12. *For a given $u_0 \in H$, calculate the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}\langle \rho(Lu_{n+1} + \lambda|u_{n+1}|) \quad &+ \quad E'(g(u_{n+1})) - E'(g(u_n)), g(v) - g(u_{n+1})\rangle \\ &= \quad \langle \rho f, g(v) - g(u_{n+1})\rangle, \forall v \in H, \end{aligned} \tag{3.25}$$

which is known as the proximal implicit point method.

Remark. *One can consider the convergence analysis of Algorithm 3.11 and Algorithm 3.12 using the technique of Noor* [21] *and Noor et al.* [25, 30, 32]. *We would like to emphasize that for appropriate choice of the operators L, g and the parameter λ, one can suggest and analyze several new iterative methods for solving general* *absolute value equations and related problems. The implementation and comparison with other techniques need further efforts.*

Conclusion

In this paper, we have considered a new class of general absolute value equations. It is shown that the third order boundary value problems can be studied in the framework of general absolute value equations. We have used the auxiliary principle technique to study the existence of the unique solution of the general absolute value equations. Some new iterative methods are suggested for solving the absolute value equations using the auxiliary principle technique. The convergence analysis of these iterative methods is investigated under suitable conditions. This is a new approach for solving the general absolute value equations. We would like to emphasize that the results obtained and discussed in this paper may motivate a number of novel applications and extensions in these areas.

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References

[1] L. Bers, F. John and M. Schechter, *Partial differential equations,* Academic Press, New York, 1966.

[2] G. Cristescu and L. Lupsa, *Non-Connected Convexities and Applications,* Kluwer Academic Publisher, Dordrechet,(2002)

[3] G. Cristescu and M. Guaianu,*Shape properties of Noors convex sets, Proceed. of the Twelfth Sym- posium of Mathematics and its Applications,* Timisoara, (2009), 1-13.

[4] W. Fechner, *Functional inequalities motivated by the Lax-Milgram Lemma,* J. Math. Anal. Appl. **402**(2013), 411-414.

[5] V. M. Filippov, *Variational Principles for Nonpotential Operators,* Amer. Math. soc. Providence, Rode Island, USA, 1989.

[6] M. Fukushima, *Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems,* Math. Program. **53** (1992), 99-110.

[7] R. Glowinski, J. L. Lions and R. Trmolires, *Numerical Analysis of Variational Inequalities,* Amsterdam: North-Holland; 1981.

[8] S. L. Hu and Z. H. Huang, *A note on absolute value equations,* Optim. Lett. *4(3)*(2010), 417-424

[9] Y. F. Ke and C. F. Ma, *SOR-like itration method for solving absolute value equations,* Appl. Math. Comput. **311** (2017), 195-202.

[10] P. D. Lax and A. N. Milgram, *Parabolic equations, Annals,* Math. Study No. 33, Princeton, N.J., (1954), 167-190.

[11] J. L. Lions and G. Stampacchia, *Variational inequalities,* Commun. Pure Appl. Math. **20**(1967), 493-512.

[12] O. L. Mangasarian, *A generalized Newton method for absolute value equations,* Optim. Lett. **3**(2009), 101-108.

[13] O. L. Mangasarian, *Absoluet value equation solution via dual complementarity,* Optim. Lett. **7**(2013), 625-630.

[14] O. L. Mangasarian, *A hybrid algorithm for solving the absolute value equation,* Optim. Lett. **9** (2015), 1469-1474.

[15] O. L. Mangasarian and R. R. Meyer,*Absolute value equations,* Lin. Alg. Appl. **419(2)**(2006), 359-367.

[16] X. -H. Miao, J.-T. Yang and S. -L. Hu, *A generalized Newton method for absolute value equations associated with circular cones,* Appl. Math. Comput. *269* (2015), 155-168.

[17] H. Moosaei, S. Ketabchi, M.A. Noor, J. Iqbal and V. Hooshyarbakhsh, *Some techniques for solving absolute value equations,* Appl. Math. Comput. **268** (2015), 696-705.

[18] M. A. Noor, *Reisz-Frechet Theorem and Monotonicity,* MS Thesis, Queens University, Kingston, Ontarior, Canad, 1971.

[19] M. A. Noor, *On Variational Inequalities,* PhD Thesis, Brunel University, London, U.K. (1975).

[20] M. A. Noor, *New approximation schemes for general variational inequalities,* J. Math. Ana. Appl. **251**(2000), 217-229.

[21] M. A. Noor, *Some developments in general variational inequalities,* Appl. Math. Comput. **152**(2004), 199-277.

[22] K. I. Noor and M. A. Noor, *A generalization of the Lax-Milgram lemma,* Canad. Math. Bull. **23 (2)**(1980), 179-184.

[23] M. A. Noor and K. I. Noor, *From representaion theorems to variational inequalities,* in: Computational Mathematics and Variational Analysis (Edits: N. J. Daras, T. M. Rassias), Springer Optimization and Its Applications, 159(2020), 261-277. https:doi.org/10.1007/978-3-030-44625-3-1526

[24] M. A. Noor and K. I. Noor, *Absolute value variational inclusions,* Earthl. J. Math. Sci, **8(1)**(2022), 121-153: DOI: https://doi.org/10.34198/ejms.8122.121153

[25] M. A. Noor and K. I. Noor, *Iterative schemes for solving new system of general equations,* U.P.B. Sci. Bull., Series A. **84(1)**(2022), 59-70.

[26] M. A. Noor, K. I. Noor and S. Batool, *On generalized absolute value equations,* U.P.B. Sci. Bull., Series A. **80(4)**(2018),63-70.

[27] M. A. Noor, J. Iqbal, S. Khattri and E. Al-Said, *A new iterative method for solving absolute value equations,* Int. J. Phys. Sci. **6**(2011), 1793-1797.

[28] M. A. Noor, J. Iqbal, K.I. Noor and E. Al-Said, *On an iterative method for solving absolute value equations,* Optim. Lett. **6** (2012), 1027-1033.

[29] M. A. Noor, J. Iqbal, K.I. Noor and E. Al-Said, *Generalized AOR method for solving absolute complementarity problems,* J. Appl. Math. **2012**(2012), doi:10.1155/ 2012/743-861.

[30] M. A. Noor, K. I. Noor and M. Th. Rassias, *New trends in general avriational inequalities,* Acat. Appl. Math. **170(1)**(2020), 981-1046.

[31] M. A. Noor, K. I. Noor, A. Hamdi and E. H. El-Shemas, *On difference of two monotone operators,* Optim. Letters, **3**(2009), 329-335.

[32] M. A. Noor, K. I. Noor and Th. M. Rassias, *Some aspects of variational inequalities,* J. Comput. Appl. Math. **47** (1933), 285-312.

[33] Th. M. Rassias, M. A. Noor and K. I. Noor, *Auxiliary principle technique for the general Lax-Milgram Lemma,* J. Nonl. Funt. Analysis, **2018**(2018), Article ID 34,1-8

[34] J. Rohn, *A theorem of the alternative for the equation* $Ax + B|x| = b$, *Linear Mult. Algebra,* **52(6)**(2004), 421-426.

[35] J. Rohn, V. Hoosyarkhsh and R. Farhadsefat, *An iterative method for solving absolute value equations and sufficient conditins for unique solvability,* Optim. Lett. **8(1)**(2014), 35-44.

[36] D. K. Salkuyeh, *The Picard-HSS iteration method for absolute value equations,* Optim. Lett. **8**(2014), 2191-2202.

[37] E. Tonti, *Variational formulation for every nonlinear problem,* Intern. J. Eng. Sci. **22(11-12)**(1984), 1343- 1371.

[38] F. Wang, Z. Yu and C. Gao, *A smoothing neural network algorithm for absolute value equations,* Engeering, **7**(2015), 567-576.

[39] H. J. Wang, D. X. Cao, H. Liu and L. Qiu,*Numerical validation for sytems of absolute value equations,* Calcol, **54**(2017), 669-683.

[40] S. -L. Wu and P. Guo, *On the unique solvability of the absolute value equation,* J. Optim. Theory Appl. **169**(2016), 705-712

[41] L. -L. Zhang, *Two-step modulus based matrix splitting iteration for linear complementarity problems,* Numer. Algorithms, **57** (2011), 83-99.

[42] J. -J. Zhang, *The relaxed nonlinear PHSS-like iteration method for absolute value equations,* Appl. Math. Comput. **265**(2015), 266-274.

[43] N. Zheng, J. -F. Yin, *Accelerated modulus-based matrix splitting iteration methods for linear complementarity problem,* Numer. Algorithms, **64**(2013), 245-262.

[44] C. Zhang, Q. -J. Wei, *Global and finite convergence of a generalized Newton method for absolute value equations,* J. Optim. Theory Appl. **143**(2009), 391-403.

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