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RELATION-THEORETIC COUPLED FIXED POINT THEOREMS

FARUK SK, FAIZAN AHMAD KHAN*, QAMRUL HAQ KHAN

Abstract. In this paper, we introduce mixed $\mathcal{R}$-monotone property of a mapping and utilize the same to investigate existence and uniqueness of coupled fixed points in a metric space endowed with a binary relation $\mathcal{R}$. Moreover, we present some coupled fixed point results for mappings without mixed monotone property using relation-theoretic approach. Our results generalize some well-known coupled fixed points theorems. Further, we give some illustrative examples in support of our results.

1. Introduction

In 1922, S. Banach [3] proved a fundamental result in metric fixed point theory called Banach contraction principle (BCP) which states that "every contraction mapping in complete metric space has a unique fixed point". Several researchers generalized this result in various directions. In 1986, Turinici [29] reported fixed point theorems in ordered metric spaces. Coupled fixed points (in short, CFP) was introduced by Opoitsev, see [21, 22, 23] and it is further studied by Guo et al.[11]. CFP theorems made a vital chapter in metric fixed theory in recent times.

Definition 1.1. [10] Consider $(\mathcal{M}, \preceq)$ as a partial ordered set. Then a mapping $\mathcal{F} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is said to satisfy mixed monotone property(in short, MMP) if

$$\xi_1 \preceq \xi_2 \implies \mathcal{F}(\xi_1, \eta) \preceq \mathcal{F}(\xi_2, \eta) \quad \forall \xi_1, \xi_2, \eta \in \mathcal{M}$$

and

$$\eta_1 \preceq \eta_2 \implies \mathcal{F}(\xi, \eta_2) \preceq \mathcal{F}(\xi, \eta_1) \quad \forall \eta_1, \eta_2, \xi \in \mathcal{M}.$$

In 2006, Bhaskar and Lakshmikantham [10] proved the following result concerning a contractive type condition:

Theorem 1.2. [10] *Consider a continuous mapping $\mathcal{F} : \mathcal{M}\times\mathcal{M} \to \mathcal{M}$ on a complete ordered metric space $(\mathcal{M}, d, \preceq)$ with the following hypotheses :*

(a) $\mathcal{F}$ satisfies mixed monotone property,

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*Correspondence: fkhan@ut.edu.sa (F.A.K).

(b) there exist $\xi_0, \eta_0 \in \mathcal{M}$ *such that*

$$\xi_0 \preceq \mathcal{F}(\xi_0, \eta_0) \quad \text{and} \quad \eta_0 \preceq \mathcal{F}(\eta_0, \xi_0),$$

(c) there eists $k \in [0, 1)$ *with*

$$d(\mathcal{F}(\xi, \eta), \mathcal{F}(r, s)) \leq \frac{k}{2}[d(\xi, r) + d(\eta, s)] \quad \forall\ \xi \succeq r, \quad \eta \preceq s.$$

Then $\mathcal{F}$ *has coupled fixed points.*

Using variants of the mixed monotone condition, many authors established numerous CFP theorems in partial ordered metric spaces, which are useful in applied mathematics, see [6, 12, 13, 15, 18, 24, 26]. Two elements ξ, η in a partial ordered set $(\mathcal{M}, \preceq)$ is said to be comparable if $\xi \preceq \eta$ or $\eta \preceq \xi$. We denote it by $\xi \asymp \eta$. In 2012, Doric et al.[8] established the following result without mixed monotone property:

Theorem 1.3. [8] *Let* $(\mathcal{M}, d, \preceq)$ *a complete partial ordered metric space and* $\mathcal{F} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ *be a mapping. Suppose that the following conditions hold:*

(a) for all $\xi, \eta, s \in \mathcal{M}$ *if* $\xi \asymp \mathcal{F}(\xi, \eta)$ *then* $\mathcal{F}(\xi, \eta) \asymp \mathcal{F}(\mathcal{F}(\xi, \eta), s)$,

(b) there exist $\xi_0, \eta_0 \in \mathcal{M}$ *such that* $\xi_0 \asymp \mathcal{F}(\xi_0, \eta_0)$ *and* $\eta_0 \asymp \mathcal{F}(\eta_0, \xi_0)$,

(c) there exists $k \in [0, 1)$ *such that*

$$d(\mathcal{F}(\xi, \eta), \mathcal{F}(r, s)) \leq k \ max\{d(\xi, r), d(\eta, s)\}$$

for all $\xi, \eta, r, s \in \mathcal{M}$ *satisfying* $\xi \asymp r$ *and* $\eta \asymp s$,

(d) $\mathcal{F}$ *is continuous or if* $\xi_n \to \xi$ *when* $n \to \infty$ *in* $\mathcal{M}$, *then* $\xi_n \asymp \xi$ *for* n *sufficiently large.*

Then $\mathcal{F}$ *has a coupled fixed point.*

This paper aims to introduce the concept of mixed $\mathcal{R}$-monotone property of a mapping and prove coupled fixed point theorems for such mappings in a complete metric space endowed with a binary relation $\mathcal{R}$. Also, we investigate existence and uniqueness of coupled fixed points for mappings without mixed monotone property in a complete metric space endowed with a binary relation $\mathcal{R}$. Further, we give examples to substantiate the usefulness of our findings where existing results cannot be applied.

2. Preliminaries

We will go over some basic definitions in this section, which will be needed to run our primary findings. We refer to $\mathbb{N} \cup \{0\}$ as $\mathbb{N}_0$ in the whole paper.

Definition 2.1. [10, 7] Consider the mapping $\mathcal{F} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, where $\mathcal{M}$ is a nonempty set. Then an element (ξ, η) is said to be "coupled fixed point"(in short, CFP) if

$$\mathcal{F}(\xi, \eta) = \xi \text{ and } \mathcal{F}(\eta, \xi) = \eta.$$

A coupled fixed point (ξ, η) is said to be a strong CFP if $\xi = \eta$.

Definition 2.2. [17] Let $\mathcal{M} \neq \emptyset$ be a set. A "binary relation" is a subset $\mathcal{R}$ of $\mathcal{M}^2$. The subsets, $\mathcal{M}^2$ and $\emptyset$ of $\mathcal{M}^2$ are termed as "universal relation" & "empty relation" respectively.

Definition 2.3. [1] Let $\mathcal{M} \neq \emptyset$ be a set with a binary relation $\mathcal{R}$. If either $(\xi, \eta) \in \mathcal{R}$ or $(\eta, \xi) \in \mathcal{R}$ for $\xi, \eta \in \mathcal{M}$, then ξ and η are called as $\mathcal{R}$-comparative. $[\xi, \eta] \in \mathcal{R}$ is the notion for it.

Definition 2.4. [17, 20, 9, 27, 28, 25] Let $\mathcal{M} \neq \emptyset$ be a set with a binary relation $\mathcal{R}$. Then, the relation $\mathcal{R}$ is called

(a) "amorphous" if $\mathcal{R}$ has no precise attribute,
(b) "reflexive" if $(\xi, \xi) \in \mathcal{R} \ \forall \ \xi \in \mathcal{M}$,
(c) "symmetric" if $(\xi, \eta) \in \mathcal{R} \implies (\eta, \xi) \in \mathcal{R}$,
(d) "anti-symmetric" if $(\xi, \eta) \in \mathcal{R}$ and $(\eta, \xi) \in \mathcal{R} \implies \xi = \eta$,
(e) "transitive" if $(\xi, \eta) \in \mathcal{R}$ and $(\eta, w) \in \mathcal{R} \implies (\xi, w) \in \mathcal{R}$,
(f) "partial order" if $\mathcal{R}$ is "reflexive", "anti-symmetric" and "transitive".

Definition 2.5. [17] Let $\mathcal{R}$ be a binary relation on a set $\mathcal{M} \neq \emptyset$. Then,

$$\mathcal{R}^{-1} = \{(\xi, \eta) \in \mathcal{M}^2 : (\eta, \xi) \in \mathcal{R}\} \text{ and } \mathcal{R}^s = \mathcal{R} \cup \mathcal{R}^{-1}.$$

Proposition 2.6. [1] *For a binary relation $\mathcal{R}$ defined on a non-empty set $\mathcal{M}$,*

$$(\xi, \eta) \in \mathcal{R}^s \implies [\xi, \eta] \in \mathcal{R}.$$

Definition 2.7. [1] Let $\mathcal{M} \neq \emptyset$ be a set with a binary relation $\mathcal{R}$. A sequence $\{\xi_{\Bbbk}\} \subset \mathcal{M}$ is called $\mathcal{R}$-preserving if

$$(\xi_{\Bbbk}, \xi_{\Bbbk+1}) \in \mathcal{R} \quad \forall \ \Bbbk \ \in \ \mathcal{K}_0.$$

Definition 2.8. [1] Let $\mathcal{M} \neq \emptyset$ be a set with a metric d together with a binary relation $\mathcal{R}$. Consider the $\mathcal{R}$-preserving sequence $\{\xi_{\Bbbk}\} \subset \mathcal{M}$ such that $\xi_{\Bbbk} \xrightarrow{d} \xi$. Then, $\mathcal{R}$ is called "d-self-closed" if there exists a subsequence $\{\xi_{\Bbbk_p}\}$ of $\{\xi_{\Bbbk}\}$ with $[\xi_{\Bbbk_p}, \xi] \in \mathcal{R} \ \forall \ p \in \mathcal{K}_0$.

Definition 2.9. [2] Let $\mathcal{M}$ be a nonempty set and $\mathcal{R}$ a binary relation on $\mathcal{M}$. Then, $\mathcal{S} \subseteq \mathcal{M}$ is called "$\mathcal{R}$-directed" if for each $\xi, \eta \in \mathcal{S}$, there exists $\rho \in \mathcal{M}$ such that $(\xi, \rho) \in \mathcal{R}$ and $(\eta, \rho) \in \mathcal{R}$.

Definition 2.10. [14] Let $\mathcal{R}$ be a binary relation defined on a nonempty set $\mathcal{M}$. Then, for $\xi, \eta \in \mathcal{M}$, a finite sequence $\{\xi_0, \xi_1, ..., \xi_p\} \subset \mathcal{M}$ satisfying the following conditions:

- $(\xi_\ell, \xi_{\ell+1}) \in \mathcal{R}$ for each ℓ $(0 \leq \ell \leq p-1)$,
- $\xi_0 = \xi$ and $\xi_p = \eta$,

is said to be a path of length p in $\mathcal{R}$ from ξ to η.

Definition 2.11. [2] Let $\mathcal{M} \neq \emptyset$ be a set equipped with a binary relation $\mathcal{R}$. Then, $\mathcal{S} \subseteq \mathcal{M}$ is called "$\mathcal{R}$-connected" if for each $\xi, \eta \in \mathcal{M}$, there exist a path in $\mathcal{R}$ from ξ to η.

Inspired by Bhaskar and Lakshmikantham, we introduce a relation-theoretic variant of mixed monotone property as follows:

Definition 2.12. Let $\mathcal{M} \neq \emptyset$ be a set equipped with a binary relation $\mathcal{R}$. Then a mapping $\mathcal{F} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is said to have a mixed $\mathcal{R}$-monotone property if

$$(\xi_1, \xi_2) \in \mathcal{R} \implies (\mathcal{F}(\xi_1, \eta), \mathcal{F}(\xi_2, \eta)) \in \mathcal{R} \quad \forall \xi_1, \xi_2, \eta \in \mathcal{M}$$

and

$$(\eta_1, \eta_2) \in \mathcal{R} \implies (\mathcal{F}(\xi, \eta_2), \mathcal{F}(\xi, \eta_1)) \in \mathcal{R} \quad \forall \eta_1, \eta_2, \xi \in \mathcal{M}.$$

Note that if a mapping $\mathcal{F}$ satisfies mixed monotone property then it automatically satisfies mixed $\mathcal{R}$-monotone property, but not conversely.

Example 2.13. Let $\mathcal{M} = [1, 4]$ be a partial ordered set with usual partial order. Consider the function $\mathcal{F} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ given by

$$\mathcal{F}(\xi, \eta) = \begin{cases} 2 & \text{if } \xi \in \{2, 4\} \\ 4 & \text{otherwise.} \end{cases}$$

Notice that $\mathcal{F}(1, 2) \geq \mathcal{F}(2, 2)$ but $1 \leq 2$. So, $\mathcal{F}$ does not satisfy MMP. Now take a binary relation $\mathcal{R}$ on $\mathcal{M}$ defined by

$$\mathcal{R} = \{(2, 4), (4, 2), (2, 2), (4, 4)\}.$$

Then, $\mathcal{F}$ satisfy the conditions of Definition 2.12. Hence, $\mathcal{F}$ satisfies mixed $\mathcal{R}$-monotone property.

3. Coupled fixed point theorems with mixed $\mathcal{R}$-monotone property

In this section, we give some results on the existence and uniqueness of CFPs for a mapping with mixed $\mathcal{R}$-monotone property.

Theorem 3.1. *Let $(\mathcal{M}, d)$ be a complete metric space endowed with a transitive binary relation $\mathcal{R}$ on $\mathcal{M}$ and let $\mathcal{F} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ be a mapping. Suppose that the following conditions hold:*

(a) there exist ξ_0, $\eta_0 \in \mathcal{M}$ such that $(\xi_0, \mathcal{F}(\xi_0, \eta_0)) \in \mathcal{R}$ and $(\mathcal{F}(\eta_0, \xi_0), \eta_0) \in \mathcal{R}$,
(b) $\mathcal{F}$ satisfies mixed $\mathcal{R}$-monotone property,
(c) either $\mathcal{F}$ is continuous or $\mathcal{R}$ is "d-self-closed",
(d) there exist $k \in [0, 1)$ such that

$$d(\mathcal{F}(\xi, \eta), \mathcal{F}(r, s)) \leq k\ max\{d(\xi, r), d(\eta, s)\}$$

for all $\xi, \eta, r, s \in \mathcal{M}$ with $(r, \xi) \in \mathcal{R}$ and $(\eta, s) \in \mathcal{R}$.

Then $\mathcal{F}$ has a CFP.

Proof. Assumption (a) confirms the availability of $\xi_0, \eta_0 \in \mathcal{M}$ such that

$$(\xi_0, \mathcal{F}(\xi_0, \eta_0)) \in \mathcal{R} \text{ and } (\mathcal{F}(\eta_0, \xi_0), \eta_0) \in \mathcal{R}. \tag{3.1}$$

Suppose $\mathcal{F}(\xi_0, \eta_0) = \xi_1$ and $\mathcal{F}(\eta_0, \xi_0) = \eta_1$. Then we can choose ξ_2, $\eta_2 \in \mathcal{M}$ such that $\mathcal{F}(\xi_1, \eta_1) = \xi_2$ and $\mathcal{F}(\eta_1, \xi_1) = \eta_2$. We denote

$$\mathcal{F}^2(\xi_0, \eta_0) = \mathcal{F}(\mathcal{F}(\xi_0, \eta_0), \mathcal{F}(\eta_0, \xi_0)) = \mathcal{F}(\xi_1, \eta_1) = \xi_2$$
$$\mathcal{F}^2(\eta_0, \xi_0) = \mathcal{F}(\mathcal{F}(\eta_0, \xi_0), \mathcal{F}(\xi_0, \eta_0)) = \mathcal{F}(\eta_1, \xi_1) = \eta_2.$$

In a similar manner, we develop inductively sequences $\{\xi_n\}$ and $\{\eta_n\}$ such that

$$\xi_n = \mathcal{F}^n(\xi_0, \eta_0) \text{ and } \eta_n = \mathcal{F}^n(\eta_0, \xi_0). \tag{3.2}$$

Now, we will show that

$$(\xi_n, \xi_{n+1}) \in \mathcal{R} \text{ and } (\eta_{n+1}, \eta_n) \in \mathcal{R} \quad \forall\ n \in \mathbb{N}_0. \tag{3.3}$$

Now, mathematical induction is to be used to prove this fact.
From (3.1), $(\xi_0, \xi_1) \in \mathcal{R}$ and $(\eta_1, \eta_0) \in \mathcal{R}$. So, (3.3) is true for $n = 0$. Suppose that for $t > 0$,

$$(\xi_t, \xi_{t+1}) \in \mathcal{R} \text{ and } (\eta_{t+1}, \eta_t) \in \mathcal{R}.$$

Now, we confirm that (3.3) remains true for $n = t+1$ also.
Since $(\xi_t, \xi_{t+1}) \in \mathcal{R}$, using assumption (b) we get,

$$(\mathcal{F}(\xi_t, \eta_t), \mathcal{F}(\xi_{t+1}, \eta_{t+1})) \in \mathcal{R}$$

$$\implies (\mathcal{F}(\mathcal{F}^t(\xi_0, \eta_0), \mathcal{F}^t(\eta_0, \xi_0)), \mathcal{F}(\mathcal{F}^{t+1}(\xi_0, \eta_0), \mathcal{F}^{t+1}(\eta_0, \xi_0))) \in \mathcal{R}$$

$$\implies (\mathcal{F}^{t+1}(\xi_0, \eta_0), \mathcal{F}^{t+2}(\xi_0, \eta_0)) \in \mathcal{R}$$

$$\implies (\xi_{t+1}, \xi_{t+2}) \in \mathcal{R}.$$

Similarly, we can show $(\eta_{r+2}, \eta_{r+1}) \in \mathcal{R}$. Hence, (3.3) holds for all $n \in \mathbb{N}_0$. Now, using assumption (d), we get

$$\begin{aligned}
d(\xi_{n+1}, \xi_n) &= d(\mathcal{F}^{n+1}(\xi_0, \eta_0), \mathcal{F}^n(\xi_0, \eta_0)) \\
&= d(\mathcal{F}(\mathcal{F}^n(\xi_0, \eta_0), \mathcal{F}^n(\eta_0, \xi_0)), \mathcal{F}(\mathcal{F}^{n-1}(\xi_0, \eta_0), \mathcal{F}^{n-1}(\eta_0, \xi_0))) \\
&\leq k\ max\{d(\mathcal{F}^n(\xi_0, \eta_0), \mathcal{F}^{n-1}(\xi_0, \eta_0)), d(\mathcal{F}^n(\eta_0, \xi_0), \mathcal{F}^{n-1}(\eta_0, \xi_0))\} \\
&\vdots \\
&\leq k^n\ max\{d(\mathcal{F}(\xi_0, \eta_0), \xi_0), d(\mathcal{F}(\eta_0, \xi_0), \eta_0))\}.
\end{aligned}$$

Similarly,

$$d(\eta_{n+1}, \eta_n) \leq k^n\ max\{d(\mathcal{F}(\eta_0, \xi_0), \eta_0), d(\mathcal{F}(\xi_0, \eta_0), \xi_0))\}.$$

For $m > n$, we obtain

$$\begin{aligned}
d(\xi_n, \xi_m) &\leq d(\xi_n, \xi_{n+1}) + d(\xi_{n+1}, \xi_{n+2}) + ... + d(\xi_{m-1}, \xi_m) \\
&\leq (k^n + k^{n+1} + ... + k^{m-1})\ max\{d(\mathcal{F}(\xi_0, \eta_0), \xi_0), d(\mathcal{F}(\eta_0, \xi_0), \eta_0)\} \\
&= \frac{k^n}{1-k}\ max\{d(\mathcal{F}(\xi_0, \eta_0), \xi_0), d(\mathcal{F}(\eta_0, \xi_0), \eta_0)\} \\
&\to 0,\ when\ n, m \to \infty.
\end{aligned}$$

Therefore, $\{\xi_n\}$ is a Cauchy sequence. Similarly, the sequence $\{\eta_n\}$ is also a Cauchy. Completeness of $\mathcal{M}$ affirms the availability of $\xi, \eta \in \mathcal{M}$ such that

$$\lim_{n\to\infty} \xi_n = \xi \quad \text{and} \quad \lim_{n\to\infty} \eta_n = \eta$$

which means that,

$$\lim_{n\to\infty} \mathcal{F}^n(\xi_0, \eta_0) = \xi \quad \text{and} \quad \lim_{n\to\infty} \mathcal{F}^n(\eta_0, \xi_0) = \eta.$$

Now, let $\epsilon > 0$. Since $\mathcal{F}$ is continuous at (ξ, η), for a given $\frac{\epsilon}{2} > 0$, there exist a $\delta > 0$ such that $\max\{d(\xi, r), d(\eta, s)\} < \delta$ implies

$$d(\mathcal{F}(\xi, \eta), \mathcal{F}(r, s)) < \frac{\epsilon}{2}.$$

Since $\{\mathcal{F}^n(\xi_0, \eta_0)\} \to \xi$ and $\{\mathcal{F}^n(\eta_0, \xi_0)\} \to \eta$, for $\epsilon' = min(\frac{\epsilon}{2}, \frac{\delta}{2}) > 0$, there exists n_0, m_0 such that for $n \geq n_0$, $m \geq m_0$,

$$d(\mathcal{F}^n(\xi_0, \eta_0), \xi) < \epsilon' \quad and \quad d(\mathcal{F}^m(\eta_0, \xi_0), \eta) < \epsilon'$$

Now, for $n \in \mathbb{N}$, $n \geq max\{n_0, m_0\}$,

$$\begin{aligned} d(\mathcal{F}(\xi,\eta),\xi) &\leq d(\mathcal{F}(\xi,\eta),\mathcal{F}^{n+1}(\xi_0,\eta_0)) + d(\mathcal{F}^{n+1}(\xi_0,\eta_0),\xi) \\ &= d(\mathcal{F}(\xi,\eta),\mathcal{F}(\mathcal{F}^{n}(\xi_0,\eta_0),\mathcal{F}^{n}(\eta_0,\xi_0))) + d(\mathcal{F}^{n+1}(\xi_0,\eta_0),\xi) \\ &< \frac{\epsilon}{2} + \epsilon^{'} \\ &\leq \epsilon. \end{aligned}$$

This proves that $\mathcal{F}(\xi,\eta) = \xi$. In this manner, we obtain $\mathcal{F}(\eta,\xi) = \eta$.
Alternately, since $\{\xi_n\}$ converges to ξ, then by assumption (c), there exists a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ such that

$$[\xi_{n_k}, \xi] \in \mathcal{R} \quad \forall\ k\ \in\ \mathbb{N}_0. \tag{3.4}$$

Similarly, for the sequence $\{\eta_n\}$, there exists a subsequence $\{\eta_{n_k}\}$ of $\{\eta_n\}$ satisfying

$$[\eta_{n_k}, \eta] \in \mathcal{R} \quad \forall\ k\ \in\ \mathbb{N}_0. \tag{3.5}$$

Let $\epsilon > 0$. Since $\{\mathcal{F}^n(\xi_0,\eta_0)\} \to \xi$ and $\{\mathcal{F}^n(\eta_0,\xi_0)\} \to \eta$, there exists $n_1 \in \mathbb{N}$, $n_2 \in \mathbb{N}$ such that ,

$$d(\mathcal{F}^n(\xi_0,\eta_0),\xi) < \frac{\epsilon}{2} \quad and \quad d(\mathcal{F}^m(\eta_0,\xi_0),\eta) < \frac{\epsilon}{2}$$

for all $n \geq n_1$ and $m \geq n_2$. Taking $n \in \mathbb{N}$, $n \geq max\{n_1, n_2\}$ and using (3.4) and (3.5), we obtain

$$\begin{aligned} d(\mathcal{F}(\xi,\eta),\xi) &\leq d(\mathcal{F}(\xi,\eta),\mathcal{F}^{n_{k+1}}(\xi_0,\eta_0)) + d(\mathcal{F}^{n_{k+1}}(\xi_0,\eta_0),\xi) \\ &= d(\mathcal{F}(\xi,\eta),\mathcal{F}(\mathcal{F}^{n_k}(\xi_0,\eta_0),\mathcal{F}^{n_k}(\eta_0,\xi_0))) + d(\mathcal{F}^{n_{k+1}}(\xi_0,\eta_0),\xi) \\ &\leq k\ max\{d(\xi,\mathcal{F}^{n_k}(\xi_0,\eta_0)), d(\eta,\mathcal{F}^{n_k}(\eta_0,\xi_0))\} + d(\mathcal{F}^{n_{k+1}}(\xi_0,\eta_0),\xi) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

This insinuates that $\mathcal{F}(\xi,\eta) = \xi$. In the same manner, we obtain that $\mathcal{F}(\eta,\xi) = \eta$ which ends the proof. □

The above theorem only guarantees the existence of coupled fixed points. For uniqueness, we require an additional condition to be hold. This is the purpose of the next theorem.

Theorem 3.2. *In addition to the conditions of Theorem 3.1, assume that*
(e) $\mathcal{F}(\mathcal{M}^2)$ is "$\mathcal{R}^s$-connected".
Then $\mathcal{F}$ has a unique CFP.

Proof. Theorem 3.1 confirms us the availability of at least one coupled fixed point. Suppose that (ξ,η), (ξ^*,η^*) are two CFPs of $\mathcal{F}$. Then we have,

$$\mathcal{F}(\xi,\eta) = \xi\ ;\ \mathcal{F}(\eta,\xi) = \eta \quad and \quad \mathcal{F}(\xi^*,\eta^*) = \xi^*\ ;\ \mathcal{F}(\eta^*,\xi^*) = \eta^*$$

We show that $\xi = \xi^*$ and $\eta = \eta^*$. By assumption (e), there exist a path $\{\rho_0, \rho_1, ..., \rho_{k_1}\}$ of length k_1 in $\mathcal{R}^s$ from ξ to ξ^* so that

$$\rho_0 = \xi, \quad \rho_{k_1} = \xi^*, \quad [\rho_i, \rho_{i+1}] \in \mathcal{R} \quad \forall\ i\ (0 \leq i \leq k_1 - 1).$$

Similarly, there exist a path $\{\rho_0^{'}, \rho_1^{'}, ..., \rho_{k_2}^{'}\}$ of length k_2 in $\mathcal{R}^s$ from η to η^* so that

$$\rho_0^{'} = \eta, \quad \rho_{k_2}^{'} = \eta^*, \quad [\rho_i^{'}, \rho_{i+1}^{'}] \in \mathcal{R} \quad \forall\ i\ (0 \leq i \leq k_2 - 1).$$

Using assumption (b) we get,

$$[\mathcal{F}^n(\rho_i,\rho_i^{'}),\mathcal{F}^n(\rho_{i+1},\rho_{i+1}^{'})] \in \mathcal{R}\ \forall\ i\ (0\leq i\leq k_1-1) \text{ and } \forall\ n\in\mathbb{N}$$

and

$$[\mathcal{F}^n(\rho_i^{'},\rho_i),\mathcal{F}^n(\rho_{i+1}^{'},\rho_{i+1})] \in \mathcal{R}\ \forall\ i\ (0\leq i\leq k_2-1) \text{ and } \forall\ n\in\mathbb{N}.$$

For $m=\min\{k_1,k_2\}$,

$$\begin{aligned} d(\xi,\xi^*) &= d(\mathcal{F}^n(\rho_0,\rho_0^{'}),\mathcal{F}^n(\rho_{k_1},\rho_{k_2}^{'})) \\ &\leq \sum_{i=0}^{m-1} d(\mathcal{F}^n(\rho_i,\rho_i^{'}),\mathcal{F}^n(\rho_{i+1},\rho_{i+1}^{'})) \\ &\leq k^n\sum_{i=0}^{m-1}[d(\rho_i,\rho_{i+1})+d(\rho_i^{'},\rho_{i+1}^{'})] \\ &\to 0 \quad as\ n\ \to\infty \end{aligned} \tag{3.6}$$

Hence, $\xi=\xi^*$. Similarly, we can show that $\eta=\eta^*$. Therefore, $\mathcal{F}$ has a unique CFP. □

Corollary 3.3. *Theorem 3.2 conveys the same if we utilize the below condition instead of using condition* (e).
(e') $\mathcal{F}(\mathcal{M}^2)$ *is "$\mathcal{R}^s$-directed".*

Proof. Suppose (e') holds. Then for each $\rho,\sigma\in\mathcal{F}(\mathcal{M}^2)$, there exists $\mu\in\mathcal{M}$ such that $[\rho,\mu]\in\mathcal{R}$ and $[\mu,\sigma]\in\mathcal{R}$, which means there exists a path $\{\rho,\sigma,\mu\}$ of length 2. Then, from Theorem 3.2, the conclusion is immediate. □

Corollary 3.4. *Along with the conditions of Theorem 3.1, if for terms of the sequences $\{\xi_n\}$, $\{\eta_n\}$ defined by $\xi_n=\mathcal{F}^n(\xi_0,\eta_0)$ and $\eta_n=\mathcal{F}^n(\eta_0,\xi_0)$, $[\xi_n,\eta_n]\in\mathcal{R}$ for n sufficiently large, then $\mathcal{F}$ will have a strong coupled fixed point.*

Proof. We need to only show that $\xi=\eta$. Suppose that $[\xi_n,\eta_n]\in\mathcal{R}$ for n sufficiently large. Notice that

$$\begin{aligned} d(\mathcal{F}(\xi_n,\eta_n),\mathcal{F}(\eta_n,\xi_n)) &\leq k\ max\{d(\xi_n,\eta_n),d(\eta_n,\xi_n)\} \\ &= kd(\xi_n,\eta_n), \end{aligned}$$

i.e.,

$$d(\mathcal{F}(\xi_n,\eta_n),\mathcal{F}(\eta_n,\xi_n))\leq kd(\xi_n,\eta_n).$$

By triangle inequality, we have

$$\begin{aligned} d(\xi,\eta) &\leq d(\xi,\xi_{n+1})+d(\xi_{n+1},\eta_{n+1})+d(\eta_{n+1},\eta) \\ &= d(\xi,\xi_{n+1})+d(\mathcal{F}(\xi_n,\eta_n),\mathcal{F}(\eta_n,\xi_n))+d(\eta_{n+1},\eta) \\ &\leq d(\xi,\xi_{n+1})+k\ max\{d(\xi_n,\eta_n),d(\eta_n,\xi_n)\}+d(\eta_{n+1},\eta) \\ &= d(\xi,\xi_{n+1})+kd(\xi_n,\eta_n)+d(\eta_{n+1},\eta). \end{aligned}$$

Now when we tend n to ∞, we get $\xi_n\to\xi$ and $\eta_n\to\eta$, implying thereby $d(\xi_n,\eta_n)\to d(\xi,\eta)$.

Therefore,

$$\begin{aligned} d(\xi,\eta) &\leq d(\xi,\xi)+kd(\xi,\eta)+d(\eta,\eta) \\ &\leq kd(\xi,\eta) \\ (1-k)d(\xi,\eta) &\leq 0 \\ d(\xi,\eta) &\leq 0 \end{aligned}$$

Hence, $d(\xi,\eta)=0$ which gives $\xi=\eta$. □

Remark 3.5. Notice that for $l,m\geq 0$ and $l+m<1$, we have

$$ld(x,u)+md(y,v)\leq \max\{d(\xi,r),d(\eta,s)\}.$$

So, Theorem 3.1 stands valid if the right-hand side of condition (d) in Theorem 3.1 is replaced by $ld(x,u)+md(y,v)$, where $x,y,u,v\in\mathcal{M}$, for some $l,m\geq 0$, $l+m<1$, which reduces to the contractive condition of Theorem 1.2 if $l=m$. Therefore, using a particular binary relation $\mathcal{R}=\{(\xi,\eta)\in\mathcal{M}\times\mathcal{M}:\xi\leq\eta\}$ with the metric space $\mathcal{M}$ in Theorem 3.1, we can obtain Theorem 1.2.

Example 3.6. Let $\mathcal{M}=[1,4]$ be a complete partial ordered set with usual metric d and endowed with partial order $\leq$. Now, define $\mathcal{F}:\mathcal{M}\times\mathcal{M}\to\mathcal{M}$ by

$$\mathcal{F}(\xi,\eta)=\begin{cases}2 & \text{if } \xi\in\{2,4\}\\ 4 & \text{otherwise.}\end{cases}$$

Notice that we cannot use the mixed monotone property of $\mathcal{F}$, since $1\leq 2$ but $\mathcal{F}(1,2)\geq\mathcal{F}(2,2)$. So, we cannot apply the existing Theorem 1.2 in this case. Now, consider the metric space $(\mathcal{M},d)$ endowed with the binary relation given by

$$\mathcal{R}=\{(2,4),(4,2),(2,2),(4,4)\}.$$

By routine calculation, we obtain the following

(a) for $2,4\in\mathcal{M}$, $(2,\mathcal{F}(2,4))\in\mathcal{R}$ and $(\mathcal{F}(4,2),4)\in\mathcal{R}$,
(b) $\mathcal{F}$ satisfies mixed $\mathcal{R}$-monotone property,
(c) $\mathcal{R}$ is d-self-closed,
(d) for $k=\frac{1}{2}$, we have

$$d(\mathcal{F}(\xi,\eta),\mathcal{F}(r,s)) \quad\leq\quad k\max\{d(\xi,r),d(\eta,s)\}$$

for all $\xi,\eta,r,s\in\mathcal{M}$ such that $(r,\xi)\in\mathcal{R}$ and $(\eta,s)\in\mathcal{R}$,
(e) $\mathcal{F}(\mathcal{M}^2)$ is $\mathcal{R}^s$ connected.

Now, applying Theorem 3.2, we confirm that $\mathcal{F}$ has a unique CFP (namely, $(2,2)$).

4. **Coupled fixed point theorems without mixed monotone property**

Inspired by Doric et al.[8], we present CFP theorems for mappings without MMP in a complete metric space endowed with a binary relation $\mathcal{R}$.

Theorem 4.1. *Let $(\mathcal{M},d)$ be a complete metric space endowed with a binary relation $\mathcal{R}$ on $\mathcal{M}$, and $\mathcal{F}:\mathcal{M}\times\mathcal{M}\to\mathcal{M}$ be a mapping. Suppose that the following conditions hold:*

(a) there exist ξ_0, $\eta_0\in\mathcal{M}$ such that $[\xi_0,\mathcal{F}(\xi_0,\eta_0)]\in\mathcal{R}$ and $[\eta_0,\mathcal{F}(\eta_0,\xi_0)]\in\mathcal{R}$,
(b) for all $\xi,\eta,s\in\mathcal{M}$, if $[\xi,\mathcal{F}(\xi,\eta)]\in\mathcal{R}$ then $[\mathcal{F}(\xi,\eta),\mathcal{F}(\mathcal{F}(\xi,\eta),v)]\in\mathcal{R}$,
(c) either $\mathcal{F}$ is continuous or if $\xi_n\to\xi$ in $\mathcal{M}$, then there exist a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ such that $[\xi_{n_k},\xi]$ $\forall\ n\in\mathbb{N}_0$,

(d) there exist $k \in [0,1)$ such that

$$d(\mathcal{F}(\xi,\eta),\mathcal{F}(r,s)) \leq k\ max\{d(\xi,r),d(\eta,s)\}$$

for all $\xi,\eta,r,s \in \mathcal{M}$ with $[\xi,r] \in \mathcal{R}$ and $[s,\eta] \in \mathcal{R}$.

Then $\mathcal{F}$ has a CFP.

Proof. Assumption (a) confirms the availability of $\xi_0,\eta_0 \in \mathcal{M}$ such that

$$[\xi_0,\mathcal{F}(\xi_0,\eta_0)] \in \mathcal{R} \text{ and } [\eta_0,\mathcal{F}(\eta_0,\xi_0)] \in \mathcal{R}. \tag{4.1}$$

On similar consideration in Theorem 3.1, we have

$$\xi_n = \mathcal{F}^n(\xi_0,\eta_0) \ and\ \eta_n = \mathcal{F}^n(\eta_0,\xi_0). \tag{4.2}$$

Now, we will show that

$$[\xi_n,\xi_{n+1}] \in \mathcal{R} \ and\ [\eta_n,\eta_{n+1}] \in \mathcal{R} \quad \forall\ n \in \mathbb{N}_0. \tag{4.3}$$

Now, mathematical induction is to be used to prove this fact.
From (4.1) we get, $[\xi_0,\xi_1] \in \mathcal{R}$ and $[\eta_0,\eta_1] \in \mathcal{R}$.
Now, suppose

$$[\xi_t,\xi_{t+1}] \in \mathcal{R} \text{ and } [\eta_t,\eta_{t+1}] \in \mathcal{R}.$$

Now we have to show that (4.3) holds for $n = t+1$ also.
From (4.2) we get,

$$[\xi_t,\xi_{t+1}] = [F^t(\xi_0,\eta_0),F^{t+1}(\xi_0,\eta_0)] \in \mathcal{R}.$$

Then by assumption (b),

$$[\mathcal{F}^{t+1}(\xi_0,\eta_0),\mathcal{F}(\mathcal{F}^{t+1}(\xi_0,\eta_0),\mathcal{F}^{t+1}(\eta_0,\xi_0))] \in \mathcal{R}$$
$$\implies [\mathcal{F}^{t+1}(\xi_0,\eta_0),\mathcal{F}^{t+2}(\xi_0,\eta_0)] \in \mathcal{R}$$
$$\implies [\xi_{t+1},\xi_{t+2}] \in \mathcal{R}.$$

Hence, (4.3) is true for $n = t+1$ also. Thus, $[\xi_n,\xi_{n+1}] \in \mathcal{R}$. Similarly, we can show that $[\eta_n,\eta_{n+1}] \in \mathcal{R}$. Now, in a similar process as in Theorem 3.1, we can claim that $\mathcal{F}$ has a CFP. □

Now, we prove that the CFPs in Theorem 4.1 is in fact unique under some additional conditions. This is the purpose of our next theorem. To do this, for the given binary relation $\mathcal{R}$ on $\mathcal{M}$, we equip the set $\mathcal{M} \times \mathcal{M}$ with the binary relation $\mathcal{R}'$ given by

$$\text{“}((\xi,\eta),(\xi',\eta')) \in \mathcal{R}' \iff (\xi,\xi') \in \mathcal{R} \ and\ (\eta',\eta) \in \mathcal{R}\text{”}$$

Theorem 4.2. *Suppose that Theorem 4.1 satisfy the following condition additionally:*
(e) for any two elements $(\xi,\eta),(\xi^*,\eta^*) \in \mathcal{M} \times \mathcal{M}$, there exists $(\rho,\sigma) \in \mathcal{M} \times \mathcal{M}$ such that

(i) $[(\mathcal{F}(\rho,\sigma),\mathcal{F}(\sigma,\rho)),(\mathcal{F}(\xi,\eta),\mathcal{F}(\eta,\xi))] \in \mathcal{R}'$,
(ii) $[(\mathcal{F}(\rho,\sigma),\mathcal{F}(\sigma,\rho)),(\mathcal{F}(\xi^*,\eta^*),\mathcal{F}(\eta^*,\xi^*))] \in \mathcal{R}'$.

Then $\mathcal{F}$ has a unique CFP.

Proof. Suppose that (ξ,η) and (ξ^*,η^*) are two CFPs of $\mathcal{F}$. We need to show that $\xi=\xi^*$ and $\eta=\eta^*$. By assumption (e), there exists $(\rho,\sigma)\in\mathcal{M}\times\mathcal{M}$ such that

$$[(\mathcal{F}(\rho,\sigma),\mathcal{F}(\sigma,\rho)),(\mathcal{F}(\xi,\eta),\mathcal{F}(\eta,\xi))]\in\mathcal{R}',$$

$$[(\mathcal{F}(\rho,\sigma),\mathcal{F}(\sigma,\rho)),(\mathcal{F}(\xi^*,\eta^*),\mathcal{F}(\eta^*,\xi^*))]\in\mathcal{R}'.$$

Using the same lines as in Theorem 3.1, we can construct sequences $\{\rho_n\}$, $\{\sigma_n\}$ such that $\rho_n=\mathcal{F}^n(\rho_0,\sigma_0)$ and $\sigma_n=\mathcal{F}^n(\sigma_0,\rho_0)$ where $\rho_0=\rho$ and $\sigma_0=\sigma$. Also, considering $\xi_0=\xi$, $\eta_0=\eta$, we can construct sequences $\{\xi_n\}$, $\{\eta_n\}$ such that

$$\xi_n=\mathcal{F}^n(\xi_0,\eta_0)\quad;\quad \eta_n=\mathcal{F}^n(\eta_0,\xi_0).$$

Since $[(\mathcal{F}(\rho,\sigma),\mathcal{F}(\sigma,\rho)),(\mathcal{F}(\xi,\eta),\mathcal{F}(\eta,\xi))]\in\mathcal{R}'$, we have

$$[(\mathcal{F}(\rho_0,\sigma_0),\mathcal{F}(\sigma_0,\rho_0)),(\xi,\eta)]\in\mathcal{R}'$$

which gives $[(\rho_1,\sigma_1),(\xi,\eta)]\in\mathcal{R}'$, i.e. $[\xi,\rho_1]\in\mathcal{R}$ and $[\eta,\sigma_1]\in\mathcal{R}$. In a similar process, $[\xi,\rho_n]\in\mathcal{R}$ and $[\eta,\sigma_n]\in\mathcal{R}$. Now, using assumption (d), we get

$$\begin{aligned}
d(\xi,\rho_n) &= d(\mathcal{F}(\xi,\eta),\mathcal{F}^n(\rho_0,\sigma_0))\\
&= d(\mathcal{F}(\xi,\eta),\mathcal{F}(\mathcal{F}^{n-1}(\rho_0,\sigma_0),\mathcal{F}^{n-1}(\sigma_0,\rho_0)))\\
&\le k\max\{d(\xi,\mathcal{F}^{n-1}(\rho_0,\sigma_0)),d(\eta,\mathcal{F}^{n-1}(\sigma_0,\rho_0))\}\\
&= k\max\{d(\xi,\rho_{n-1}),d(\eta,\sigma_{n-1})\}
\end{aligned}$$

$$\begin{aligned}
d(\eta,\sigma_n) &= d(\mathcal{F}(\eta,\xi),\mathcal{F}^n(\sigma_0,\rho_0))\\
&= d(\mathcal{F}(\eta,\xi),\mathcal{F}(\mathcal{F}^{n-1}(\sigma_0,\rho_0),\mathcal{F}^{n-1}(\rho_0,\sigma_0)))\\
&\le k\max\{d(\eta,\mathcal{F}^{n-1}(\sigma_0,\rho_0)),d(\xi,\mathcal{F}^{n-1}(\rho_0,\sigma_0))\}\\
&= k\max\{d(\eta,\sigma_{n-1}),d(\xi,\rho_{n-1})\}.
\end{aligned}$$

Hence, $\max\{d(\xi,\rho_n),d(\eta,\sigma_n)\}\le k\max\{d(\xi,\rho_{n-1}),d(\eta,\sigma_{n-1})\}$,
and by induction

$$\max\{d(\xi,\rho_n),d(\eta,\sigma_n)\}\quad\le\quad k^n\max\{d(\xi,\rho_0),d(\eta,\sigma_0)\}.$$

Now, when we tend n to ∞, we get

$$\lim_{n\to\infty}d(\xi,\rho_n)=0\quad;\quad\lim_{n\to\infty}d(\eta,\sigma_n)=0.$$

Similarly, we can get

$$\lim_{n\to\infty}d(\xi^*,\rho_n)=0\quad;\quad\lim_{n\to\infty}d(\eta^*,\sigma_n)=0.$$

Using triangle inequality on the above four relations, we get $\xi=\xi^*$ and $\eta=\eta^*$. Therefore, $\mathcal{F}$ has a unique CFP. □

Now we will use an example to demonstrate the importance of our results in a situation where we can't use the previous ones.

Example 4.3. Let $\mathcal{M}=\mathbb{R}$ with usual metric. Now, equip $\mathcal{M}$ with the binary relation $\mathcal{R}=\{(\xi,\eta)\in\mathbb{R}\times\mathbb{R}:\xi<\eta\}$. Consider a mapping $\mathcal{F}:\mathcal{M}\times\mathcal{M}\to\mathcal{M}$ by

$$\mathcal{F}(\xi,\eta)=\frac{\xi}{7}+\frac{\eta}{9}.$$

Here, $\mathcal{R}$ is not a partial order. Moreover, $\mathcal{F}$ does not satisfy MMP. So, Theorem 1.2 and Theorem 1.3 cannot be utilized here to comment on the availability of CFPs

of the mapping.
Now,

$$\begin{aligned}
d(\mathcal{F}(\xi,\eta),\mathcal{F}(r,s)) &= \left|\frac{\xi}{7}+\frac{\eta}{9}-\frac{r}{7}-\frac{s}{9}\right| \\
&= \left|\frac{1}{7}(\xi-r)+\frac{1}{9}(\eta-s)\right| \\
&\leq \frac{1}{7}(\mid \xi-r \mid + \mid \eta-s \mid) \\
&\leq \frac{2}{7}\max\{\mid \xi-r \mid, \mid \eta-s \mid\} \\
&= \frac{2}{7}\max\{d(\xi,r),d(\eta,s)\}.
\end{aligned}$$

So, the contractive condition of Theorem 4.1 is satisfied with $k=\frac{2}{7}$. Also, all the hypotheses of Theorem 4.1 and Theorem 4.2 are met. Hence, we can conclude that $\mathcal{F}$ has a unique CFP which is $(0,0)$.

References

[1] A. Alam, M. Imdad, *Relation-theoretic contraction principle*, J. Fixed Point Theory Appl., 17(4) (2015) 693-702.

[2] A. Alam, M. Imdad, *Nonlinear contractions in metric spaces under locally T-transitive binary relations*, Fixed Point Theory, 19(1) (2018) 13-24.

[3] S. Banach, *Sur les operations dans les ensembles abstraints er leur application aux equations intgrales*, Fund. Math., 3 (1922) 133-181.

[4] V. Berinde, *Coupled coincidence point theorems for mixed monotone nonlinear operators,* Computers and Mathematics with Applications, 64(6) 2012 1770-1777.

[5] V. Berinde, *Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces*, Nonlinear Anal. 74(2011) 7347-7355.

[6] B.S. Choudhury, A. Kundu, *A coupled coincident point results in partially ordered metric spaces for compatible mappings*, Nonlinear Anal. TMA (2010) 2524-2531.

[7] B.S. Choudhury, P. Maity, Cyclic coupled fixed point result using Kannan type contractions, Journal Of Operators. 2014(2014), Article ID 876749, 5 pages.

[8] D. Doric, Z. Kadelburg, S. Redonovič, *Coupled fixed point results for mappings without mixed monotone property*, Applied Mathematics Letters 25 (2012) 1803-1808.

[9] V. Flaška, J. Ježek, T. Kepka and J. Kortelainen, *Transitive closures of binary relations I*, Acta Univ. Carolin. Math. Phys. 48(1)(2007) 55-69.

[10] T. Gnana Bhaskar, V. Lakshikantham, *Fixed point theorems in partially ordered metric space and applications*, Nonlinear Anal. TMA 65 (2006) 1379-1393.

[11] D. Guo, V. Lakshmikantham, *Coupled fixed points of nonlinear operators with applications*, Nonlinear Anal. TMA 11 (1987) 623-632.

[12] J. Harjani, B. Lopez, K. Sadarangani, *Fixed point theorems for mixed monotone operator and application to integral equations*, Nonlinear Anal. TMA 74(2011) 1749-1760.

[13] Q. H. Khan, T. Rashid, Hassen Aydi, *On strong coupled coincidence point of g-coupling and an application,* Journal of Function Spaces Vol 2018 Article ID 4034535.

[14] B. Kolman, R.C. Busby, S. Ross, *Discrete mathematical structures*, Third Edition, PHI Pvt. Ltd., New Delhi, 2000.

[15] V. Lakshmikantham, Lj. Ciric, *Coupled fixed point theorems for nonlinear contraction in partially ordered metric spaces*, Nonlinear Anal. TMA 70 (2009) 4341-4349.

[16] V. Lakshmikantham, Ljubomir iri, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Analysis 70(12) 2009 4341-4349

[17] S. Lipschutz, *Schaums Outlines of Theory and Problems of Set Theory and Related Topics.* McGraw-Hill, New York, 1964.

[18] N.V. Luong, N.X. Thuan, *Coupled fixed points in partially ordered metric spaces and application*, Nonlinear Anal. TMA 74 (2011) 983-992.

[19] N.V. Luong, N.X. Thuan, *Coupled fixed points in partially ordered metric spaces and application*, Nonlinear Anal. 74(2011) 983-992.

[20] R. D. Maddux, *Relation algebras, Studies in Logic and the Foundations of Mathematics*, vol. 150, Elsevier B. V., Amsertdam(2006).

[21] V. I. Opoitsev, *Heterogenic and combined-concave operators*, Syber. Math. J. 16(1975) 781-792(in Russian).

[22] V. I. Opoitsev, *Dynamics of collective. behaviour-III. Heterogenic systems*, Avtomat. i. Telemekh. 36 (1975), 124-138(in Russian).

[23] V. I. Opoitsev, T.A. Khurodze, *Nonlinear operator in spaces with a cone*, Tibilis. Gos. Univ., Tbilisi (1984) 271 (in Russian).

[24] B. Samet, *Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces*, Nonlinear Anal. TMA 72 (2010) 4508-4517.

[25] B. Samet and M. Turinici, *Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications*, Commun. Math. Anal. 13(2) (2012) 82-97.

[26] W. Shatanawi, *Partially ordered metric spaces and coupled fixed point results*, Comput. Math. Appl. 60 (2010) 2508-2515.

[27] H. L. Skala, *Trellis theory*, Algebra Universalis 1 (1971) 218-233.

[28] A. Stouti and A. Maaden, *Fixed point and common fixed point theorems in pseudo-ordered sets*, Proyecciones 32(4) (2013) 409-418.

[29] M. Turinici, *Abstract comparison principle and multivariable Gronwall-Bellman inequalities*, J. Math. Anal. Appl. 117(1) (1986) 100-127.

Faruk Sk
Department of Mathematics, Aligarh Muslim University, India
E-mail address: sk.faruk.amu@gmail.com

Faizan Ahmad Khan
Department of Mathematics, University of Tabuk, Tabuk-71491, KSA
E-mail address: fkhan@ut.edu.sa

Qamrul Haq Khan
Department of Mathematics, Aligarh Muslim University, India
E-mail address: qhkhan.ssitm@gmail.com