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A GENERAL AND OPTIMAL DECAY RESULT FOR A VISCOELASTIC EQUATION WITH A STRONG TIME DEPENDENT DELAY

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Abstract. In this paper, we establish an optimal and general decay result for the energy of a viscoelastic equation exhibiting a strong time-dependent delay. This is achieved by considering a minimal condition on the relaxation function g. The exponential and polynomial decay rates are obtained as special cases.The theoretical computations are supported with a numerical analysis of the problem under consideration. This work extends and generalizes some recent results in the literature.

1. Introduction

In this work, we consider the following problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-s)\Delta u(s)ds - \mu_1 \Delta u_t - \mu_2 \Delta u_t(\cdot, t-\tau(t)) = 0, \\ \hfill \text{in } \ \Omega \times (0, +\infty), \\ u\big|_{\partial\Omega} = 0, \ u\big|_{t=0} = u_0, \ u_t\big|_{t=0} = u_1, \\ u_t(x,t) = f_0(x,t), \hfill t \in [-\tau(0), 0), \ x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$, with a smooth boundary $\partial\Omega$, μ_1, μ_2 are constants, $\tau(t) > 0$ is the time-dependent delay and g is the relaxation function to be specified. In the past decade, various researchers have studied the effect of delay damping in the wave equation. For instance, Nacaise and Pignotti [17] considered the following delay equation with internal feedback

$$u_{tt} - \Delta u + a(x)[\mu_1 u_t + \mu_2 u_t(t-\tau)] = 0 \tag{1.2}$$

and established an exponential decay result when $0 < \mu_2 < \mu_1$. Nacaise and Pignotti [18, 19] investigated an abstract evolution equation and established similar results as in [17]. Kafini et al. [10] looked at the nonlinear wave equation

$$u_{tt} + Au + G(u_t) + \mu G(u_t(t-\tau)) = F(u) \tag{1.3}$$

and proved that under suitable conditions, solutions blow up in finite time. For wave equation with strong delay, Messaoudi et al. [15] considered

$$u_{tt} - \Delta u - \mu_1 \Delta u_t - \mu_2 \Delta u_t(t-\tau) = 0 \tag{1.4}$$

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and established a well posedness and exponential stability result. Liu [12] looked at the time-dependent delay equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + \mu_1 u_t + \mu_2 u_t(t-\tau(t)) = 0 \tag{1.5}$$

and proved a general decay result when $|\mu_2| < \sqrt{1-d}\mu_1$. The decay result in [12] was improved by Dia and Yang [5] under weaker conditions on the relaxation function. Kirane and Said- Houari [11] investigated the delay equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0 \tag{1.6}$$

and established the well-posedness as well as a general decay estimate when $0 < \mu_2 \leq \mu_1$. In [9], Feng investigated a wave equation similar to (1.6), however with strong time dependent delay term and viscoelastic memory, that is

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + \mu_1 \Delta u_t + \mu_2 \Delta u_t(t-\tau(t)) = 0. \tag{1.7}$$

The well posedness of equation (1.7) was established, furthermore, an exponential stability result for the associated energy functional was proved, under the assumption that $|\mu_2| < \sqrt{1-d}\mu_1$ holds. Benaissa et al.[3] studied the delay equation

$$u_{tt} - \Delta u + \int_0^t h(t-s)\Delta u(s)ds + \mu_1 g_1(u_t) + \mu_2 g_2(u_t(t-\tau)) = 0 \tag{1.8}$$

and proved the well-posedness in addition to a general decay result for the corresponding energy functional. Liu and Zhang [14] looked at the nonlinear wave equation with infinite memory and delay

$$u_{tt} - \alpha\Delta u + \int_{-\infty}^t \mu(t-s)\Delta u(s)ds + \mu_1 u_t + \mu_2 u_t(t-\tau) + f(u) = h \tag{1.9}$$

and established the well-posedness without any restrictions on μ_1 and μ_2. Moreover, they showed that the energy functional decay exponentially when $0 < |\mu_2| < \mu_1$. Alabau-Boussouira et al. [1] studied the nonlinear wave equation

$$u_{tt} - \Delta u + \int_0^{+\infty} g(s)\Delta u(t-s)ds + k u_t(t-\tau) = 0 \tag{1.10}$$

and proved that the equation is exponentially stable for k small enough. For more related results concerning the wave equation with a weak time delay term under appropriate assumption on μ_1 and μ_2, we refer the reader to Mukiawa [7], Enyi and Mukiawa [8], Benaissa et al. [4], Datko et al.[6], Liu [13], Nicaise and Valein [20, 21] and references therein.

Remark. The problem (1.1) we considered in this paper is an improvement and generalization over the problem (1.7) of Feng [9]. This is obvious because if in particular, we take M to be the identity map in our assumption (A2) (2.3) for the kernel g in the finite memory term, we get the assumption (1.2) of Feng [9]. Furthermore, we have presented a numerical analysis of the problem to validate our theoretical analysis, this was also lacking in the work of Feng [9].

This work is organized as follows: In Section 2, we set the problem and state some basic assumptions. In Section 3, we present some strategic lemmas needed. Again, in section 4 we state and prove our main results. Also, section 5 is devoted to giving numerical results concerning our considered problem. Finally, in section 6 we give a conclusion statement.

2. Setting of the problem

We consider the following problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-s)\Delta u(s)ds - \mu_1 \Delta u_t - \mu_2 \Delta u_t(\cdot, t-\tau(t)) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \ \Omega \times (0,+\infty), \\ u|_{\partial\Omega} = 0, \ u|_{t=0} = u_0, \ u_t|_{t=0} = u_1, \\ u_t(x,t) = f_0(x,t), \qquad\qquad\qquad\qquad t \in [-\tau(0), 0), \ x \in \Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$, with smooth boundary $\partial\Omega$.

Assumptions:

(A1) The relaxation function $g : [0,\infty) \to (0,\infty)$ is a $\mathcal{C}^1$ increasing function and satisfies

$$g(0) > 0, \qquad\qquad 1 - \int_0^\infty g(s)ds = l > 0. \tag{2.2}$$

(A2) There exists a $\mathcal{C}^1-$function $M : [0,\infty) \to (0,\infty)$, which is either linear or is a strictly increasing and strictly convex $\mathcal{C}^2$ function on $[0,\alpha]$, $\alpha > 0$, $\alpha \leq g(0)$, with $M(0) = M'(0) = 0$, such that

$$g'(t) \leq -\xi(t) M(g(t)), \qquad \forall t \geq 0, \tag{2.3}$$

where ξ is a positive nonincreasing differentiable function.

(A3) There exist $\tau_0, \tau_1 > 0$ such that

$$0 < \tau_0 \leq \tau(t) \leq \tau_1, \ \forall t > 0, \tag{2.4}$$

(A4)

$$\tau(t) \in W^{2,\infty}(0,T) \text{ and } \tau'(t) \leq d < 1, \ \forall t, T > 0. \tag{2.5}$$

We can deduce from (A1) and (A2) the following:

(I) From (A1), it follows that $\lim_{t\to\infty} g(t) = 0$. Thus, there exists $t_0 \geq 0$ large enough, such that

$$g(t_0) = \alpha \quad \text{and} \quad g(t_0) \leq \alpha, \quad \forall t \geq t_0. \tag{2.6}$$

(II) Since g and ξ are positive, nonincreasing and continuous functions, in addition to M being a positive continuous function, it follows that, for all $t \in [0, t_0]$,

$$\left.\begin{array}{l} 0 < g(t_0) \leq g(t) \leq g(0) \\ 0 < \xi(t_0) \leq \xi(t) \leq \xi(0) \end{array}\right\} \Rightarrow a \leq \xi(t) M(g(t)) \leq b$$

for some positive constants a and b. Hence,

$$g'(t) \leq -\xi(t) M(g(t)) \leq -\frac{a}{g(0)} g(0) \leq -\frac{a}{g(0)} g(t), \ \forall t \in [0, t_0]. \tag{2.7}$$

(III) M has an extension $\overline{M}$, which is a strictly increasing and strictly convex $\mathcal{C}^2$ function on $(0,\infty)$. As an example, given that $M(\alpha) = a_1$, $M'(\alpha) = a_2$ and $M''(\alpha) = a_3$, then we can define $\overline{M}$ by

$$\overline{M}(t) = \frac{a_3}{2} t^2 + (a_2 - a_3\alpha)t + \left(a_1 + \frac{a_3}{2}\alpha^2 - a_2\alpha\right), \ \forall t > \alpha. \tag{2.8}$$

We will as well make use of the Jensen's inequality:

Given that G is a convex function on $[a,b]$, $f : \Omega \to [a,b]$ and h are integrable functions on Ω, $h(x) \geq 0$, and $\int_\Omega h(x)dx = \varrho > 0$, then

$$G\left[\frac{1}{\varrho}\int_\Omega f(x)h(x)dx\right] \leq \frac{1}{\varrho}\int_\Omega G[f(x)]h(x)dx.$$

We have the following well-possedness result, which is obtained by using the Classical Faedo-Galerkin method, see , e.g [15].

Theorem 2.1. *Assume that $\mu_2 \leq \mu_1$ and assumptions (A1)-(A4) hold. If $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $f_0 \in H^1(\Omega \times (-\tau(0), 0))$, then (2.1) possesses a unique weak solution $(u, u_t) \in \mathcal{C}([0, +\infty), H_0^1(\Omega) \times L^2(\Omega))$.*

3. Strategic lemmas

For convenience, we will denote the norm $\|\cdot\|_2$ of the Lebesgue space $L^2(\Omega)$ by $\|\cdot\|$. The constants $c > 0$ and $C > 0$ are generic constants which may change in value from one line to the other or within the same line. We define the energy functional of problem (2.1) as

$$\begin{aligned} E(t) =& \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|^2 + \frac{1}{2}(g \circ \nabla u) \\ &+ \frac{\zeta}{2}\int_{t-\tau(t)}^t \int_\Omega e^{-\lambda(t-s)}\|\nabla u_t(s)\|^2 dxds, \end{aligned} \tag{3.1}$$

where $\zeta, \lambda > 0$ are constants satisfying, (see [9, 18])

$$\frac{\mu_2^2 e^{\lambda\tau_1}}{\mu_1(1-d)} < \zeta < \mu_1. \tag{3.2}$$

and

$$0 < \lambda < \frac{2}{\tau_1}\log_e\left(\frac{\mu_1}{|\mu_2|}\sqrt{1-d}\right), \tag{3.3}$$

while

$$(g \circ v)(t) = \int_\Omega \int_0^t g(t-s)|v(t) - v(s)|^2 dsdx.$$

Lemma 3.1. *Assume that $|\mu_2| < \mu_1\sqrt{1-d}$, then the energy functional satisfies, along with the solution of Problem (2.1), the inequality*

$$\begin{aligned} E'(t) \leq& \frac{1}{2}(g' \circ \nabla u) - \frac{1}{2}g(t)\|\nabla u\|^2 + \left(\frac{\zeta}{2} - \frac{\mu_1}{2}\right)\|\nabla u_t\|^2 \\ &+ \left[\frac{\mu_2^2}{2\mu_1} - \frac{\zeta}{2}e^{-\lambda\tau_1}(1-d)\right]\|\nabla u_t(t-\tau(t))\|^2 \\ &- \frac{\lambda\zeta}{2}\int_{t-\tau(t)}^t \int_\Omega e^{-\lambda(t-s)}\|\nabla u_t(s)\|^2 dxds \ \leq 0, \quad \forall t \geq 0. \end{aligned} \tag{3.4}$$

Proof. Differentiating the energy functional, we have

$$\begin{aligned} E'(t) =& \int_\Omega u_t u_{tt} dx + \frac{1}{2}\frac{d}{dt}\left[\left(1 - \int_0^t g(s)ds\right)\int_\Omega |\nabla u(t)|^2 dx\right] + \frac{1}{2}\frac{d}{dt}(g \circ \nabla u) \\ &+ \frac{\zeta}{2}\|\nabla u_t\|^2 - \frac{\zeta}{2}e^{-\lambda\tau(t)}(1 - \tau'(t))\|\nabla u_t(t-\tau(t))\|^2 \\ &- \frac{\lambda\zeta}{2}\int_{t-\tau(t)}^t \int_\Omega e^{-\lambda(t-s)}\|\nabla u_t(s)\|^2 dxds. \end{aligned} \tag{3.5}$$

Also, multiplying $(2.1)_1$ by u_t and integrating over Ω yields

$$\begin{aligned} &\int_\Omega u_t u_{tt} dx + \frac{1}{2}\frac{d}{dt}\left[\left(1 - \int_0^t g(s)ds\right)\int_\Omega |\nabla u(t)|^2 dx\right] + \frac{1}{2}\frac{d}{dt}(g \circ \nabla u) \\ =& \frac{1}{2}(g' \circ \nabla u) - \frac{1}{2}g(t)\int_\Omega |\nabla u(t)|^2 dx - \mu_1 \int_\Omega |\nabla u_t(t)|^2 \\ &- \mu_2 \int_\Omega \nabla u_t(t) \cdot \nabla u_t(t-\tau(t))dx. \end{aligned} \tag{3.6}$$

By Young's inequality, we have

$$-\mu_2 \int_\Omega \nabla u_t(t) \cdot \nabla u_t(t-\tau(t))dx \le \frac{\mu_1}{2}\|\nabla u_t\|^2 + \frac{\mu_2^2}{2\mu_1}\|\nabla u(t-\tau(t))\|^2. \tag{3.7}$$

Now, substituting (3.6) into (3.5), then making use of (3.7), assumptions (A3) and (A4), we obtain

$$\begin{aligned} E'(t) \le & \frac{1}{2}(g' \circ \nabla u) - \frac{1}{2}g(t)\|\nabla u\|^2 + \left(\frac{\zeta}{2} - \frac{\mu_1}{2}\right)\|\nabla u_t\|^2 \\ & + \left[\frac{\mu_2^2}{2\mu_1} - \frac{\zeta}{2}e^{-\lambda\tau_1}(1-d)\right]\|\nabla u_t(t-\tau(t))\|^2 \\ & - \frac{\lambda\zeta}{2}\int_{t-\tau(t)}^t \int_\Omega e^{-\lambda(t-s)}\|\nabla u_t(s)\|^2 dxds. \end{aligned} \tag{3.8}$$

Therefore (3.4) follows from (3.8) and (3.2). □

Lemma 3.2. *Let u be the solution of Problem* (2.1)*. The functional defined by*

$$\Phi(t) = \int_\Omega u u_t dx, \tag{3.9}$$

satisfies

$$\begin{aligned} \Phi'(t) \le & -\frac{l}{2}\|\nabla u\|^2 + \|u_t\|^2 + \frac{3\mu_1}{2l}\|\nabla u_t\|^2 + \frac{3\mu_2}{2l}\|\nabla u_t(t-\tau(t))\|^2 \\ & + \frac{3C_\kappa}{2l}(\eta \circ \nabla u)(t), \end{aligned} \tag{3.10}$$

for any $\kappa \in (0,1)$, where

$$C_\kappa = \int_0^\infty \frac{g^2(s)}{\kappa g(s) - g'(s)}ds \quad \text{and} \quad \eta(t) = \kappa g(t) - g'(t). \tag{3.11}$$

Proof. From (3.9), by taking into account $(2.1)_1$, (2.2) and Young's inequality, we obtain

$$\begin{aligned} \Phi'(t) = & \|u_t\|^2 - \|\nabla u\|^2 - \int_\Omega u\left(\int_0^t g(t-s)\Delta u(s)ds\right)dx - \mu_1\int_\Omega \nabla u \cdot \nabla u_t dx \\ & - \mu_2 \int_\Omega \nabla u \cdot \nabla u_t(t-\tau(t))dx \\ = & \|u_t\|^2 - \left(1 - \int_0^t g(s)ds\right)\|\nabla u\|^2 + \int_\Omega \nabla u \cdot \left(\int_0^t g(t-s)(\nabla u(s) - \nabla u(t))ds\right)dx \\ & - \mu_1 \int_\Omega \nabla u \cdot \nabla u_t dx - \mu_2 \int_\Omega \nabla u \cdot \nabla u_t(t-\tau(t))dx \\ \le & -\frac{l}{2}\|\nabla u\|^2 + \|u_t\|^2 + \frac{3\mu_1}{2l}\|\nabla u_t\|^2 + \frac{3\mu_2}{2l}\|\nabla u_t(t-\tau(t))\|^2 \\ & + \frac{3}{2l}\int_\Omega\left(\int_0^t g(t-s)|\nabla u(s) - \nabla u(t)|ds\right)^2 dx. \end{aligned} \tag{3.12}$$

Now, using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} & \int_\Omega\left(\int_0^t g(t-s)|\nabla u(s) - \nabla u(t)|ds\right)^2 dx \\ = & \int_\Omega\left(\int_0^t \frac{g(t-s)}{\sqrt{\kappa g(t-s) - g'(t-s)}}\sqrt{\kappa g(t-s) - g'(t-s)}|\nabla u(s) - \nabla u(t)|ds\right)^2 dx \\ \le & \left(\int_0^t \frac{g^2(s)}{\kappa g(s) - g'(s)}ds\right)\int_\Omega\int_0^t \Big(\kappa g(t-s) - g'(t-s)\Big)|\nabla u(s) - \nabla u(t)|^2 dsdx \\ \le & C_\kappa(\eta \circ \nabla u)(t). \end{aligned} \tag{3.13}$$

Substituting (3.13) into (3.12), we get (3.10). □

Lemma 3.3. *Let u be the solution of Problem* (2.1)*. The functional defined by*

$$\Psi(t) = -\int_\Omega u_t \int_0^t g(t-s)(u(t)-u(s))dsdx, \tag{3.14}$$

satisfies

$$\begin{aligned}\Psi'(t) \leq & -\left(\int_0^t g(s) - \sigma\right)\|u_t\|^2 + \sigma\|\nabla u\|^2 + \sigma\|\nabla u_t\|^2 + \sigma\|\nabla u_t(t-\tau(t))\|^2 \\ & + \frac{C_0}{\sigma}(C_\kappa + 1)(\eta \circ \nabla u)(t),\end{aligned} \tag{3.15}$$

for any $\sigma \in (0,1)$.

Proof. Using $(2.1)_1$ and integration by parts, we obtain

$$\begin{aligned}\Psi'(t) = & \underbrace{\left(1 - \int_0^t g(s)ds\right)\int_\Omega \nabla u \cdot \left(\int_0^t g(t-s)\big(\nabla u(t) - \nabla u(s)\big)ds\right)dx}_{I_1} \\ & + \underbrace{\int_\Omega \left(\int_0^t g(t-s)\big(\nabla u(t) - \nabla u(s)\big)ds\right)^2 ds}_{I_2} - \left(\int_0^t g(s)ds\right)\int_\Omega u_t^2 dx \\ & + \underbrace{\mu_1 \int_\Omega \nabla u_t \cdot \left(\int_0^t g(t-s)\big(\nabla u(t) - \nabla u(s)\big)ds\right)}_{I_3} \\ & + \underbrace{\mu_2 \int_\Omega \nabla u_t(t-\tau(t)) \cdot \left(\int_0^t g(t-s)\big(\nabla u(t) - \nabla u(s)\big)ds\right)dx}_{I_4} \\ & - \underbrace{\int_\Omega u_t \left(\int_0^t g'(t-s)\big(u(t)-u(s)\big)ds\right)dx}_{I_5}.\end{aligned} \tag{3.16}$$

Using Young's inequality, Poincaré's inequality, Cauchy-Schwatz inequality and similar calculations as in (3.13), we estimate the terms $I_1 - I_5$ as follows:

$$\begin{aligned}I_1 \leq & \sigma\|\nabla u\|^2 + \frac{C_\kappa}{4\sigma}(\eta \circ \nabla u)(t) \\ I_2 \leq & C_\kappa(\eta \circ \nabla u)(t) \\ I_3 \leq & \sigma\|\nabla u_t\|^2 + \frac{\mu_1^2}{4\sigma}C_\kappa(\eta \circ \nabla u)(t) \\ I_4 \leq & \sigma\|\nabla u_t(t-\tau(t))\|^2 + \frac{\mu_2^2}{4\sigma}C_\kappa(\eta \circ \nabla u)(t) \\ I_5 = & \int_\Omega u_t \int_0^t \eta(t-s)\big(u(t)-u(s)\big)dsdx - \int_\Omega u_t \int_0^t \kappa g(t-s)\big(u(t)-u(s)\big)dsdx \\ \leq & \sigma\|u_t\|^2 + \frac{\left(\int_0^t \eta(s)ds\right)}{\sigma}(\eta \circ \nabla u)(t) + \frac{\kappa^2}{2\sigma}C_\kappa(\eta \circ \nabla u)(t) \\ \leq & \sigma\|u_t\|^2 + \frac{c}{\sigma}(\eta \circ \nabla u)(t) + \frac{c}{\sigma}C_\kappa(\eta \circ \nabla u)(t).\end{aligned}$$

Now, making use of the estimates above on $I_1 - I_5$, (3.16) yields

$$\begin{aligned}\Psi'(t) \leq & -\left(\int_0^t g(s)ds - \sigma\right)\|u_t\|^2 + \sigma\|\nabla u\|^2 + \sigma\|\nabla u_t\|^2 + \sigma\|\nabla u_t(t-\tau(t))\|^2 \\ & + \frac{1}{\sigma}\left[\frac{C_\kappa}{4} + \sigma C_\kappa + \frac{\mu_1^2 C_\kappa}{4} + \frac{\mu_2^2 C_\kappa}{4} + c(1+C_\kappa)\right](\eta \circ \nabla u)(t).\end{aligned} \tag{3.17}$$

Estimate (3.15) follows from (3.17), where $C_0 \geq \max\left\{\frac{1}{4}, \sigma, \frac{\mu_1^2}{4}, \frac{\mu_2^2}{4}, c\right\}$. □

Lemma 3.4. *Let u be the solution of Problem (2.1). The functional defined by*

$$\Theta(t) = \int_\Omega \int_0^t h(t-s)|\nabla u(s)|^2 ds dx, \tag{3.18}$$

where $h(t) = \int_t^\infty g(s)ds$ satisfies

$$\Theta'(t) \leq -\frac{3}{4}(g \circ \nabla u)(t) + 5(1-l)\|\nabla u\|^2. \tag{3.19}$$

Proof. We notice that $h'(t) = -g(t)$, therefore

$$\begin{aligned}\Theta'(t) =& h(0)\int_\Omega |\nabla u|^2 dx - \int_\Omega \int_0^t g(t-s)|\nabla u(s)|^2 ds dx \\ \leq & h(0)\|\nabla u\|^2 - \int_\Omega \int_0^t g(t-s)|\nabla u(s) - \nabla u(t)|^2 ds dx \\ & - 2\int_\Omega \nabla u(t) \cdot \int_0^t g(t-s)\big(\nabla u(s) - \nabla u(t)\big) ds dx.\end{aligned} \tag{3.20}$$

It follows from Young's inequality, Cauchy-Schwatz inequality as well as (2.2), we have

$$\begin{cases} -2\int_\Omega \nabla u(t) \cdot \int_0^t g(t-s)\big(\nabla u(s) - \nabla u(t)\big) ds dx \leq 4(1-l)\|\nabla u\|^2 + \frac{1}{4}(g \circ \nabla u)(t), \\ \\ h(0) = 1-l, \end{cases}$$

then substituting this in (3.20), we obtain (3.19). □

Lemma 3.5. *Given $t_0 > 0$. Then, the functional $\mathscr{L}$ defined by*

$$\mathscr{L}(t) := KE(t) + \varepsilon_1 \Phi(t) + \varepsilon_2 \Psi(t)$$

with $K, \varepsilon_1, \varepsilon_2 > 0$ appropriately chosen, satisfies for all $t \geq t_0$,

$$\begin{aligned}\mathscr{L}'(t) \leq & -\frac{21}{4}(1-l)\|\nabla u\|^2 - \frac{1}{4}\|u_t\|^2 + \frac{1}{2}(g \circ \nabla u)(t) \\ & - \frac{\lambda\zeta}{4\kappa_0}\int_{t-\tau(t)}^t \int_\Omega e^{-\lambda(t-s)}\|\nabla u_t\|^2 dx ds.\end{aligned} \tag{3.21}$$

In addition, there exist β_1, $\beta_2 > 0$ such that

$$\beta_1 E(t) \leq \mathscr{L}(t) \leq \beta_2 E(t), \tag{3.22}$$

i.e., $\mathscr{L}(t) \sim E(t)$.

Proof. We set $g_0 = \int_0^{t_0} g(s)ds$ and recall that $g' = \kappa g - \eta$. Making use of (3.4), (3.10) and (3.15), in addition to taking $\sigma = \dfrac{l}{4\varepsilon_2}$, we find that for all $t \geq t_0$,

$$\begin{aligned}
\mathscr{L}'(t) \leq &-\left(\varepsilon_1\frac{l}{2} - \frac{l}{4}\right)\|\nabla u\|^2 - \left(\varepsilon_2 g_0 - \frac{l}{4} - \varepsilon_1\right)\|u_t\|^2 \\
&+ \left[\frac{l}{4} + \varepsilon_1\frac{3\mu_1}{2l} + K\left(\frac{\zeta}{2} - \frac{\mu_1}{2}\right)\right]\|\nabla u_t\|^2 \\
&+ \left[\frac{l}{4} + \varepsilon_1\frac{3\mu_2}{2l} + K\left(\frac{\mu_2^2}{2\mu_1} - \frac{\zeta}{2}e^{-\lambda\tau_1}(1-d)\right)\right]\|\nabla u_t(t-\tau(t))\|^2 \\
&+ K\frac{\kappa}{2}(g\circ\nabla u)(t) - \left[\frac{K}{2} - \frac{4C_0}{l}\varepsilon_2^2 - C_\kappa\left(\varepsilon_1\frac{3}{2l} + \varepsilon_2^2\frac{4C_0}{l}\right)\right](\eta\circ\nabla u)(t) \\
&- K\frac{\lambda\zeta}{2}\int_{t-\tau(t)}^{t}\int_\Omega e^{-\lambda(t-s)}\|\nabla u_t(s)\|^2 dxds. \qquad (3.23)
\end{aligned}$$

Now, we choose ε_1 large enough so that

$$\varepsilon_1\frac{l}{2} - \frac{l}{4} > \frac{21}{4}(1-l), \qquad (3.24)$$

then choose ε_2 large enough so that

$$\varepsilon_2 g_0 - \frac{l}{4} - \varepsilon_1 > \frac{1}{4}. \qquad (3.25)$$

Observe that $\dfrac{\kappa g^2(s)}{\kappa g(s) - g'(s)} < g(s)$ and $\lim\limits_{\kappa\to 0}\dfrac{\kappa g^2(s)}{\kappa g(s) - g'(s)} = 0$, hence by the Lebesgue dominated convergence theorem, we have that

$$\kappa C_\kappa = \int_0^\infty \frac{\kappa g^2(s)}{\kappa g(s) - g'(s)}ds \to 0 \quad \text{as} \quad \kappa \to 0.$$

Therefore, there exists $\kappa_0 \in (0,1)$ such that for all $\kappa < \kappa_0$, we have

$$\kappa C_\kappa < \frac{1}{2\left(\varepsilon_1\frac{3}{2l} + \varepsilon_2^2\frac{4C_0}{l}\right)}.$$

We now choose K large enough and choose κ satisfying

$$\frac{K}{6} - \frac{4C_0}{l}\varepsilon_2^2 > 0 \quad \text{and} \quad \kappa = \frac{1}{K}, \qquad (3.26)$$

which yields, on account of (3.2) and (3.3), that

$$\frac{l}{4} + \varepsilon_1\frac{3\mu_1}{2l} + K\left(\frac{\zeta}{2} - \frac{\mu_1}{2}\right) < 0, \qquad (3.27)$$

$$\frac{l}{4} + \varepsilon_1\frac{3\mu_2}{2l} + K\left(\frac{\mu_2^2}{2\mu_1} - \frac{\zeta}{2}e^{-\lambda\tau_1}(1-d)\right) < 0 \qquad (3.28)$$

and

$$\frac{K}{2} - \frac{4C_0}{l}\varepsilon_2^2 - C_\kappa\left(\varepsilon_1\frac{3}{2l} + \varepsilon_2^2\frac{4C_0}{l}\right) > 0. \qquad (3.29)$$

Combining (3.24)-(3.29), we obtain

$$\begin{aligned}
\mathscr{L}'(t) \leq &-\frac{21}{4}(1-l)\|\nabla u\|^2 - \frac{1}{4}\|u_t\|^2 + \frac{1}{2}(g\circ\nabla u)(t) \\
&- \frac{\lambda\zeta}{4\kappa_0}\int_{t-\tau(t)}^{t}\int_\Omega e^{-\lambda(t-s)}\|\nabla u_t(2)\|^2 dxds.
\end{aligned}$$

Now, we establish that $\mathscr{L} \sim E$. On account of Poincarè inequality and Cauch-Schwars inequality, we have

$$
\begin{aligned}
&|\mathscr{L}(t) - KE(t)| \\
&\leq \varepsilon_1 |\Phi(t)| + \varepsilon_2 |\Psi(t)| \\
&\leq c \int_\Omega \left[|\nabla u|^2 + |u_t|^2 + \left(\int_0^t g(t-s)|u(t) - u(s)| ds \right)^2 \right] dx \\
&\leq c \int_\Omega \left[|\nabla u|^2 + |u_t|^2 + \left(\int_0^t g(s) ds \right) \int_0^t g(t-s)|\nabla u(t) - \nabla u(s)|^2 ds \right] dx \\
&\leq c \left[\left(1 - \int_0^t g(s) ds \right) \int_\Omega |\nabla u|^2 dx + \int_\Omega |u_t|^2 dx + (g \circ \nabla u)(t) \right] \\
&\quad + \frac{c\zeta}{2} \int_{t-\tau(t)}^t \int_\Omega e^{-\lambda(t-s)} \|\nabla u_t(s)\|^2 dx ds \\
&\leq cE(t).
\end{aligned}
$$

Therefore, (3.22) follows immediately. □

4. Main results

Lemma 4.1. *There exists $\beta_3 > 0$ such that, the functional defined by*

$$\mathcal{F}(t) = \mathscr{L}(t) + \beta_3 E(t),$$

satisfies

$$\mathcal{F}'(t) \leq -\beta_4 E(t) + \int_{t_0}^t g(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dx ds, \quad \forall t \geq t_0, \tag{4.1}$$

for some $\beta_4 > 0$. Moreover, $\mathcal{F} \sim E$.

Proof. From (2.7) and (3.4), we deduce that for any $t \geq t_0$,

$$
\begin{aligned}
&\int_0^{t_0} g(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dx ds \\
&\leq -\frac{g(0)}{a} \int_0^t g'(t-s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dx ds \\
&\leq -\beta_3 E'(t).
\end{aligned} \tag{4.2}
$$

Now, from (3.21), we have

$$
\begin{aligned}
\mathscr{L}'(t) \leq& -\beta_4 E(t) + (g \circ \nabla u)(t) \\
\leq& -\beta_4 E(t) - \beta_3 E'(t) + \int_{t_0}^t g(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dx ds.
\end{aligned}
$$

Hence (4.1) follows immediately, and since $\mathscr{L} \sim E$, then $\mathcal{F} \sim E$. □

Now, we state and prove our main decay result.

Theorem 4.2. *Assume that $|\mu_2| < \mu_1\sqrt{1-d}$. There exist constants $\omega_1 \in (0,1]$ and $\omega_2 > 0$ such that the energy functional satisfies*

$$E(t) \leq \omega_2 M_2^{-1} \left(\omega_1 \int_{g^{-1}(\alpha)}^t \xi(s) ds \right), \tag{4.3}$$

where $M_2(t) = \int_t^\alpha \frac{1}{sM'(s)} ds$ and M_2 is convex and strictly decreasing on $(0, \alpha]$. In addition, $\lim_{t \to 0} M_2(t) = +\infty$.

Proof. To prove Theorem 4.2, we shall adopt the method of [16] and consider two cases.

Case 1. M is linear.

We multiply (4.1) by $\xi(t)$, then on account of (2.3) and (3.4), we get

$$\begin{aligned}\xi(t)\mathcal{F}'(t) \leq & -\beta_4\xi(t)E(t) + \int_{t_0}^{t}\xi(s)g(s)\int_\Omega |\nabla u(t)-\nabla u(s)|^2 dxds\\ \leq & -\beta_4\xi(t)E(t) - \int_{t_0}^{t} g'(s)\int_\Omega |\nabla u(t)-\nabla u(s)|^2 dxds\\ \leq & -\beta_4\xi(t)E(t) - \int_{0}^{t} g'(s)\int_\Omega |\nabla u(t)-\nabla u(s)|^2 dxds\\ \leq & -\beta_4\xi(t)E(t) - E'(t).\end{aligned}$$

Hence,

$$(\xi\mathcal{F}+E)'(t) \leq -\beta_4\xi(t)E(t), \quad \forall t \geq t_0. \tag{4.4}$$

Since $\xi > 0, \mathcal{F} > 0$ and $\mathcal{F} \sim E$, there exists $\beta_5 > 0$ such that

$$L_1'(t) \leq -\beta_5\xi(t)L_1(t), \quad \forall t \geq t_0, \tag{4.5}$$

where $L_1 = \xi\mathcal{F} + E$. Combining (4.4) and (4.5), there exists $\beta_6 > 0$ such that,

$$E(t) \leq ce^{-\beta_6\int_{t_0}^{t}\xi(s)ds} = \omega_2 M_2^{-1}\left(\omega_1\int_{t_0}^{t}\xi(s)ds\right), \quad \forall t \geq t_0.$$

Case 2. M is nonlinear.

We define the functional

$$\mathcal{L}(t) = \mathscr{L}(t) + \Theta(t).$$

Clearly, $\mathcal{L} \geq 0$, moreover by (3.19) and (3.21) there exists $\beta_7 > 0$ such that

$$\begin{aligned}\mathcal{L}(t) \leq & -\frac{1}{4}\left[(1-l)\|\nabla u\|^2 + \|u_t\|^2 + (g\circ\nabla u)(t) + \frac{\lambda\zeta}{\kappa_0}\int_{t-\tau(t)}^{t} e^{-\lambda(t-s)}\|\nabla u_t(s)\|^2 ds\right]\\ \leq & -\beta_7 E(t).\end{aligned} \tag{4.6}$$

Integrating (4.6) over (t_0, t), we get

$$\beta_7\int_{t_0}^{t} E(s)ds \leq \mathcal{L}(t_0) - \mathcal{L}(t) \leq \mathcal{L}(t_0),$$

and we deduce that

$$\int_0^\infty E(s)ds < \infty. \tag{4.7}$$

Now, we define the functional

$$\varphi(t) := \gamma\int_0^t\int_\Omega |\nabla u(t)-\nabla u(s)|^2 dxds,$$

and observe that on account of (4.7), we can choose $0 < \gamma < \dfrac{1}{2\int_0^\infty E(s)ds}$ such that

$$0 < \varphi(t) < 1, \quad \forall t \geq t_0. \tag{4.8}$$

We also define the functional

$$\psi(t) := -\int_{t_0}^{t} g'(s)\int_\Omega |\nabla u(t)-\nabla u(t-s)|^2 dxds.$$

Clearly, $\psi(t) \leq -cE'(t)$. Recalling that $M(0) = 0$ and M is strictly convex on $[0,\alpha]$, $r > 0$, then for any $\rho \in (0,1)$ and $t \in (0,\alpha]$

$$M(\rho t) < \rho M(t). \tag{4.9}$$

Using assumptions (A_1)-(A_2), (4.8), (4.9) and Jensen's inequality, as well as (2.8), we have

$$\begin{aligned}
\psi(t) =& \frac{1}{\gamma\varphi(t)}\int_{t_0}^{t}\varphi(t)(-g'(s))\int_\Omega \gamma|\nabla u(t)-\nabla u(t-s)|^2dxds\\
\geq& \frac{1}{\gamma\varphi(t)}\int_{t_0}^{t}\varphi(t)\xi(s)M(g(s))\int_\Omega \gamma|\nabla u(t)-\nabla u(t-s)|^2dxds\\
\geq& \frac{\xi(t)}{\gamma\varphi(t)}\int_{t_0}^{t}M(\varphi(t)g(s))\int_\Omega \gamma|\nabla u(t)-\nabla u(t-s)|^2dxds\\
\geq& \frac{\xi(t)}{\gamma}M\left(\frac{1}{\varphi(t)}\int_{t_0}^{t}\varphi(t)g(s)\int_\Omega \gamma|\nabla u(t)-\nabla u(t-s)|^2dxds\right)\\
=& \frac{\xi(t)}{\gamma}M\left(\gamma\int_{t_0}^{t}g(s)\int_\Omega |\nabla u(t)-\nabla u(t-s)|^2dxds\right)\\
=& \frac{\xi(t)}{\gamma}\overline{M}\left(\gamma\int_{t_0}^{t}g(s)\int_\Omega |\nabla u(t)-\nabla u(t-s)|^2dxds\right),
\end{aligned}$$

where $\overline{M}$ is the extension of M, see (2.8). Therefore,

$$\int_{t_0}^{t}g(s)\int_\Omega |\nabla u(t)-\nabla u(t-s)|^2dxds \leq \frac{1}{\gamma}\overline{M}^{-1}\left(\frac{\gamma\psi(t)}{\xi(t)}\right),$$

then due to (4.1), we get

$$\mathcal{F}'(t) \leq -\beta_4 E(t) + c\overline{M}^{-1}\left(\frac{\gamma\psi(t)}{\xi(t)}\right), \quad \forall t\geq t_0. \tag{4.10}$$

Let $\alpha_0 < \alpha$, and define the functional $\mathscr{F}_1$ by

$$\mathscr{F}_1(t) := \overline{M}'(t)\left(\alpha_0\frac{E(t)}{E(0)}\right)\mathcal{F}(t) + E(t).$$

Recalling that $E' \leq 0$, $\overline{M}' > 0$, $\overline{M}'' > 0$ as well as making use of (4.10), we obtain that $\mathscr{F}_1 \sim E$ and

$$\begin{aligned}
\mathscr{F}_1'(t) =& \overline{M}'\left(\alpha_0\frac{E(t)}{E(0)}\right)\mathcal{F}'(t) + \alpha_0\frac{E'(t)}{E(0)}\overline{M}''\left(\alpha_0\frac{E(t)}{E(0)}\right)\mathcal{F}(t) + E\\
\leq& -\beta_4 E(t)\overline{M}'\left(\alpha_0\frac{E(t)}{E(0)}\right) + C\overline{M}'\left(\alpha_0\frac{E(t)}{E(0)}\right)\overline{M}^{-1}\left(\frac{\gamma\psi(t)}{\xi(t)}\right) + E'(t).
\end{aligned} \tag{4.11}$$

We denote by $\overline{M}^\star$ the convex conjugate of $\overline{M}$ in the sense of Young. Thus,

$$\overline{M}^\star(y) = y(\overline{M}')^{-1}(y) - \overline{M}[(\overline{M}')^{-1}(y)] \tag{4.12}$$

and $\overline{M}^\star$ satisfies the following Young inequality

$$pq \leq \overline{M}^\star(p) + \overline{M}(q). \tag{4.13}$$

We set $p = \overline{M}'\left(\alpha_0\frac{E(t)}{E(0)}\right)$ and $q = \overline{M}^{-1}\left(\frac{\gamma\psi(t)}{\xi(t)}\right)$, then on account of (4.11)-(4.13), we obtain

$$\begin{aligned}
\mathscr{F}_1'(t) \leq& -\beta_4 E(t)\overline{M}'\left(\alpha_0\frac{E(t)}{E(0)}\right) + c\overline{M}^\star\left(\overline{M}'\left(\alpha_0\frac{E(t)}{E(0)}\right)\right) + c\frac{\gamma\psi(t)}{\xi(t)}\\
\leq& -\beta_4 E(t)\overline{M}'\left(\alpha_0\frac{E(t)}{E(0)}\right) + c\alpha_0\frac{E(t)}{E(0)}\overline{M}'\left(\alpha_0\frac{E(t)}{E(0)}\right) + c\frac{\gamma\psi(t)}{\xi(t)}.
\end{aligned} \tag{4.14}$$

Now, using (3.4) we have $\alpha_0 \dfrac{E(t)}{E(0)} < \alpha$ which implies by (2.8) that

$$\overline{M}'\left(\alpha_0 \frac{E(t)}{E(0)}\right) = M'\left(\alpha_0 \frac{E(t)}{E(0)}\right).$$

Hence, multiplying (4.14) by $\xi(t)$, and making use of $\psi(t) \le -cE'(t)$, we get

$$\begin{aligned}
\xi(t)\mathscr{F}_1'(t) \le &- \beta_4 \xi(t) E(t) M'\left(\alpha_0 \frac{E(t)}{E(0)}\right) + c\alpha_0 \frac{E(t)}{E(0)} \xi(t) M'\left(\alpha_0 \frac{E(t)}{E(0)}\right) \\
&+ c\gamma\psi(t). \\
\le &- \beta_4 \xi(t) E(t) M'\left(\alpha_0 \frac{E(t)}{E(0)}\right) + c\alpha_0 \frac{E(t)}{E(0)} \xi(t) M'\left(\alpha_0 \frac{E(t)}{E(0)}\right) - cE'(t).
\end{aligned} \tag{4.15}$$

Again, defining $\mathscr{F}_2 = \xi\mathscr{F}_1 + cE$, then since $\mathscr{F}_1 \sim E$, there exist $\varpi_1, \varpi_2 > 0$ such that

$$\varpi_1 \mathscr{F}_2 \le E(t) \le \varpi_2 \mathscr{F}_2. \tag{4.16}$$

Now, from (4.15) and choosing $\alpha_0 < \dfrac{\beta_4 E(0)}{c}$, there exists $\beta_8 > 0$ such that for all $t \ge t_0$, we have

$$\mathscr{F}_2'(t) \le -\beta_8 \xi(t) \left(\frac{E(t)}{E(0)}\right) M'\left(\frac{E(t)}{E(0)}\right) = -\beta_8 \xi(t) M_1\left(\frac{E(t)}{E(0)}\right). \tag{4.17}$$

Let

$$H(t) = \frac{\varpi_1 \mathscr{F}_2(t)}{E(0)},$$

then recalling, from (A_2), that M is strictly increasing and strictly convex on $(0, \alpha]$ and that $M_1'(t) = M'(\alpha_0 t) + \alpha_0 t M''(\alpha_0 t)$, we deduce that $M_1(t), M_1'(t) > 0$. Again, from (4.16), we obtain that

$$H(t) \sim E(t). \tag{4.18}$$

It follows due to (4.17) that there exists $\omega_1 > 0$, such that

$$H'(t) \le -\omega_1 \xi(t) M_1(H(t)), \quad \forall t \ge t_0. \tag{4.19}$$

We finally define

$$M_2(t) = \int_t^\alpha \frac{1}{sM'(s)} ds,$$

M_2 is strictly decreasing on $(0, \alpha]$ and $\lim_{t\to 0} M_2(t) = \infty$. We integrate (4.19) over (t_0, t), to obtain

$$M_2(\alpha_0 H(t)) - M_2(\alpha_0 H(t_0)) = \int_{\alpha_0 H(t_0)}^{\alpha_0 H(t)} \frac{1}{sM'(s)} ds \ge \omega_1 \int_{t_0}^t \xi(s) ds,$$

which implies that

$$M_2(\alpha_0 H(t)) \ge \omega_1 \int_{t_0}^t \xi(s) ds.$$

It follows that

$$H(t) \le \frac{1}{\alpha_0} M_2^{-1}\left(\omega_1 \int_{g^{-1}(\alpha)}^t \xi(s) ds\right), \tag{4.20}$$

hence, using (4.18) and (4.20), estimate (4.3) follows immediately. □

5. Numerical study

In this section, we illustrate numerically the result in Theorem 4.2. We establish a numerical scheme for our problem (2.1) using finite difference method in time and finite element method in space.

To discretize in time, we truncate the interval $(0,\infty)$ into $(0,T]$ where T is large enough. Divide $[0,T]$ uniformly into N subintervals with size k each and nodes $\{t_n\}_{n=0}^N$, i.e., $t_n = nk$ for $0 \le n \le N$, where $k = T/N$. For the grid function w^n, let

$$\delta_t w^n = \frac{w^n - w^{n-1}}{k}, \quad \delta_{tt} w^n = \frac{w^{n+1} - 2w^n + w^{n-1}}{k^2}.$$

For the spatial discretization, we choose $\Omega = (a,b)\times(c,d)$ and then divide both (a,b) (in the x-direction) and (c,d) (in the y-direction) into a family of uniform cells. Let $x_i = i\,h_x$ for $0 \le i \le M_x$ with $h_x = (b-a)/M_x$ and let $y_j = j\,h_y$ for $0 \le j \le M_y$ with $h_y = (d-c)/M_y$. Then, the $\mathcal{C}^2$ Galerkin finite dimensional space $S_h := S_{h_x} \otimes S_{h_y}$, where

$$S_{h_x} = \{v \in H^1(a,b) : \ v|_{[x_{i-1},x_i]} \in P_3 \text{ for } 1 \le i \le N_x, \text{ with } v(x)|_{x=a,b} = 0\},$$

where P_3 is the space of polynomials of degree at most 3 in x, S_{h_y} is defined similarly.

Usually, continuous Galerkin finite element schemes are motivated by the weak formulation of the model problem. So, we take the inner product of (2.1) with $\phi \in H_0^1(\Omega)$ then using Green's formula. This leads to

$$\langle u'', \phi\rangle + \langle \nabla u, \nabla\phi\rangle - \int_0^t g(t-s)\langle \nabla u(s), \nabla\phi\rangle\, ds + \mu_1\langle \nabla u', \nabla\phi\rangle + \mu_2\langle \nabla u'(t-\tau(t)), \nabla\phi\rangle = 0. \tag{5.1}$$

Replacing $u'(t-\tau(t))$ by $u'(t) - \int_{t-\tau(t)}^t u''(s)ds$, we have

$$\langle u'', \phi\rangle + \langle \nabla u, \nabla\phi\rangle - \int_0^t g(t-s)\langle \nabla u(s), \nabla\phi\rangle\, ds + \mu_1\langle \nabla u', \nabla\phi\rangle + \mu_2\left\langle \nabla\left(u' - \int_{t-\tau(t)}^t u''(s)ds\right), \nabla\phi\right\rangle = 0. \tag{5.2}$$

Consequently, for each $t>0$, the semi-discrete finite element solution $u_h(t) \in S_h$ is defined by

$$\langle u_h'', \phi\rangle + \langle \nabla u_h, \nabla\phi\rangle - \int_0^t g(t-s)\langle \nabla u_h(s), \nabla\phi\rangle\, ds + \mu_1\langle \nabla u_h', \nabla\phi\rangle + \mu_2\left\langle \nabla\left(u_h' - \int_{t-\tau(t)}^t u_h''(s)ds\right), \nabla\phi\right\rangle = 0. \tag{5.3}$$

Our fully-discrete numerical solution U_h^n approximates $u(t_n)$ is defined by

$$\langle \delta_{tt}U_h^n, \phi\rangle + \langle \nabla U_h^n, \nabla\phi\rangle - \int_0^{t_{n+1}} g(t_{n+1}-s)\langle \nabla U_h^n(s), \nabla\phi\rangle ds + \mu_1\langle \nabla\delta_t U_h^n, \nabla\phi\rangle + \mu_2\left\langle \nabla\left(\delta_t U_h^n - \int_{t-\tau(t)}^t \delta_{tt}U_h^n(s)ds\right), \nabla\phi\right\rangle = 0, \tag{5.4}$$

$\forall \phi \in S_h$, and for $1 \le n \le N-1$.

For computing purposes, we need to write our scheme in a matrix form. let $d_{hx} := \dim S_{hx} = N_x - 1$ and let $\{\phi_{p_x}\}_{p=1}^{d_{hx}}$ denote the basis functions of S_{h_x}. We define $d_{hx}\times d_{hx}$ matrices:

$$\boldsymbol{M}_x = \left[\int_a^b \phi_q\phi_p dx\right] \text{ and } \boldsymbol{G}_x = \left[\int_a^b \phi_q'\phi_p' dx\right].$$

In the y-direction, we use similar notations but with y in place of x.

The $(d_{hx} \times d_{hy})$-dimensional column vector $\mathbf{b}^n$ is the transpose of the vector

$$[b^n_{1,1}, b^n_{1,2}, \cdots, b^n_{1,d_{hy}}, \cdots, b^n_{d_{hx},1}, \cdots, b^n_{d_{hx},d_{hy}}].$$

Therefore, through tensor products of one-dimensional $\mathcal{C}^2$ splines, the fully-discrete scheme (5.4) has the following matrix representation, for $1 \leq n \leq N-1$,

$$\begin{aligned}
&(\boldsymbol{M}_x \otimes \boldsymbol{M}_y - \mu_2\tau(t_n)\boldsymbol{G}_x \otimes \boldsymbol{G}_y)\,\mathbf{b}^{n+1} = \\
&\quad \big(2\boldsymbol{M}_x \otimes \boldsymbol{M}_y - k^2\boldsymbol{G}_x \otimes \boldsymbol{G}_y - k(\mu_1+\mu_2)\boldsymbol{G}_x \otimes \boldsymbol{G}_y - 2\mu_2\tau(t_n)\boldsymbol{G}_x \otimes \boldsymbol{G}_y\big)\,\mathbf{b}^n \\
&+k^2\boldsymbol{G}_x \otimes \boldsymbol{G}_y \sum_{j=0}^{n} g^j_{n+1}\mathbf{b}^j + (k(\mu_1+\mu_2)\boldsymbol{G}_x \otimes \boldsymbol{G}_y + \mu_2\tau(t_n)\boldsymbol{G}_x \otimes \boldsymbol{G}_y - \boldsymbol{M}_x \otimes \boldsymbol{M}_y)\,\mathbf{b}^{n-1},
\end{aligned}$$

with $g^j_{n+1} := \int_{t_j}^{t_{j+1}} g(t_{n+1}-s)\,ds$.

Therefore, at each time level t_{n+1}, we solve a finite square linear system, where the unknown is the column vector $\mathbf{b}^{n+1}$.

Furthermore, from the matrix form, it is clear that our scheme (5.4) is a three-time level scheme, so the approximate solutions U^0_h and U^1_h need to be determined first, and then U^j_h for $2 \leq j \leq N$ can be computed by solving the above linear system recursively. We choose $U^0_h \in S_h$ to be the bicubic spline polynomial interpolates u_0 at the interior nodal nodes. However, we choose $U^1_h \in S_h$ to be the bicubic spline polynomial interpolates $u_0 + t_1 u_1$ at the interior nodal nodes.

For the computer implementation of the linear system, it is important to consider discretization of spatial Galerkin-type integrals in the scheme. To this end, on each cell of our two-dimensional partition, the integrals are approximated using 2-point Gauss quadrature rule in each direction.

In our test problem, we choose $\Omega = (0,1) \times (0,1)$, the time interval is $(0,80)$, the initial data $u_0(x,y) = 2^{10}xy(1-x)(1-y)$, $u_1(x,y) = 0$, the relaxation function $g(t) = \mathrm{e}^{-t}$, $\tau(t) = \frac{1}{2}\mathrm{e}^{-2t}$ and the coefficients $\mu_1 = 1, \mu_2 = 2$. The spatial mesh consists of 400 (square) cells of equal areas, while the time domain consists of 80000 subintervals. In this example, we expect that the energy decays exponentially. On the other hand, if the relaxation function $g(t) = \frac{1}{(1+t)^2}$, we expect that the energy decays polynomially, which is confirmed Figuers 1–4.

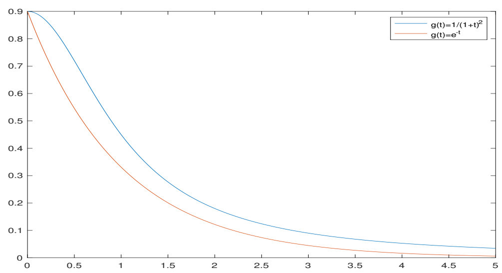

FIGURE 1. The graphical plot of the approximated energy $E(t)$ against t in the interval $[0,5]$.

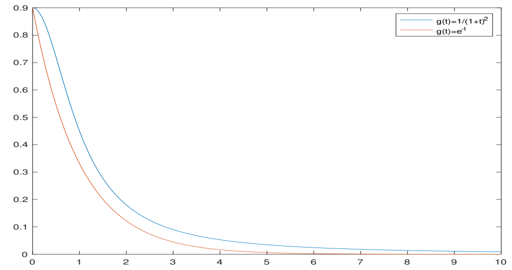

FIGURE 2. The graphical plot of the approximated energy $E(t)$ against t in the interval $[0, 10]$.

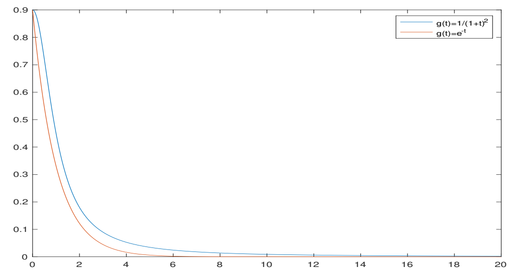

FIGURE 3. The graphical plot of the approximated energy $E(t)$ against t in the interval $[0, 20]$.

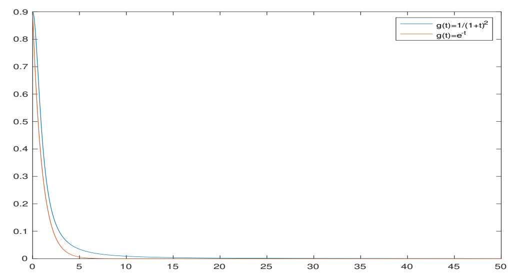

FIGURE 4. The graphical plot of the approximated energy $E(t)$ against t in the interval $[0, 50]$.

6. Conclusion

In the present work, we have established an exponential decay result for a wave equation with finite memory and strong time dependent delay, with a more general condition on the kernel in the memory term. The recent work of Feng [9] is a particular case. We also presented numerical analysis of the problem in other to validate our theoretical result.

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