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TWO-WEIGHTED INEQUALITIES FOR GENERALIZED FRACTIONAL INTEGRAL OPERATOR AND ITS COMMUTATORS IN GENERALIZED WEIGHTED MORREY SPACES

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Abstract. In this paper we find the conditions for the boundedness of generalized fractional integrals I_ρ and its commutators from the generalized weighted Morrey spaces $\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)$ to the generalized weighted Morrey spaces $\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)$, where $\rho : (0,\infty) \to (0,\infty)$ is a measurable function, $(\omega_1,\omega_2) \in A_{p,q}(\mathbb{R}^n)$ and $b \in BMO(\mathbb{R}^n)$.

1. Introduction

V. S. Guliyev gave a concept of generalized weighted Morrey space $\mathcal{M}^{p,\varphi}_\omega$ in 2012 ([16]), defined by the norm

$$\|f\|_{\mathcal{M}^{p,\varphi}_\omega} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \|\omega\|^{-1}_{L_p(B(x,r))} \|f\|_{L_{p,\omega}(B(x,r))},$$

where $1 \le p < \infty$, φ is a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, ω is a non-negative measurable function on $\mathbb{R}^n$, $B(x,r)$ is an open ball in $\mathbb{R}^n$ and $f \in L^{loc}_{p,\omega}(\mathbb{R}^n)$. Here, $L_{p,\omega}(B(x,r))$ denotes the weighted L_p -space of measurable functions f for which

$$\|f\|_{L_{p,\omega}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{L_{p,\omega}(\mathbb{R}^n)} = \left(\int_{B(x,r)} |f(y)|^p \omega(y) dy \right)^{\frac{1}{p}}.$$

if $\omega(x) = \chi_{B(x,r)}$, then $\mathcal{M}^{p,\varphi}_\omega(\mathbb{R}^n) = \mathcal{M}^{p,\varphi}(\mathbb{R}^n)$ is the generalized Morrey space and if $\varphi(x,r) = r^{\frac{n-\lambda}{p}}$, then $\mathcal{M}^{p,\varphi}_\omega(\mathbb{R}^n) = L_{p,\lambda}(\omega)$ is the weighted Morrey space. Therefore, $\mathcal{M}^{p,\varphi}_\omega$ spaces could be viewed as extension of both generalized Morrey spaces and weighted Morrey spaces. In [16], the boundedness of sublinear

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operators, their higher order commutators generated by Calderón-Zygmund operators and Riesz potentials in the spaces $\mathcal{M}_{\omega}^{p,\varphi}$ were obtained (see also [20, 27]).

For a measurable function $\rho : (0,\infty) \to (0,\infty)$ the generalized fractional maximal operator M_ρ and the generalized fractional integral operator I_ρ are defined by

$$M_\rho f(x) = \sup_{t>0} \frac{\rho(t)}{t^n} \int_{B(x,t)} |f(y)|dy, \quad I_\rho f(x) = \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} f(y)dy$$

for any suitable function f on $\mathbb{R}^n$. If $\rho(t) = t^\alpha$, then $M^\alpha = M_{t^\alpha}$ is the fractional maximal operator and $I^\alpha = I_{t^\alpha}$ is the Riesz potential defined as

$$M^\alpha f(x) = \sup_{r>0} |B(x,r)|^{-1+\alpha/n} \int_{B(x,r)} |f(y)|dy, \; 0 \le \alpha < n,$$

and

$$I^\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\alpha}}, \qquad 0 < \alpha < n,$$

respectively, where $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$.

The generalized fractional integral operator I_ρ was initially investigated in [13]. Nowadays many authors have been culminating important observations about the operators M_ρ and I_ρ especially in connection with Morrey spaces. It is not possible to mention all the literature on this subject, but the authors who contributed the most in this field can be summarized as [7, 8, 9, 21, 19, 27, 28, 33, 41, 42, 43].

In this paper we investigate two-weight inequalities for generalized fractional integral operator and its commutators in generalized weighted Morrey spaces. Two-weight norm inequalities for the operators of harmonic analysis on various function spaces were widely studied (see, for example [5, 14, 26, 31]). The weighted norm inequalities with different types of weights on Morrey spaces were also studied (see, for example [22, 35, 37]). The two-weight norm inequality for the Hardy-Littlewood maximal function on Morrey spaces was obtained in [44]. Two-weight norm inequalities on weighted Morrey spaces for fractional maximal operators and fractional integral operators were obtained in [36]. Two-weight norm inequalities on generalized weighted Morrey spaces for maximal, Calderón-Zygmund operators and their commutators were obtained in [1]. Two-weight norm inequalities on generalized weighted Morrey spaces for Riesz potential and its commutators were obtained in [2].

Throughout the paper we use the letter C for a positive constant, independent of appropriate parameters and not necessary the same at each occurrence. If $p \in [1,\infty]$, the conjugate number p' is defined by $pp' = p + p'$. $\mathfrak{M}(\mathbb{R}_+)$, $\mathfrak{M}^+(\mathbb{R}_+)$ and $\mathfrak{M}^+(\mathbb{R}_+;\uparrow)$ stand for the set of Lebesgue-measurable functions on $\mathbb{R}_+$, and its subspaces of non-negative and non-negative non-decreasing functions, respectively.

2. Preliminaries

Although the A_p class is well-known, we offer the definition of A_p weight functions.

Definition 2.1. *The weight function* ω *belongs to the class* $A_p(\mathbb{R}^n)$ *for* $1 \leq p < \infty$ *, if the following statement*

$$\sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \left(\int\limits_{B(x,r)} \omega^p(y)dy \right)^{\frac{1}{p}} \left(\int\limits_{B(x,r)} \omega^{-p'}(y)dy \right)^{\frac{1}{p'}}$$

is finite and ω *belongs to* $A_1(\mathbb{R}^n)$ *, if there exists a positive constant* C *such that for any* $x \in \mathbb{R}^n$ *and* $r > 0$

$$|B(x,r)|^{-1} \int\limits_{B(x,r)} \omega(y)dy \leq C \operatorname*{ess\,sup}_{y\in B(x,r)} \frac{1}{\omega(y)}.$$

Definition 2.2. *The weight function* (ω_1, ω_2) *belongs to the class* $\widetilde{A}_p(\mathbb{R}^n)$ *for* $1 \leq p < \infty$ *, if the following statement*

$$\sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \left(\int\limits_{B(x,r)} \omega_2^p(y)dy \right)^{\frac{1}{p}} \left(\int\limits_{B(x,r)} \omega_1^{-p'}(y)dy \right)^{\frac{1}{p'}}$$

is finite.

Definition 2.3. *The weight function* (ω_1, ω_2) *belongs to the class* $A_{p,q}(\mathbb{R}^n)$ *for* $1 \leq p, q < \infty$ *, if the following statement*

$$\sup_{x\in\mathbb{R}^n, r>0} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}-1} \left(\int\limits_{B(x,r)} \omega_2^q(y)dy \right)^{\frac{1}{q}} \left(\int\limits_{B(x,r)} \omega_1^{-p'}(y)dy \right)^{\frac{1}{p'}}$$

is finite.

The following theorem was proved in [32].

Theorem 2.4. *Let* $1 \leq p < \infty$ *, then*
1) $M : L_{p,\varphi}(\mathbb{R}^n) \to L_{p,\varphi}(\mathbb{R}^n)$ *if and only if* $\varphi \in A_p(\mathbb{R}^n)$ *,*
2) $M : L_{1,\varphi}(\mathbb{R}^n) \to WL_{1,\varphi}(\mathbb{R}^n)$ *if and only if* $\varphi \in A_1(\mathbb{R}^n)$ *.*

Let $M^\sharp$ be the sharp maximal function defined by

$$M^\sharp f(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y) - f_{B(x,r)}|dy,$$

where $f_{B(x,r)}(x) = |B(x,r)|^{-1} \int_{B(x,r)} f(y)dy$.

Definition 2.5. *We define the* $BMO(\mathbb{R}^n)$ *space as the set of all locally integrable functions* f *such that*

$$\|f\|_{BMO} = \sup_{x\in\mathbb{R}^n,\, r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y) - f_{B(x,r)}|dy < \infty$$

or

$$\|f\|_{BMO} = \inf_C \sup_{x\in\mathbb{R}^n,\, r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y) - C|dy < \infty.$$

Definition 2.6. *We define the* $BMO_{p,\omega}(\mathbb{R}^n)$ $(1 \le p < \infty)$ *space as the set of all locally integrable functions* f *such that*

$$\|f\|_{BMO_{p,\omega}} = \sup_{x\in\mathbb{R}^n,\, r>0} \frac{\|(f(\cdot) - f_{B(x,r)})\chi_{B(x,r)}\|_{L_{p,\omega}(\mathbb{R}^n)}}{\|\omega\|_{L_p(B(x,r))}}$$

or

$$\|f\|_{BMO_{p,\omega}} = \sup_{x\in\mathbb{R}^n,\, r>0} \frac{1}{|B(x,r)|} \|(f(\cdot)-f_{B(x,r)})\chi_{B(x,r)}\|_{L_{p,\omega}(\mathbb{R}^n)} \|\omega^{-1}\|_{L_{p'}(B(x,r))} < \infty.$$

Theorem 2.7. [29] *Let* $1 \le p < \infty$ *and* ω *be a Lebesgue measurable function. If* $\omega \in A_p(\mathbb{R}^n)$ *, then the norms* $\|\cdot\|_{BMO_{p,\omega}}$ *and* $\|\cdot\|_{BMO}$ *are mutually equivalent.*

Before proving the main theorems, we need the following lemma.

Lemma 2.8. [24] *Let* $b \in BMO(\mathbb{R}^n)$ *. Then there is a constant* $C > 0$ *such that*

$$\left|b_{B(x,r)} - b_{B(x,t)}\right| \le C\|b\|_{BMO} \ln\frac{t}{r} \quad \text{for} \quad 0 < 2r < t,$$

where C *is independent of* b *,* x *,* r*, and* t *.*

Let $L_{\infty,v}(\mathbb{R}_+)$ be the weighted L_∞ -space with the norm

$$\|g\|_{L_{\infty,v}(\mathbb{R}_+)} = \operatorname*{ess\,sup}_{t>0} v(t)g(t).$$

We denote

$$\mathbb{A} = \left\{\varphi \in \mathfrak{M}^+(\mathbb{R}_+;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0\right\}.$$

Let u be a continuous and non-negative function on $\mathbb{R}_+$. We define the supremal operator $\overline{S}_u$ by

$$(\overline{S}_u g)(t) := \|u\, g\|_{L_\infty(0,t)},\ t \in (0,\infty).$$

The following theorem was proved in [3].

Theorem 2.9. [3] *Suppose that* v_1 *and* v_2 *are nonnegative measurable functions such that* $0 < \|v_1\|_{L_\infty(0,t)} < \infty$ *for every* $t > 0$ *. Let* u *be a continuous nonnegative function on* $\mathbb{R}$ *. Then the operator* $\overline{S}_u$ *is bounded from* $L^\infty_{v_1}(\mathbb{R}_+)$ *to* $L^\infty_{v_2}(\mathbb{R}_+)$ *on the cone* $\mathbb{A}$ *if and only if*

$$\left\| v_2 \overline{S}_u \left(\|v_1\|^{-1}_{L_\infty(0,\cdot)} \right) \right\|_{L_\infty(\mathbb{R}_+)} < \infty.$$

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w g(t) := \int_0^t g(s)w(s)ds,\ \ H^*_w g(t) := \int_t^\infty g(s)w(s)ds, \quad 0 < t < \infty,$$

where w is a weight.

The following theorem was proved in [17].

Theorem 2.10. [17] *Let* v_1 *,* v_2 *and* w *be weights on* $(0,\infty)$ *and* $v_1(t)$ *be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H^*_w g(t) \le C \sup_{t>0} v_1(t) g(t)$$

holds for some $C > 0$ *for all non-negative and non-decreasing* g *on* $(0,\infty)$ *if and only if*

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Theorem 2.11. [17, 18] *Let* v_1 , v_2 *and* w *be weights on* $(0,\infty)$ *and* $v_1(t)$ *be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H_w g(t) \le C \sup_{t>0} v_1(t) g(t) \tag{2.1}$$

holds for some $C > 0$ *for all non-negative and non-decreasing* g *on* $(0,\infty)$ *if and only if*

$$B := \sup_{t>0} v_2(t) \int_0^t \frac{w(s)ds}{\sup_{0<\tau<s} v_1(\tau)} < \infty.$$

Moreover, the value $C = B$ *is the best constant for* (2.1).

3. Two-weight inequalities for generalized fractional integrals in generalized weighted Morrey spaces

In this section we prove the two-weight inequalities for generalized fractional integral in generalized weighted Morrey spaces $\mathcal{M}^{p,\varphi}_{\omega}$.
We assume that

$$\int_1^\infty \frac{\rho(t)}{t^n} \frac{dt}{t} < \infty, \tag{3.1}$$

so that the generalized fractional integral operator I_ρ is well defined, at least for characteristic functions $1/|x|^{2n}$ of complementary balls:

$$f(x) = \frac{\chi_{\mathbb{R}^n \setminus B(0,1)}(x)}{|x|^{2n}}.$$

Furthermore, we also assume that ρ satisfies the growth condition: there exist constants $C > 0$ and $0 < 2k_1 < k_2 < \infty$ such that

$$\sup_{r/2<t\le 3r/2} \frac{\rho(t)}{t^n} \le C_1 \int_{k_1 r}^{k_2 r} \frac{\rho(t)}{t^n} \frac{dt}{t}, \ r > 0. \tag{3.2}$$

This condition is weaker than the usual doubling condition for the function $\frac{\rho(t)}{t^n}$: there exists a constant $C > 0$ such that

$$\frac{1}{C_1} \frac{\rho(t)}{t^n} \le \frac{\rho(r)}{r^n} \le C_1 \frac{\rho(t)}{t^n}, \ r > 0. \tag{3.3}$$

whenever r and t satisfy $r,t > 0$ and $\frac{1}{2}t \le r \le 2t$. In the sequel for the generalized fractional integral operator I_ρ we always assume that ρ satisfies the condition (3.2).

The following theorem was proved in [11].

Theorem 3.1. *Let* $1 < p < q < \infty$. *Then the operator* I_ρ *is bounded from* $L_p(\mathbb{R}^n)$ *to* $L_q(\mathbb{R}^n)$ *if and only if there exists* $C > 0$ *such that for all*

$$\rho(r) \le C r^{\frac{n}{p} - \frac{n}{q}}. \tag{3.4}$$

Theorem 3.2. *Let* $1 < p < q < \infty$, *the function* ρ *satisfies the conditions*(3.1)-(3.3), *if the condition* (3.4) *is fulfill and* $(\omega_1, \omega_2) \in A_{p,q}(\mathbb{R}^n)$. *Then there exists a constant* $C > 0$ *such that for an arbitrary* $f \in L_{p,\omega_1}(B(x,t))$ *the inequality*

$$\|I_\rho f\|_{L_{q,\omega_2}(B(x,t))} \leq C \|\omega_2\|_{L_q(B(x,t))} \int_t^\infty \frac{\|f\|_{L_{p,\omega}(B(x,s))}}{\|\omega_2\|_{L_q(B(x,s))}} \frac{ds}{s} \tag{3.5}$$

is hold.

Proof. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{B(x,2t)}(y), \quad f_2(y) = f(y)\chi_{\mathbb{R}^n \setminus B(x,2t)}(y), \quad t > 0, \tag{3.6}$$

and have

$$I_\rho f(x) = I_\rho f_1(x) + I_\rho f_2(x).$$

First we estimate $|I_\rho f_1(x)|$. By using Hölder's inequality we have

$$\begin{aligned}
|I_\rho f_1(x)| &= \int\limits_{B(x,t)} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)| dy \\
&\leq \sum_{j=-\infty}^{-1} \frac{\rho(2^j t)}{(2^j t)^n} \int\limits_{B(x,2^{j+1}t)\setminus B(x,2^j t)} |f(y)| dy \\
&\leq \sum_{j=-\infty}^{-1} \frac{\rho(2^j t)}{(2^j t)^n} \|f\|_{L_{p,\omega_1}(B(x,2^{j+1}t)\setminus B(x,2^j t))} \|\omega_1^{-1}\|_{L_{p'}(B(x,2^{j+1}t)\setminus B(x,2^j t))} \\
&\leq C \sum_{j=-\infty}^{-1} \|f\|_{L_{p,\omega_1}(B(x,2^{j+1}t)\setminus B(x,2^j t))} \|\omega_2\|^{-1}_{L_q(B(x,2^{j+1}t)\setminus B(x,2^j t))} \\
&\leq C \|f\|_{L_{p,\omega_1}(B(x,t))} \|\omega_2\|^{-1}_{L_q(B(x,t))}.
\end{aligned} \tag{3.7}$$

Then

$$\|I_\rho f_1\|_{L_{q,\omega_2}(B(x,t))} \leq C \|f\|_{L_{p,\omega_1}(B(x,2t))},$$

where the constant C is independent of f .

Taking into account that, we get

$$\|I_\rho f_1\|_{L_{q,\omega_2}(B(x,t))} \leq C \|\omega_2\|_{L_q(B(x,t))} \int_t^\infty \frac{\|f\|_{L_{p,\omega}(B(x,s))}}{\|\omega_2\|_{L_q(B(x,s))}} \frac{ds}{s}. \tag{3.8}$$

When $|x-z| \leq t$, $|z-y| \geq 2t$, we have $\frac{1}{2}|z-y| \leq |x-y| \leq \frac{3}{2}|z-y|$, and therefore

$$|I_\rho f_2(z)| \leq \int_{\mathbb{R}^n \setminus B(x,2t)} \frac{\rho(|z-y|)}{|z-y|^n} |f(y)| dy \leq C \int_{\mathbb{R}^n \setminus B(x,2t)} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)| dy.$$

We choose $\beta > \frac{n}{q}$ and obtain

$$\int_{\mathbb{R}^n \setminus B(x,2t)} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)| dy$$

$$= \beta \int_{\mathbb{R}^n \setminus B(x,2t)} |f(y)| \left(\int_{|x-y|}^\infty \frac{\rho(s)}{s^{n+1}} ds \right) dy$$

$$= \beta \int_{2t}^{\infty} \frac{\rho(s)}{s^n} \left(\int_{\{y\in\mathbb{R}^n : 2t\le|x-y|\le s\}} |f(y)|dy \right) \frac{ds}{s}$$

$$\le C \int_{2}^{\infty} \frac{\rho(s)}{s^n} \|\chi_{B(x,s)}\omega_1^{-1}\|_{L_{p'}(\mathbb{R}^n)} \|f\|_{L_{p,\omega_1}(B(x,s))} \frac{ds}{s}$$

$$\le C \int_{t}^{\infty} \frac{\rho(s)}{s^n} \|f\|_{L_{p,\omega_1}(B(x,s))} \|\omega_1^{-1}\|_{L_{p'}(B(x,s))} \frac{ds}{s}.$$

Hence

$$\|I_\rho f_2\|_{L_{q,\omega_2}(B(x,t))} \le C\|\omega_2\|_{L_q(B(x,t))} \int_t^{\infty} \frac{\rho(s)}{s^{\frac{n}{p}-\frac{n}{q}}} \|f\|_{L_{p,\omega_1}(B(x,s))} \|\omega_2\|^{-1}_{L_q(B(x,s))} \frac{ds}{s}.$$

Therefore we get

$$\|I_\rho f_2\|_{L_{q,\omega_2}(B(x,t))} \le C\|\omega_2\|_{L_q(B(x,t))} \int_t^{\infty} \frac{\|f\|_{L_{p,\omega}(B(x,s))}}{\|\omega_2\|_{L_q(B(x,s))}} \frac{ds}{s} \tag{3.9}$$

which together with (3.8) yields (3.5).

□

Theorem 3.3. *Let* $1 < p < q < \infty$, *the function* ρ *satisfies the conditions* (3.1)-(3.3), *if the condition* (3.4) *is fulfill and* $(\omega_1, \omega_2) \in A_{p,q}(\mathbb{R}^n)$. *Let the functions* $\varphi_1(x,r)$ *and* $\varphi_2(x,r)$ *fulfill the condition*

$$\int_t^{\infty} \frac{\operatorname{ess\,inf}_{s<r<\infty} \varphi_1(x,r)\|\omega_1\|_{L^{p(\cdot)}(B(x,r))}}{\|\omega_2\|_{L^{q(\cdot)}(B(x,s))}} \frac{ds}{s} \le C\varphi_2(x,t). \tag{3.10}$$

Then the operator I_ρ *is bounded from* $\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)$ *to* $\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)$.

Proof. Let $f \in \mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)$. From the definition of the norm of generalized weighted Morrey spaces we write

$$\|I_\rho f\|_{\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, t>0} \frac{1}{\varphi_2(x,t)\|\omega_2\|_{L_q(B(x,t))}} \|I_\rho f \chi_{B(x,t)}\|_{L_{q,\omega_2}(\mathbb{R}^n)}. \tag{3.11}$$

We estimate $\|I_\rho f \chi_{B(x,t)}\|_{L_{q,\omega_2}(\mathbb{R}^n)}$ in (3.11) by means of Theorems 3.2, 2.11 and obtain

$$\|I_\rho f\|_{\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)}$$

$$\le C \sup_{x\in\mathbb{R}^n,\, t>0} \frac{\|\omega_2\|_{L_q(B(x,t))}}{\varphi_2(x,t)\|\omega_2\|_{L_q(B(x,t))}} \int_t^{\infty} \frac{\|f\|_{L_{p,\omega}(B(x,s))}}{\|\omega_2\|_{L_q(B(x,s))}} \frac{ds}{s}$$

$$\le C \sup_{x\in\mathbb{R}^n,\, t>0} \frac{1}{\varphi_1(x,t)\|\omega_1\|_{L_p(B(x,t))}} \|f\|_{L_{p,\omega_1}(B(x,t))} = C\|f\|_{\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)}.$$

It remains to make use of condition (3.10).

□

Corollary 3.4. *Let* $1 < p < q < \infty$, *the function* ρ *satisfies the conditions* (3.1)-(3.3), *if the condition* (3.4) *is fulfill and* $(\omega_1, \omega_2) \in A_{p,q}(\mathbb{R}^n)$. *Then the operator* I_ρ *is bounded from* $L_{p,\omega_1}(\mathbb{R}^n)$ *to* $L_{q,\omega_2}(\mathbb{R}^n)$.

From the inequality $M_\rho f(x) \le C(I_\rho)|f|(x)$, we get the following corollary.

Corollary 3.5. *Let* $1 < p < q < \infty$, *the function* ρ *satisfies the conditions* (3.1)-(3.3), *if the condition* (3.4) *is fulfill and* $(\omega_1, \omega_2) \in A_{p,q}(\mathbb{R}^n)$. *Then the operator* M_ρ *is bounded from* $L_{p,\omega_1}(\mathbb{R}^n)$ *to* $L_{q,\omega_2}(\mathbb{R}^n)$.

4. Two-weight inequalities for commutators of generalized fractional integrals in generalized weighted Morrey spaces

In this section we prove the two-weight inequalities for the commutators of generalized fractional integrals in generalized weighted Morrey spaces $\mathcal{M}_{\omega}^{p,\varphi}$.

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. In this section we consider commutators of generalized fractional integral operator defined by the following equality

$$[b, I_\rho]f(x) = \int\limits_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} (b(x) - b(y))\, f(y)dy.$$

Given a measurable function b the operator $|b, I_\rho|$ is defined by

$$|b, I_\rho|f(x) = \int\limits_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n}\, |b(x) - b(y)|\, |f(y)|dy.$$

The maximal commutator is defined by

$$M_b(f)(x) := \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |b(x) - b(y)||f(y)|dy$$

for all $x \in \mathbb{R}^n$.

Theorem 4.1. [1] *Let* $1 < p < \infty$, $(\omega_1, \omega_2) \in \widetilde{A}_p(\mathbb{R}^n)$ *and the function* $\varphi_1(x,r)$ *and* $\varphi_2(x,r)$ *satisfy the condition*

$$\sup_{t>r} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) \|\omega_1\|_{L_p(B(x,s))}}{\|\omega_2\|_{L_p(B(x,t))}} \le C\varphi_2(x,r), \tag{4.1}$$

where C *does not depend on* x *and* t .

Then the operator M *is bounded from the space* $\mathcal{M}_{\omega_1}^{p,\varphi_1}(\mathbb{R}^n)$ *to the space* $\mathcal{M}_{\omega_2}^{p,\varphi_2}(\mathbb{R}^n)$.

Theorem 4.2. [1] *Let* $1 < p < \infty$, $b \in BMO(\mathbb{R}^n)$ *and* $(\omega_1, \omega_2) \in \widetilde{A}_p(\mathbb{R}^n)$, $\omega_1 \in A_p(\mathbb{R}^n)$, *then the operator* M_b *is bounded from* $L_{p,\omega_1}(\mathbb{R}^n)$ *to* $L_{p,\omega_2}(\mathbb{R}^n)$.

Theorem 4.3. [1] *Let* $1 < p < \infty$, $b \in BMO(\mathbb{R}^n)$ *and* $(\omega_1, \omega_2) \in \widetilde{A}_p(\mathbb{R}^n)$, $\omega_1, \omega_2 \in A_p(\mathbb{R}^n)$. *If the functions* $\varphi_1(x,r)$ *and* $\varphi_2(x,r)$ *satisfy the condition*

$$\sup_{t>r} \left(1 + \ln\frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) \|\omega_1\|_{L_p(B(x,s))}}{\|\omega_2\|_{L_p(B(x,t))}} \le C\varphi_2(x,r), \tag{4.2}$$

where C *does not depend on* x *and* t .

Then the operator M_b *is bounded from the space* $\mathcal{M}_{\omega_1}^{p,\varphi_1}(\mathbb{R}^n)$ *to the space* $\mathcal{M}_{\omega_2}^{p,\varphi_2}(\mathbb{R}^n)$.

Theorem 4.4. *Let* $1 < p < q < \infty$, $b \in BMO(\mathbb{R}^n)$, *the function* ρ *satisfies the conditions* (3.1)-(3.3), *if the condition* (3.4) *is fulfill and* $(\omega_1, \omega_2) \in A_{p,q}(\mathbb{R}^n)$, $\omega_1 \in A_p(\mathbb{R}^n)$, $\omega_2 \in A_q(\mathbb{R}^n)$.

Then

$$\| |b, I_\rho| f\|_{L_{q,\omega_2}(B(x,t))} \le C\, \|b\|_{BMO}\, \|\omega_2\|_{L_q(B(x,t))}$$

$$\times \int_t^{\infty} \Big(1+\ln \frac{s}{t}\Big)\, \|f\|_{L_{p,\omega_1}(B(x,s))} \|\omega_2\|^{-1}_{L_q(B(x,s))} \frac{ds}{s}, \tag{4.3}$$

where $t>0$, C *does not depend on* f , x *and* t.

Proof. We represent f as (3.6) and have

$$|b, I_\rho| f(x) \le |b, I_\rho| f_1(x) + |b, I_\rho| f_2(x).$$

First we estimate $|b, I_\rho| f_1(x)$. By using Hölder's inequality we have

$$\begin{aligned} |b, I_\rho| f_1(x) &= \int\limits_{B(x,t)} \frac{\rho(|x-y|)}{|x-y|^n}\, |b(y)-b(x)|\, |f(y)| dy \\ &\le |b(x)-b_{B(x,t)}| \int\limits_{B(x,t)} \frac{\rho(|x-y|)}{|x-y|^n}\, |f(y)| dy \\ &+ \int\limits_{B(x,t)} \frac{\rho(|x-y|)}{|x-y|^n}\, |b(y)-b_{B(x,t)}|\, |f(y)| dy = F_1(x)+F_2(x). \end{aligned}$$

By using inequality (3.7) we have

$$\begin{aligned} F_1(x) &= |b(x)-b_{B(x,t)}| \int\limits_{B(x,t)} \frac{\rho(|x-y|)}{|x-y|^n}\, |f(y)| dy \le C M_b \chi_{B(x,t)}(x) |I_\rho f_1(x)| \\ &\le C M_b \chi_{B(x,t)}(x) \|f\|_{L_{p,\omega_1}(B(x,t))} \|\omega_2\|^{-1}_{L_q(B(x,t))}. \end{aligned} \tag{4.4}$$

By using Hölder's inequality we have

$$\begin{aligned} F_2(x) &= \int\limits_{B(x,t)} \frac{\rho(|x-y|)}{|x-y|^n}\, |b(y)-b_{B(x,t)}|\, |f(y)| dy \\ &\le \sum_{j=-\infty}^{-1} \frac{\rho(2^j t)}{(2^j t)^n} \int\limits_{B(x,2^{j+1}t)\setminus B(x,2^j t)} |b(y)-b_{B(x,t)}|\, |f(y)| dy \\ &\le \sum_{j=-\infty}^{-1} \frac{\rho(2^j t)}{(2^j t)^n} |b_{B(x,2^{j+1}t)} - b_{B(x,t)}| \int\limits_{B(x,2^{j+1}t)\setminus B(x,2^j t)} |f(y)| dy \\ &+ \sum_{j=-\infty}^{-1} \frac{\rho(2^j t)}{(2^j t)^n} \int\limits_{B(x,2^{j+1}t)\setminus B(x,2^j t)} |b(y)-b_{B(x,2^{j+1}t)}|\, |f(y)| dy \\ &\le C\, \|b\|_{BMO} \sum_{j=-\infty}^{-1} \frac{\rho(2^j t)}{(2^j t)^n} \|f\|_{L_{p,\omega_1}(B(x,2^{j+1}t)\setminus B(x,2^j t))} \|\omega_1^{-1}\|_{L_{p'}(B(x,2^{j+1}t)\setminus B(x,2^j t))} \\ &\le C\, \|b\|_{BMO} \sum_{j=-\infty}^{-1} \|f\|_{L_{p,\omega_1}(B(x,2^{j+1}t)\setminus B(x,2^j t))} \|\omega_2\|^{-1}_{L_q(B(x,2^{j+1}t)\setminus B(x,2^j t))} \\ &\le C\, \|b\|_{BMO}\, \|f\|_{L_{p,\omega_1}(B(x,t))} \|\omega_2\|^{-1}_{L_q(B(x,t))}. \end{aligned} \tag{4.5}$$

Then by inequality (4.4), (4.5) and Theorem 4.2 we obtain

$$\| |b, I_\rho| f_1\|_{L_{q,\omega_2}(B(x,t))} \le C\, \|b\|_{BMO}\, \|f\|_{L_{p,\omega_1}(B(x,2t))},$$

where the constant C is independent of f .

Taking into account that, we get

$$\| |b, I_\rho| f_1 \|_{L_{q,\omega_2}(B(x,t))} \le C \, \|b\|_{BMO} \, \|\omega_2\|_{L_q(B(x,t))}$$

$$\times \int_t^\infty \Big(1 + \ln \frac{s}{t}\Big) \frac{\|f\|_{L_{p,\omega_1}(B(x,s))}}{\|\omega_2\|_{L_q(B(x,s))}} \frac{ds}{s}. \tag{4.6}$$

When $|x-z| \le t$, $|z-y| \ge 2t$, we have $\frac{1}{2}|z-y| \le |x-y| \le \frac{3}{2}|z-y|$. Therefore we get

$$|b, I_\rho| f_2(z) \le \int_{\mathbb{R}^n \setminus B(x,2t)} \frac{\rho(|z-y|)}{|z-y|^n} \, |b(y) - b(z)| \, |f(y)| dy$$

$$\le C \int_{\mathbb{R}^n \setminus B(x,2t)} \frac{\rho(|x-y|)}{|x-y|^n} \, |b(y) - b(z)| \, |f(y)| dy.$$

We obtain

$$\int_{\mathbb{R}^n \setminus B(x,2t)} \frac{\rho(|x-y|)}{|x-y|^n} \, |b(y) - b(z)| \, |f(y)| dy$$

$$= \int_{\mathbb{R}^n \setminus B(x,2t)} |b(y) - b(z)||f(y)| \left(\int_{|x-y|}^\infty \frac{\rho(s)}{s^{n+1}} \right) dy$$

$$\le C \int_{2t}^\infty \frac{\rho(s)}{s^{n+1}} \left(\int_{\{y \in \mathbb{R}^n : 2t \le |x-y| \le s\}} |b(y) - b(z)||f(y)| dy \right) ds$$

$$\le C \int_{2t}^\infty \frac{\rho(s)}{s^n} \left(\int_{\{y \in \mathbb{R}^n : 2t \le |x-y| \le s\}} |b(y) - b_{B(x,t)}||f(y)| dy \right) \frac{ds}{s}$$

$$+ C|b(z) - b_{B(x,t)}| \int_{2t}^\infty \frac{\rho(s)}{s^n} \left(\int_{\{y \in \mathbb{R}^n : 2t \le |x-y| \le s\}} |f(y)| dy \right) \frac{ds}{s} = J_1 + J_2.$$

To estimate J_1 :

$$J_1 = C \int_{2t}^\infty \frac{\rho(s)}{s^n} \left(\int_{\{y \in \mathbb{R}^n : 2t \le |x-y| \le s\}} |b(y) - b_{B(x,t)}||f(y)| dy \right) \frac{ds}{s}$$

$$\le C \int_t^\infty \frac{\rho(s)}{s^n} \|b(\cdot) - b_{B(x,s)}\|_{L_{p',\omega^{-1}}(B(x,s))} \|f\|_{L_{p,\omega}(B(x,s))} \frac{ds}{s}$$

$$+ C \int_t^\infty \frac{\rho(s)}{s^n} |b_{B(x,t)} - b_{B(x,s)}| \left(\int_{B(x,s)} |f(y)| dy \right) \frac{ds}{s}$$

$$\le C \|b\|_{BMO} \int_t^\infty \frac{\rho(s)}{s^n} \|\omega^{-1}\|_{L_{p'}(B(x,s))} \|f\|_{L_{p,\omega}(B(x,s))} \frac{ds}{s}$$

$$+ C \|b\|_{BMO} \int_t^\infty \frac{\rho(s)}{s^n} \ln \frac{s}{t} \|\omega^{-1}\|_{L_{p'}(B(x,s))} \|f\|_{L_{p,\omega}(B(x,s))} \frac{ds}{s}$$

$$\le C \|b\|_{BMO} \int_t^\infty \Big(1 + \ln \frac{s}{t}\Big) \frac{\rho(s)}{s^{\frac{n}{p} - \frac{n}{q}}} \|\omega_2\|^{-1}_{L_q(B(x,s))} \|f\|_{L_{p,\omega}(B(x,s))} \frac{ds}{s}. \tag{4.7}$$

To estimate J_2 :

$$
\begin{aligned}
J_2 =& C|b(z) - b_{B(x,t)}| \int_{2t}^{\infty} \frac{\rho(s)}{s^n} \left(\int_{\{y\in\mathbb{R}^n: 2t\le|x-y|\le s\}} |f(y)|dy \right) \frac{ds}{s} \\
&\le C|B(x,t)|^{-1} \int_{B(x,t)} |b(z)-b(y)|dy \int_t^{\infty} \frac{\rho(s)}{s^n} \|\omega^{-1}\|_{L_{p'}(B(x,s))} \|f\|_{L_{p,\omega}(B(x,s))} \frac{ds}{s} \\
&\le C M_b \chi_{B(x,t)}(z) \int_t^{\infty} \left(1+\ln\frac{s}{t}\right) \frac{\rho(s)}{s^{\frac{n}{p}-\frac{n}{q}}} \|\omega_2\|^{-1}_{L_q(B(x,s))} \|f\|_{L_{p,\omega}(B(x,s))} \frac{ds}{s},
\end{aligned}
\tag{4.8}
$$

where C does not depend on x,t . Then by Theorem 4.2 and (3.4), (4.7), (4.8) we have

$$
\begin{aligned}
&\||b,I_\rho|f_2\|_{L_{q,\omega_2}(B(x,t))} \le \|J_1\|_{L_{q,\omega_2}(B(x,t))} + \|J_2\|_{L_{q,\omega_2}(B(x,t))} \\
&\le C\|b\|_{BMO}\|\chi_{B(x,t)}\|_{L_{q,\omega_2}} \int_t^{\infty} \left(1+\ln\frac{s}{t}\right) \|\omega_2\|^{-1}_{L_q(B(x,s))} \|f\|_{L_{p,\omega_1}(B(x,s))} \frac{ds}{s} \\
&\quad + C\|M_b\chi_{B(x,t)}\|_{L_{q,\omega_2}} \int_t^{\infty} \left(1+\ln\frac{s}{t}\right) \|\omega_2\|^{-1}_{L_q(B(x,s))} \|f\|_{L_{p,\omega_1}(B(x,s))} \frac{ds}{s} \\
&\le C\|b\|_{BMO}\|\omega_2\|_{L_q(B(x,t))} \int_t^{\infty} \left(1+\ln\frac{s}{t}\right) \|\omega_2\|^{-1}_{L_q(B(x,s))} \|f\|_{L_{p,\omega_1}(B(x,s))} \frac{ds}{s} \\
&\quad + C\|b\|_{BMO}\|\omega_2\|_{L_q(B(x,t))} \int_t^{\infty} \left(1+\ln\frac{s}{t}\right) \|\omega_2\|^{-1}_{L_q(B(x,s))} \|f\|_{L_{p,\omega_1}(B(x,s))} \frac{ds}{s} \\
&\le C\|b\|_{BMO}\|\omega_2\|_{L_q(B(x,t))} \int_t^{\infty} \left(1+\ln\frac{s}{t}\right) \|\omega_2\|^{-1}_{L_q(B(x,s))} \|f\|_{L_{p,\omega_1}(B(x,s))} \frac{ds}{s}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\||b,I_\rho|f_2\|_{L_{q,\omega_2}(B(x,t))} \le C\|b\|_{BMO}\|\omega_2\|_{L_q(B(x,t))} \\
&\times \int_t^{\infty} \left(1+\ln\frac{s}{t}\right) \|\omega_2\|^{-1}_{L_q(B(x,s))} \|f\|_{L_{p,\omega_1}(B(x,s))} \frac{ds}{s},
\end{aligned}
$$

which together with (4.6) yields (4.3). □

In the following theorem we prove the boundedness of commutators of generalized fractional integral operator $|b,I_\rho|$ from the generalized weighted Morrey spaces $\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)$ to the generalized weighted Morrey spaces $\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)$. We find conditions on the functions $\varphi_1(x,r)$ and $\varphi_2(x,r)$ for the boundedness of $|b,I_\rho|$ from $\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)$.

Theorem 4.5. *Let* $1<p<q<\infty$, $b\in BMO(\mathbb{R}^n)$, *the function* ρ *satisfies the conditions* (3.1)-(3.3)*, if the condition* (3.4) *is fulfill,* $(\omega_1,\omega_2)\in A_{p,q}(\mathbb{R}^n)$, $\omega_1\in A_p(\mathbb{R}^n)$, $\omega_2\in A_q(\mathbb{R}^n)$ *and the functions* $\varphi_1(x,r)$ *and* $\varphi_2(x,r)$ *fulfill the condition*

$$
\int_t^{\infty} \left(1+\ln\frac{s}{t}\right) \frac{\operatorname*{ess\,inf}_{s<t<1} \varphi_1(x,r)\|\omega_1\|_{L_p(B(x,r))}}{\|\omega_2\|_{L_q(B(x,s))}} \frac{ds}{s} \le C\varphi_2(x,t). \tag{4.9}
$$

Then the operator $|b,I_\rho|$ *is bounded from* $\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)$ *to* $\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)$.

Proof. Let $f \in \mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)$. From the definition of norm of generalized weighted Morrey spaces we write

$$\||b, I_\rho|f\|_{\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} \frac{1}{\varphi_2(x,t)\|\omega_2\|_{L_q(B(x,t))}} \||b, I_\rho|f\chi_{B(x,r)}\|_{L_{q,\omega_2}(\mathbb{R}^n)}. \tag{4.10}$$

We estimate $\||b, I_\rho|f\chi_{B(x,r)}\|_{L_{q,\omega_2}(\mathbb{R}^n)}$ in (4.10) by means of Theorems 4.4, 2.11 and obtain

$$\||b, I_\rho|f\|_{\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)}$$

$$\leq C\,\|b\|_{BMO} \sup_{x\in\mathbb{R}^n,\, t>0} \frac{\|\omega_2\|_{L_q(B(x,t))}}{\varphi_2(x,t)\|\omega_2\|_{L_q(B(x,t))}} \int_t^\infty \Big(1+\ln\frac{s}{t}\Big) \frac{\|f\|_{L_{p,\omega}(B(x,s))}}{\|\omega_2\|_{L_{p'}(B(x,s))}} \frac{ds}{s}$$

$$\leq C\,\|b\|_{BMO} \sup_{x\in\mathbb{R}^n,\, t>0} \frac{1}{\varphi_1(x,t)\|\omega_1\|_{L_p(B(x,t))}} \|f\|_{L_{p,\omega_1}(B(x,t))} = C\,\|b\|_{BMO}\,\|f\|_{\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)}.$$

It remains to make use of condition (4.9). □

Corollary 4.6. *Let* $1<p<q<\infty$ *,* $b\in BMO(\mathbb{R}^n)$ *, the function* ρ *satisfies the conditions* (3.1)-(3.3)*, if the condition* (3.4) *is fulfill and* $(\omega_1,\omega_2)\in A_{p,q}(\mathbb{R}^n)$ *,* $\omega_1\in A_p(\mathbb{R}^n)$ *,* $\omega_2\in A_q(\mathbb{R}^n)$ *. The operator* $|b, I_\rho|$ *is bounded from* $L_{p,\omega_1}(\mathbb{R}^n)$ *to* $L_{q,\omega_2}(\mathbb{R}^n)$ *.*

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