

## NEW DISCRETE FORM OF HILBERT INEQUALITY FOR THREE VARIABLES

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ABSTRACT. In this paper, we introduce a new form of discrete Hilbert inequality for three variables, also, we prove that the constant which appears on the right hand of inequality is the best constant. Also, the reverse form and equivalent forms are, obtained.

### 1. INTRODUCTION

The well-known Hardy-Hilbert inequality for two nonnegative functions  $f, g$  and two positive parameters  $p$  and  $q$  such that  $p > 1, \frac{1}{p} + \frac{1}{q} = 1$  is given as

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left( \int_0^\infty f^p(x) dx \right)^{\frac{1}{p}} \left( \int_0^\infty g^q(y) dy \right)^{\frac{1}{q}}, \quad (1.1)$$

where the constant  $C = \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$  is the best possible.

The corresponding discrete form of inequality (1.1) for two nonnegative sequences of real numbers  $\{a_m\}$  and  $\{b_n\}$  is given as:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left( \sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}, \quad (1.2)$$

where  $\{a_m\} \in l_p$  and  $\{b_n\} \in l_q$ . The constant factor  $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$  is the best possible. The inequalities (1.1) and (1.2) have several applications, see results in [9] and [13].

In the last twenty years the inequalities (1.1) and (1.2) have many extensions and equivalent forms in many different ways see [3], [10] and [11].

Hilbert integral inequality for three variables can be generalized [12], as

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$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{f(x,y)g(z)}{(x+y+z)^\lambda} dx dy dz < C \left( \int_0^\infty \int_0^\infty \frac{f^p(x,y)dx dy}{(x+y)^{-2p+\lambda+p\xi+2}} \right)^{\frac{1}{p}} \left( \int_0^\infty \frac{g^q(z)dz}{z^{-q-q\xi+\lambda+1}} \right)^{\frac{1}{q}} \quad (1.3)$$

where  $\lambda > 0$ ,  $\xi \in (-\frac{\lambda}{p}, \frac{\lambda}{q})$ ,  $f(x,y)$  is non-negative function defined on  $(0; \infty) \times (0; \infty)$ ; and  $g(z) > 0$  on  $(0; \infty)$ , also, the constant  $C = B(\frac{\lambda}{p} - \xi, \frac{\lambda}{q} + \xi)$  is the best possible.

Also, the discrete form of (1.3) was obtained in [13] namely

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \sum_{r=1}^\infty \frac{a_{m,n} b_r}{(m+n+r)^\lambda} < C \left( \sum_{m=1}^\infty \sum_{n=1}^\infty (m+n)^{2p-\lambda-p\xi-2} a_{m,n}^p \right)^{\frac{1}{p}} \left( \sum_{r=1}^\infty r^{q+q\xi-\lambda-1} b_r^q \right)^{\frac{1}{q}} \quad (1.4)$$

where the constant  $C = B(\frac{\lambda}{p} - \xi, \frac{\lambda}{q} + \xi)$  is the best possible, and  $a_{m,n}$  is non-negative double sequence and  $b_r$  is non-negative sequence

## 2. PRELIMINRIES AND LEMMAS

We use the most common representation of gamma and beta function, which are defined by improper integrals respectively:

$$\Gamma(\mu) = \int_0^\infty t^{\mu-1} e^{-t} dt, \quad \mu > 0, \quad (2.1)$$

$$B(\psi, \phi) = \int_0^\infty \frac{t^{\psi-1}}{(t+1)^{\psi+\phi}} dt, \quad \psi, \phi > 0. \quad (2.2)$$

Also, for gamma and beta functions, we will use the useful representations for them as follows:

$$\frac{1}{x^\mu} = \frac{1}{\Gamma(\mu)} \int_0^\infty t^{\mu-1} e^{-xt} dt \quad (2.3)$$

$$B(\psi, \phi) = \frac{\Gamma(\psi)\Gamma(\phi)}{\Gamma(\psi+\phi)} \quad (2.4)$$

$$B(\psi, \phi) = B(\phi, \psi) \quad (2.5)$$

Next, we introduce the following lemmas which are necessary tools to prove our result:

**Lemma 2.1.** Let  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\lambda > 0$ , and  $b_r > 0$ , then for  $t > 0$  and  $0 \leq \varphi < \lambda + \frac{1}{p}$ , we have

$$\sum_{r=1}^\infty r^\lambda e^{-rt} b_r \leq t^{\varphi-\lambda-\frac{1}{p}} (\Gamma(\lambda p - p\varphi + 1))^{\frac{1}{p}} \left( \sum_{r=1}^\infty r^{q\varphi} e^{-rt} b_r^q \right)^{\frac{1}{q}} \quad (2.6)$$

*Proof :* Using Hölder inequality, we get

$$\begin{aligned}
\sum_{r=1}^{\infty} r^{\lambda} e^{-rt} b_r &= \sum_{r=1}^{\infty} (r^{\lambda-\varphi} e^{-\frac{rt}{p}}) (r^{\varphi} e^{-\frac{rt}{q}} b_r) \\
&\leq \left( \sum_{r=1}^{\infty} r^{\lambda p - \varphi p} e^{-rt} \right)^{\frac{1}{p}} \left( \sum_{r=1}^{\infty} r^{q\varphi} e^{-rt} b_r^q \right)^{\frac{1}{q}} \\
&\leq \left( \int_0^{\infty} x^{\lambda p - \varphi p} e^{-xt} dx \right)^{\frac{1}{p}} \left( \sum_{r=1}^{\infty} r^{q\varphi} e^{-rt} b_r^q \right)^{\frac{1}{q}} \\
&= t^{\varphi - \lambda - \frac{1}{p}} \Gamma^{\frac{1}{p}} (\lambda p - p\varphi + 1) \left( \sum_{r=1}^{\infty} r^{q\varphi} e^{-rt} b_r^q \right)^{\frac{1}{q}}
\end{aligned}$$

**Lemma 2.2.** Let  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\lambda > 0$ , and  $a_{m,n} > 0$ , then for  $t > 0$  and  $0 \leq \psi < \lambda + \frac{1}{q}$ , we have

$$\left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} \left( \frac{mn}{m+n} \right)^{\lambda} e^{-\left(\frac{mn}{m+n}\right)t} \right)^p \leq t^{p\psi - \lambda p} \Gamma^{\frac{p}{q}} (q\lambda - q\psi) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\left(\frac{mn}{m+n}\right)^{p\psi}}{(m+n)^{-\frac{2p}{q}}} e^{-\left(\frac{mn}{m+n}\right)t} a_{m,n}^p \quad (2.7)$$

*Proof :* Using Hölder inequality, using the substitutions  $y = xu$  and  $x = \frac{1+u}{u}v$ , (to evaluate the integral on the righthand of the inequality), we have

$$\begin{aligned}
&\left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} \left( \frac{mn}{m+n} \right)^{\lambda} e^{-\left(\frac{mn}{m+n}\right)t} \right)^p \\
&= \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \left( \frac{mn}{m+n} \right)^{\lambda} e^{-\left(\frac{mn}{m+n}\right)\frac{t}{q}} \right\} \left\{ e^{-\left(\frac{mn}{m+n}\right)\frac{t}{p}} a_{m,n} \right\} \right)^p \\
&= \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{\left(\frac{mn}{m+n}\right)^{\lambda-\psi}}{(m+n)^{\frac{2}{q}}} e^{-\left(\frac{mn}{m+n}\right)\frac{t}{q}} \right\} \left\{ \frac{\left(\frac{mn}{m+n}\right)^{\psi}}{(m+n)^{-\frac{2}{q}}} e^{-\left(\frac{mn}{m+n}\right)\frac{t}{p}} a_{m,n} \right\} \right)^p \\
&\leq \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\left(\frac{mn}{m+n}\right)^{q\lambda-q\psi}}{(m+n)^2} e^{-\left(\frac{mn}{m+n}\right)t} \right)^{\frac{p}{q}} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\left(\frac{mn}{m+n}\right)^{\psi p}}{(m+n)^{-\frac{2p}{q}}} e^{-\left(\frac{mn}{m+n}\right)t} a_{m,n}^p \\
&\leq \left( \iint_0^{\infty} \frac{\left(\frac{xy}{x+y}\right)^{q\lambda-q\psi}}{(x+y)^2} e^{-\left(\frac{xy}{x+y}\right)t} dx dy \right)^{\frac{p}{q}} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\left(\frac{mn}{m+n}\right)^{\psi p}}{(m+n)^{-\frac{2p}{q}}} e^{-\left(\frac{mn}{m+n}\right)t} a_{m,n}^p
\end{aligned}$$

$$\begin{aligned}
&= \left( \int_0^\infty \int_0^\infty \frac{\left(\frac{x(xu)}{x+(xu)}\right)^{q\lambda-q\psi}}{(x+(xu))^2} e^{-\left(\frac{x(xu)}{x+(xu)}\right)t} (xdu) dx \right)^{\frac{p}{q}} \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{\left(\frac{mn}{m+n}\right)^{\psi p}}{(m+n)^{-\frac{2p}{q}}} e^{-\left(\frac{mn}{m+n}\right)t} a_{m,n}^p \\
&= \left( \int_0^\infty \int_0^\infty \frac{\left(\frac{x}{1+u}\right)^{q\lambda-q\psi}}{x(1+u)^2} e^{-x\frac{u}{1+u}t} du dx \right)^{\frac{p}{q}} \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{\left(\frac{mn}{m+n}\right)^{\psi p}}{(m+n)^{-\frac{2p}{q}}} e^{-\left(\frac{mn}{m+n}\right)t} a_{m,n}^p \\
&= \left( \int_0^\infty \int_0^\infty \frac{v^{q\lambda-q\psi-1}}{(1+u)^2} e^{-vt} du dv \right)^{\frac{p}{q}} \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{\left(\frac{mn}{m+n}\right)^{\psi p}}{(m+n)^{-\frac{2p}{q}}} e^{-\left(\frac{mn}{m+n}\right)t} a_{m,n}^p \\
&= t^{p\psi-p\lambda} \Gamma^{\frac{p}{q}}(q\lambda - q\psi) \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{\left(\frac{mn}{m+n}\right)^{\psi p}}{(m+n)^{-\frac{2p}{q}}} e^{-\left(\frac{mn}{m+n}\right)t} a_{m,n}^p
\end{aligned}$$

### 3. Main Result

**Theorem 3.1.** Let  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $0 < \lambda < \frac{1}{q}$ ,  $-\frac{\lambda}{p} < \varsigma < \frac{\lambda}{q}$ , suppose that  $a_{m,n}$  is a non-negative double sequence and  $b_r$  is a non-negative sequence.

If  $\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{(mn)^{\lambda p - \lambda - \varsigma p}}{(m+n)^{\lambda p - \lambda - \varsigma p - \frac{2p}{q}}} a_{m,n}^p < \infty$  and  $\sum_{r=1}^\infty r^{q\lambda + q\varsigma - \lambda + \frac{q}{p}} b_r^q < \infty$ , then

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \sum_{r=1}^\infty \frac{a_{m,n} b_r}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^\lambda} \leq C \left( \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{(mn)^{\lambda p - \lambda - \varsigma p} a_{m,n}^p}{(m+n)^{\lambda p - \lambda - \varsigma p - \frac{2p}{q}}} \right)^{\frac{1}{p}} \left( \sum_{r=1}^\infty r^{q\lambda + q\varsigma - \lambda + \frac{q}{p}} b_r^q \right)^{\frac{1}{q}} \quad (3.1)$$

where  $C = B(\frac{\lambda}{p} + \varsigma, \frac{\lambda}{q} - \varsigma)$  is the best possible.

**Proof :** Let

$$\begin{aligned}
I &= \sum_{m=1}^\infty \sum_{n=1}^\infty \sum_{r=1}^\infty \frac{a_{m,n} b_r}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^\lambda} \\
&= \sum_{m=1}^\infty \sum_{n=1}^\infty \sum_{r=1}^\infty \frac{a_{m,n} b_r (mnr)^\lambda}{(mn + r(m+n))^\lambda} \\
&= \left( \frac{1}{\Gamma(\lambda)} \sum_{m=1}^\infty \sum_{n=1}^\infty \sum_{r=1}^\infty a_{m,n} b_r (mn)^\lambda (r)^\lambda \int_0^\infty t^{\lambda-1} e^{-[mn+r(m+n)]t} dt \right) \quad (3.2)
\end{aligned}$$

By substituting  $t = \frac{\omega}{m+n}$  in (3.2), applying Hölder inequality, and replace  $\omega$  by  $t$ , we get:

$$\begin{aligned}
I &= \frac{1}{\Gamma(\lambda)} \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} a_{m,n} b_r (mn)^{\lambda} r^{\lambda} \int_0^{\infty} \left( \frac{\omega}{m+n} \right)^{\lambda-1} e^{-[mn+r(m+n)]\frac{\omega}{m+n}} \frac{d\omega}{m+n} \right) \\
&= \frac{1}{\Gamma(\lambda)} \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} a_{m,n} b_r (mn)^{\lambda} r^{\lambda} \int_0^{\infty} \left( \frac{\omega}{m+n} \right)^{\lambda-1} e^{-[\frac{mn}{m+n}+r]\omega} \frac{d\omega}{m+n} \right) \\
&= \frac{1}{\Gamma(\lambda)} \int_0^{\infty} \left( \omega^{\frac{\lambda-1}{p}+\varsigma} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} \left( \frac{mn}{m+n} \right)^{\lambda} e^{-\frac{mn}{m+n}\omega} \right) \left( \omega^{\frac{\lambda-1}{q}-\varsigma} \sum_{r=1}^{\infty} r^{\lambda} e^{-r\omega} b_r \right) d\omega \\
&\leq \frac{1}{\Gamma(\lambda)} \left( \int_0^{\infty} t^{\lambda-1+p\varsigma} \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} \left( \frac{mn}{m+n} \right)^{\lambda} e^{-\frac{mn}{m+n}t} \right)^p dt \right)^{\frac{1}{p}} \\
&\quad \times \left( \int_0^{\infty} t^{\lambda-1-q\varsigma} \left( \sum_{r=1}^{\infty} r^{\lambda} e^{-rt} b_r \right)^q dt \right)^{\frac{1}{q}}. \tag{3.3}
\end{aligned}$$

Substitute (2.6) and (2.7) in (3.3), we get:

$$\begin{aligned}
I &\leq \frac{1}{\Gamma(\lambda)} \left( \int_0^{\infty} t^{\lambda-1+p\varsigma} \left( t^{p\psi-\lambda p} \Gamma^{\frac{p}{q}} (q\lambda - q\psi) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\left( \frac{mn}{m+n} \right)^{p\psi}}{(m+n)^{-\frac{2p}{q}}} e^{-\left( \frac{mn}{m+n} \right)t} a_{m,n}^p \right) dt \right)^{\frac{1}{p}} \\
&\quad \times \left( \int_0^{\infty} t^{\lambda-1-q\varsigma} \left( t^{\varphi-\lambda-\frac{1}{p}} \Gamma^{\frac{1}{p}} (\lambda p - p\varphi + 1) \left( \sum_{r=1}^{\infty} r^{q\varphi} e^{-rt} b_r^q \right)^{\frac{1}{q}} \right)^q dt \right)^{\frac{1}{q}} \\
&= \frac{\Gamma^{\frac{1}{q}} (q\lambda - q\psi) \Gamma^{\frac{1}{p}} (\lambda p - p\varphi + 1)}{\Gamma(\lambda)} \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\left( \frac{mn}{m+n} \right)^{p\psi}}{(m+n)^{-\frac{2p}{q}}} a_{m,n}^p \int_0^{\infty} t^{\lambda-1+p\varsigma+p\psi-\lambda p} e^{-\left( \frac{mn}{m+n} \right)t} dt \right)^{\frac{1}{p}} \\
&\quad \times \left( \sum_{r=1}^{\infty} r^{q\varphi} b_r^q \int_0^{\infty} t^{\lambda-1-q\varsigma+\varphi q-\lambda q-\frac{q}{p}} e^{-rt} dt \right)^{\frac{1}{q}} \\
&= \frac{\Gamma^{\frac{1}{q}} (q\lambda - q\psi) \Gamma^{\frac{1}{p}} (\lambda p - p\varphi + 1) \Gamma^{\frac{1}{p}} (\lambda + p\varsigma + p\psi - \lambda p) \Gamma^{\frac{1}{q}} (\lambda - q\varsigma + \varphi q - \lambda q - \frac{q}{p})}{\Gamma(\lambda)} \\
&\quad \times \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(mn)^{\lambda p - \lambda - p\varsigma}}{(m+n)^{\lambda p - \lambda - p\varsigma - \frac{2p}{q}}} a_{m,n}^p \right)^{\frac{1}{p}} \left( \sum_{r=1}^{\infty} r^{q\lambda+q\varsigma-\lambda+\frac{q}{p}} b_r^q \right)^{\frac{1}{q}} \\
&= C_{\phi,\vartheta} \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(mn)^{\lambda p - \lambda - p\varsigma}}{(m+n)^{\lambda p - \lambda - p\varsigma - \frac{2p}{q}}} a_{m,n}^p \right)^{\frac{1}{p}} \cdot \left( \sum_{r=1}^{\infty} r^{q\lambda+q\varsigma-\lambda+\frac{q}{p}} b_r^q \right)^{\frac{1}{q}} \tag{3.4}
\end{aligned}$$

If we set  $\phi = \frac{pq\lambda - \lambda - ps}{pq}$  and  $\vartheta = \frac{pq\lambda - \lambda + ps + q}{pq}$ , we can obtain inequality (3.1) with constant  $C_{\frac{pq\lambda - \lambda - ps}{pq}, \frac{pq\lambda - \lambda + ps + q}{pq}} = C_{\phi, \vartheta} = C$ .

To show that the constant  $C$  in (3.1) is the best possible. Define the sequence  $\tilde{a}_{m,n} = \frac{q^{\frac{1}{q}}(mn)^{\varsigma - \frac{\lambda}{q}}}{(m+n)^{2+\frac{\epsilon}{p}+\varsigma-\frac{\lambda}{q}}} \tilde{a}_{m,n}^p$  for  $m, n \geq 1$  and the sequence  $\tilde{b}_r = r^{\frac{\lambda-q\epsilon}{q}-\varsigma-\lambda-1}$  for  $r \geq 1$ . Suppose that  $C$  is not the best possible, then there exist a constant  $M$ , where  $0 < M < C$ , such that

$$\begin{aligned}
I &\leq M \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(mn)^{\lambda p - \lambda - ps}}{(m+n)^{\lambda p - \lambda - ps - \frac{2p}{q}}} \tilde{a}_{m,n}^p \right)^{\frac{1}{p}} \left( \sum_{r=1}^{\infty} r_r^{q\lambda + q\varsigma - \lambda + \frac{q}{p}} \tilde{b}_r^q \right)^{\frac{1}{q}} \\
&= M \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(mn)^{\lambda p - \lambda - ps}}{(m+n)^{\lambda p - \lambda - ps - \frac{2p}{q}}} \left( \frac{q^{\frac{1}{q}}(mn)^{\varsigma - \frac{\lambda}{q}}}{(m+n)^{2+\frac{\epsilon}{p}+\varsigma-\frac{\lambda}{q}}} \right)^p \right)^{\frac{1}{p}} \left( \sum_{r=1}^{\infty} r_r^{q\lambda + q\varsigma - \lambda + \frac{q}{p}} \left( r^{\frac{\lambda-q\epsilon}{q}-\varsigma-\lambda-1} \right)^q \right)^{\frac{1}{q}} \\
&= M \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{q^{\frac{1}{q}}(mn)^0}{(m+n)^{\epsilon+2}} \right)^{\frac{1}{p}} \left( \sum_{r=1}^{\infty} r^{-q\epsilon-1} \right)^{\frac{1}{q}} \\
&< q^{\frac{1}{q}} M \left( \sum_{m=1}^{\infty} \frac{1}{(m+1)^{\epsilon+2}} + \sum_{n=2}^{\infty} \frac{1}{(n+1)^{\epsilon+2}} + \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{1}{(m+n)^{\epsilon+2}} \right)^{\frac{1}{p}} \left( \sum_{r=1}^{\infty} r^{-q\epsilon-1} \right)^{\frac{1}{q}} \\
&< q^{\frac{1}{q}} M \left( \int_0^{\infty} \frac{1}{(x+1)^{\epsilon+2}} dx + \int_1^{\infty} \frac{1}{(y+1)^{\epsilon+2}} dy + \int_1^{\infty} \int_1^{\infty} \frac{1}{(x+y)^{\epsilon+2}} dxdy \right)^{\frac{1}{p}} \left( 1 + \int_1^{\infty} \frac{1}{z^{q\epsilon+1}} dz \right)^{\frac{1}{q}} \\
&= q^{\frac{1}{q}} M \left( \frac{1}{(\epsilon+1)} + \frac{1}{2^{\epsilon+1}(\epsilon+1)} + \frac{1}{2^{\epsilon}\epsilon(\epsilon+1)} \right)^{\frac{1}{p}} \left( 1 + \frac{1}{q\epsilon} \right)^{\frac{1}{q}} \\
&= q^{\frac{1}{q}} M \left( \frac{1}{\epsilon} \right)^{\frac{1}{p}} \left( \frac{\epsilon}{(\epsilon+1)} + \frac{\epsilon}{2^{\epsilon+1}(\epsilon+1)} + \frac{1}{2^{\epsilon}(\epsilon+1)} \right)^{\frac{1}{p}} \left( \frac{1}{q\epsilon} \right)^{\frac{1}{q}} (q\epsilon+1)^{\frac{1}{q}}
\end{aligned}$$

$$= \frac{M}{\epsilon} H(\epsilon), \tag{3.5}$$

it is clear that  $H(\epsilon) \rightarrow 1$  as  $\epsilon \rightarrow 0^+$ .

On the other side, if we estimate the left hand side of (3.1), let  $u = x \left( \frac{m+n}{mn} \right)$ , we find

$$\begin{aligned}
I &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{\tilde{a}_{m,n} \tilde{b}_r}{\left( \frac{1}{m} + \frac{1}{n} + \frac{1}{r} \right)^{\lambda}} \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{\frac{q^{\frac{1}{q}}(mn)^{\varsigma - \frac{\lambda}{q}}}{(m+n)^{2+\frac{\epsilon}{p}+\varsigma-\frac{\lambda}{q}}} r^{\frac{\lambda-q\epsilon}{q}-\varsigma-\lambda-1}}{\left( \frac{1}{m} + \frac{1}{n} + \frac{1}{r} \right)^{\lambda}} \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{q^{\frac{1}{q}}(mn)^{\varsigma - \frac{\lambda}{q}} r^{\frac{\lambda-q\epsilon}{q}-\varsigma-\lambda-1} (mnr)^{\lambda}}{(m+n)^{2+\frac{\epsilon}{p}+\varsigma-\frac{\lambda}{q}} (mn+r(m+n))^{\lambda}}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{q^{\frac{1}{q}} (mn)^{\varsigma - \frac{\lambda}{q}} (mn)^{\lambda} r^{\frac{\lambda - q\epsilon}{q} - \varsigma - 1}}{(m+n)^{2 + \frac{\epsilon}{p} + \varsigma - \frac{\lambda}{q}} (mn + r(m+n))^{\lambda}} \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{q^{\frac{1}{q}} (mn)^{\varsigma - \frac{\lambda}{q}} (mn)^{\lambda} \sum_{r=1}^{\infty} \frac{r^{\frac{\lambda - q\epsilon}{q} - \varsigma - 1}}{(mn + r(m+n))^{\lambda}}}{(m+n)^{2 + \frac{\epsilon}{p} + \varsigma - \frac{\lambda}{q}}} \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{q^{\frac{1}{q}} (mn)^{\varsigma - \frac{\lambda}{q}} (mn)^{\lambda} \sum_{r=1}^{\infty} \frac{r^{\frac{\lambda - q\epsilon}{q} - \varsigma - 1}}{(mn)^{\lambda} (1 + r(\frac{m+n}{mn}))^{\lambda}}}{(m+n)^{2 + \frac{\epsilon}{p} + \varsigma - \frac{\lambda}{q}}} \\
&\geq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{q^{\frac{1}{q}} (mn)^{\varsigma - \frac{\lambda}{q}}}{(m+n)^{2 + \frac{\epsilon}{p} + \varsigma - \frac{\lambda}{q}}} \int_1^{\infty} \frac{x^{\frac{\lambda - q\epsilon}{q} - \varsigma - 1}}{(1 + x(\frac{m+n}{mn}))^{\lambda}} dx \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{q^{\frac{1}{q}} (mn)^{\varsigma - \frac{\lambda}{q}}}{(m+n)^{2 + \frac{\epsilon}{p} + \varsigma - \frac{\lambda}{q}}} \int_{\frac{m+n}{mn}}^{\infty} \frac{(\frac{mn}{m+n} u)^{\frac{\lambda - q\epsilon}{q} - \varsigma - 1}}{(1+u)^{\lambda}} \left( \frac{mn}{m+n} du \right) \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{q^{\frac{1}{q}} (mn)^{\varsigma - \frac{\lambda}{q}}}{(m+n)^{2 + \frac{\epsilon}{p} + \varsigma - \frac{\lambda}{q}}} \left( \frac{mn}{m+n} \right)^{\frac{\lambda - q\epsilon}{q} - \varsigma} \int_{\frac{m+n}{mn}}^{\infty} \frac{u^{\frac{\lambda - q\epsilon}{q} - \varsigma - 1}}{(1+u)^{\lambda}} du \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{q^{\frac{1}{q}} (mn)^{-\epsilon}}{(m+n)^{2 + \frac{\epsilon}{p} - \epsilon}} \int_{\frac{m+n}{mn}}^{\infty} \frac{u^{\frac{\lambda - q\epsilon}{q} - \varsigma - 1}}{(1+u)^{\lambda}} du,
\end{aligned}$$

since  $\epsilon \rightarrow 0^+$  (so small), it follows  $q^{\frac{1}{q}} (mn)^{-\epsilon} = \frac{q^{\frac{1}{q}}}{(mn)^{\epsilon}} > 1$ , we get:

$$\begin{aligned}
I &> \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(m+n)^{2+\epsilon}} \int_{\frac{m+n}{mn}}^{\infty} \frac{(u)^{\frac{\lambda - q\epsilon}{q} - \varsigma - 1}}{(1+u)^{\lambda}} du \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(m+n)^{2+\epsilon}} \left( \int_0^{\infty} \frac{(u)^{\frac{\lambda - q\epsilon}{q} - \varsigma - 1}}{(1+u)^{\lambda}} du - \int_0^{\frac{m+n}{mn}} \frac{(u)^{\frac{\lambda - q\epsilon}{q} - \varsigma - 1}}{(1+u)^{\lambda}} du \right) \\
&\geq \int_1^{\infty} \int_1^{\infty} \frac{1}{(x+y)^{2+\epsilon}} dx dy \left( B\left(\frac{\lambda}{q} + \epsilon - \varsigma, \frac{\lambda}{p} - \epsilon + \varsigma\right) - \int_0^{\frac{m+n}{mn}} (u)^{\frac{\lambda + \epsilon q \lambda - \epsilon}{q} - \varsigma - 1} du \right) \\
&= \frac{1}{2^\epsilon \epsilon (1+\epsilon)} \left( B\left(\frac{\lambda}{q} + \epsilon - \varsigma, \frac{\lambda}{p} - \epsilon + \varsigma\right) - \int_0^{\frac{m+n}{mn}} (u)^{\frac{\lambda + \epsilon q \lambda - \epsilon}{q} - \varsigma - 1} du \right) \\
&= \frac{B\left(\frac{\lambda}{q} + \epsilon - \varsigma, \frac{\lambda}{p} - \epsilon + \varsigma\right)}{2^\epsilon \epsilon (1+\epsilon)} - O(1),
\end{aligned} \tag{3.6}$$

when  $\epsilon \rightarrow 0^+$  in (3.5) and (3.6), we obtain a contradiction. By this the proof of theorem is completed.

The next part in this paper , we give the reverse form of our main inequality (3.1). Also the constant  $C$  here is the best as in the first Theorem.

**Theorem 3.2.** Let  $0 < p < 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $0 < \lambda < \frac{1}{q}$ ,  $-\frac{\lambda}{p} < \varsigma < \frac{\lambda}{q}$ , suppose that  $a_{m,n}$  is a non-negative double sequence and  $b_r$  is a non-negative sequence. If  $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(mn)^{\lambda p - \lambda - \varsigma p}}{(m+n)^{\lambda p - \lambda - \varsigma p - \frac{2p}{q}}} a_{m,n}^p < \infty$  and  $\sum_{r=1}^{\infty} r^{q\lambda + q\varsigma - \lambda + \frac{p}{q}} b_r^q < \infty$ , then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{a_{m,n} b_r}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \geq C \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(mn)^{\lambda p - \lambda - \varsigma p}}{(m+n)^{\lambda p - \lambda - \varsigma p - \frac{2p}{q}}} a_{m,n}^p \right)^{\frac{1}{p}} \left( \sum_{r=1}^{\infty} r^{q\lambda + q\varsigma - \lambda + \frac{p}{q}} b_r^q \right)^{\frac{1}{q}} \quad (3.7)$$

Where  $C = B(\frac{\lambda}{p} + \varsigma, \frac{\lambda}{q} - \varsigma)$  is the best possible.

**proof:** In order to prove this theorem , we will follow the same steps as in the proof of the first Theorem, but we use the reverse Hölder inequality instead of Hölder inequality,

so,we leave it. In particular (for theorem 3.2)

a) If  $\lambda = \frac{1}{2}$ ,  $\varsigma = 0$ ,  $p = \frac{1}{2}$ , substitute these values in (3.2), we get:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{a_{m,n} b_r}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\frac{1}{2}}} \geq -2 \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(m+n)^{\frac{3}{4}}}{(mn)^{\frac{1}{4}}} a_{m,n}^{\frac{1}{2}} \right)^2 \left( \sum_{r=1}^{\infty} r^2 b_r^{-1} \right)^{-1}$$

b) If  $\lambda = \frac{1}{2}$ ,  $\varsigma = -\frac{1}{2}$ ,  $p = \frac{1}{3}$ , substitute these values in (3.2) we get:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{a_{m,n} b_r}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\frac{1}{2}}} \geq \frac{4}{3} \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(m+n)}{(mn)^{\frac{1}{2}}} a_{m,n}^{\frac{1}{3}} \right)^3 \left( \sum_{r=1}^{\infty} r^{-2} b_r^{-\frac{1}{2}} \right)^{-2}$$

#### 4. Equivalent Forms

In this part , we give some equivalent forms of the inequality (3.1).

**Theorem 4.1 :** Using the same conditions of Theorem 3.1, we have the following inequalities:

$$\sum_{r=1}^{\infty} r^{-\lambda - p\varsigma - 1} \left( \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{a_{m,n}}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \right)^p \leq C^p \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(mn)^{\lambda p - \lambda - \varsigma p}}{(m+n)^{\lambda p - \lambda - \varsigma p - \frac{2p}{q}}} a_{m,n}^p \quad (4.1)$$

and

$$\sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{(mn)^{\varsigma q - \lambda}}{(m+n)^{\varsigma q - \lambda + 2}} \left( \sum_{r=1}^{\infty} \frac{b_r}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \right)^q \leq C^q \sum_{r=1}^{\infty} r^{q\lambda + q\varsigma - \lambda + \frac{q}{p}} b_r^q \quad (4.2)$$

Both of (4.1) and (4.2) are equivalent to (3.1) and  $C^p$  and  $C^q$  are the best possible.

Proof: To prove (4.1) , set

$$b_r = r^{-\lambda - p\varsigma - 1} \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{m,n}}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \right)^{p-1}$$

By using (3.1), we obtain

$$\begin{aligned}
& \sum_{r=1}^{\infty} r^{-\lambda-p\varsigma-1} \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{m,n}}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \right)^p \\
&= \sum_{r=1}^{\infty} r^{-\lambda-p\varsigma-1} \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{m,n}}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \right)^{p-1} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{m,n}}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \\
&\leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{a_{m,n} b_r}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \\
&\leq C \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(mn)^{\lambda p - \lambda - \varsigma p}}{(m+n)^{\lambda p - \lambda - \varsigma p - \frac{2p}{q}}} a_{m,n}^p \right)^{\frac{1}{p}} \left( \sum_{r=1}^{\infty} r^{q\lambda + q\varsigma - \lambda + \frac{p}{q}} b_r^q \right)^{\frac{1}{q}} \\
&= C \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(mn)^{\lambda p - \lambda - \varsigma p}}{(m+n)^{\lambda p - \lambda - \varsigma p - \frac{2p}{q}}} a_{m,n}^p \right)^{\frac{1}{p}} \\
&\quad \times \left( \sum_{r=1}^{\infty} r^{q\lambda + q\varsigma - \lambda + \frac{p}{q}} \left( r^{-\lambda-p\varsigma-1} \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{m,n}}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \right)^{p-1} \right)^q \right)^{\frac{1}{q}} \\
&= C \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(mn)^{\lambda p - \lambda - \varsigma p}}{(m+n)^{\lambda p - \lambda - \varsigma p - \frac{2p}{q}}} a_{m,n}^p \right)^{\frac{1}{p}} \left( \sum_{r=1}^{\infty} r^{q\lambda + q\varsigma - \lambda + \frac{p}{q}} (r^{-\lambda-p\varsigma-1})^q \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{m,n}}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \right)^p \right)^{\frac{1}{q}} \\
&= C \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(mn)^{\lambda p - \lambda - \varsigma p}}{(m+n)^{\lambda p - \lambda - \varsigma p - \frac{2p}{q}}} a_{m,n}^p \right)^{\frac{1}{p}} \left( \sum_{r=1}^{\infty} r^{-\lambda-p\varsigma-1} \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{m,n}}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \right)^p \right)^{\frac{1}{q}}
\end{aligned}$$

It means

$$\begin{aligned}
& \sum_{r=1}^{\infty} r^{-\lambda-p\varsigma-1} \left( \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{a_{m,n}}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \right)^p \\
&\leq C \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(mn)^{\lambda p - \lambda - \varsigma p}}{(m+n)^{\lambda p - \lambda - \varsigma p - \frac{2p}{q}}} a_{m,n}^p \right)^{\frac{1}{p}} \left( \sum_{r=1}^{\infty} r^{-\lambda-p\varsigma-1} \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{m,n}}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \right)^p \right)^{\frac{1}{q}}
\end{aligned}$$

If we multiply both sides of the inequality (4.3) by  $\left( \sum_{r=1}^{\infty} r^{-\lambda-p\varsigma-1} \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{m,n}}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \right)^p \right)^{-\frac{1}{q}}$   
we get (4.1). Also we can get (3.1) from (4.1) and by using Hölder inequality as  
following

$$\begin{aligned}
& \sum_{r=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{m,n} b_r}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \\
&= \sum_{r=1}^{\infty} \left( r^{-\frac{\lambda}{p} - \varsigma - \frac{1}{p}} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(mn)^{\lambda p - \lambda - \varsigma p}}{(m+n)^{\lambda p - \lambda - \varsigma p - \frac{2p}{q}}} a_{m,n}^p \right) \left( r^{\frac{\lambda}{p} + \varsigma + \frac{1}{p}} b_r \right)
\end{aligned}$$

$$\begin{aligned}
&\leq \left( \sum_{r=1}^{\infty} r^{-\lambda-p\varsigma-1} \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{m,n}}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \right)^p \right)^{\frac{1}{p}} \left( \sum_{r=1}^{\infty} r^{\lambda q - \lambda + q\lambda + \frac{q}{b}} b_r \right) \\
&\leq C \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(mn)^{\lambda p - \lambda - \varsigma p}}{(m+n)^{\lambda p - \lambda - \varsigma p - \frac{2p}{q}}} a_{m,n}^p \right)^{\frac{1}{p}} \left( \sum_{r=1}^{\infty} r^{\lambda q - \lambda + q\lambda + \frac{q}{b}} b_r \right)
\end{aligned}$$

To prove (4.2), set

$$a_{m,n} = \frac{(mn)^{q\varsigma-\lambda}}{(m+n)^{q\varsigma-\lambda+2}} \left( \sum_{r=1}^{\infty} \frac{b_r}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \right)^{q-1}$$

By using (3.1), we get

$$\begin{aligned}
&\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(mn)^{q\varsigma-\lambda}}{(m+n)^{q\varsigma-\lambda+2}} \left( \sum_{r=1}^{\infty} \frac{b_r}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \right)^q \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(mn)^{q\varsigma-\lambda}}{(m+n)^{q\varsigma-\lambda+2}} \left( \sum_{r=1}^{\infty} \frac{b_r}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \right)^{q-1} \left( \sum_{r=1}^{\infty} \frac{b_r}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \right) \\
&= \sum_{r=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{m,n} b_r}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \\
&\leq C \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(mn)^{\lambda p - \lambda - \varsigma p}}{(m+n)^{\lambda p - \lambda - \varsigma p - \frac{2p}{q}}} a_{m,n}^p \right)^{\frac{1}{p}} \left( \sum_{r=1}^{\infty} r^{q\lambda + q\varsigma - \lambda + \frac{p}{q}} b_r^q \right)^{\frac{1}{q}} \\
&= C \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(mn)^{q\varsigma-\lambda}}{(m+n)^{q\varsigma-\lambda+2}} \left( \sum_{r=1}^{\infty} \frac{b_r}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \right)^q \right)^{\frac{1}{p}} \left( \sum_{r=1}^{\infty} r^{q\lambda + q\varsigma - \lambda + \frac{p}{q}} b_r^q \right)^{\frac{1}{q}}
\end{aligned}$$

It means that

$$\begin{aligned}
&\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(mn)^{q\varsigma-\lambda}}{(m+n)^{q\varsigma-\lambda+2}} \left( \sum_{r=1}^{\infty} \frac{b_r}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \right)^q \\
&\leq C \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(mn)^{q\varsigma-\lambda}}{(m+n)^{q\varsigma-\lambda+2}} \left( \sum_{r=1}^{\infty} \frac{b_r}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \right)^q \right)^{\frac{1}{p}} \left( \sum_{r=1}^{\infty} r^{q\lambda + q\varsigma - \lambda + \frac{p}{q}} b_r^q \right)^{\frac{1}{q}}
\end{aligned} \tag{4.4}$$

By multiply (4.4) by  $\left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(mn)^{q\varsigma-\lambda}}{(m+n)^{q\varsigma-\lambda+2}} \left( \sum_{r=1}^{\infty} \frac{b_r}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \right)^q \right)^{-\frac{1}{p}}$ , we get (4.2).  
Also we can get (3.1) from (4.2) and by using Höder inequality as following:

$$\begin{aligned}
& \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{a_{m,n} b_r}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{(mn)^{-\frac{qs-\lambda}{q}}}{(m+n)^{-\frac{qs-\lambda+2}{q}}} a_{m,n} \right) \left( \frac{(mn)^{\frac{qs-\lambda}{q}}}{(m+n)^{\frac{qs-\lambda+2}{q}}} \sum_{r=1}^{\infty} \frac{b_r}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \right) \\
&\leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{(mn)^{\lambda p - \lambda - ps}}{(m+n)^{-\lambda p - \lambda - ps - \frac{2p}{q}}} a_{m,n}^p \right)^{\frac{1}{p}} \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(mn)^{qs-\lambda}}{(m+n)^{qs-\lambda+2}} \left( \sum_{r=1}^{\infty} \frac{b_r}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \right)^q \right)^{\frac{1}{q}} \\
&\leq C \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(mn)^{\lambda p - \lambda - sp}}{(m+n)^{\lambda p - \lambda - sp - \frac{2p}{q}}} a_{m,n}^p \right)^{\frac{1}{p}} \left( \sum_{r=1}^{\infty} r^{\lambda q - \lambda + q\lambda + \frac{q}{b}} b_r^q \right)
\end{aligned}$$

Thus ,wo proved the equivalence relation between (4.2) and (3.1).

**Theorem 4.2 :** Using the same conditions of Theorem 3.2, we have the following inequalities:

$$\sum_{r=1}^{\infty} r^{-\lambda - ps - 1} \left( \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{a_{m,n}}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \right)^p \geq C^p \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(mn)^{\lambda p - \lambda - sp}}{(m+n)^{\lambda p - \lambda - sp - \frac{2p}{q}}} a_{m,n}^p \quad (4.5)$$

and

$$\sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{(mn)^{qs-\lambda}}{(m+n)^{qs-\lambda+2}} \left( \sum_{r=1}^{\infty} \frac{b_r}{(\frac{1}{m} + \frac{1}{n} + \frac{1}{r})^{\lambda}} \right)^q \geq C^q \sum_{r=1}^{\infty} r^{q\lambda + qs - \lambda + \frac{q}{b}} b_r^q \quad (4.6)$$

Both of (4.5) and (4.6) are equivalent to (3.2) and  $C^p$  and  $C^q$  are the best possible.

**Proof :** Since our proofs of (4.5) and (4.6) are similar to proofs of (4.1) and (4.2), so we leave it.

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