Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 13 Issue 3 (2022), Pages 14-26
https://doi.org/10.54379/jma-2022-3-2

DYNAMIC ANALYSIS OF A CHAOTIC 3D QUADRATIC SYSTEM USING PLANAR PROJECTION.

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Abstract. The theory of dynamical systems is one of the most important theorems of scientific research because it relies heavily on most of the major fields of applied mathematics to give a sufficiently broad view of reality, but it still poses some problems, especially with regard to the modeling of certain physical phenomena. Since most of these systems are designed as continuous or discrete dynamic systems with large dimensions and multiple bifurcation parameters, researchers face major problems in qualitative study. In this paper, we propose a method to study bifurcations of continuous three-dimensional dynamic systems in general and chaotic systems in particular, which contains many bifurcation parameters. This method is mainly based on the projection on the plane and on the appropriate bifurcation parameter.

1. Introduction

The development of chaos theory begins in the late XXe century. This represents a new approach to problems scientists in all disciplines, as well in mathematics or physics as in medicine or biology. Indeed, the traditional science is based on notions such as determinism and seeks first and foremost, predictability. Chaos theory, for its part, aims to find an order in the apparent chaos. To do this, she relies on theconcepts of non-linearity and auto-similarity and has as an experimental tool computer predilection. Chaos theory, with its new approach, not only helps us to better understand the world around us, but also provides applications concrete and topical in fields as varied as physics, biology, astronomy, medicine, radio communication and computer science.

The purpose of this paper is to provide a new method for the study of a three-dimensional continuous quadratic chaotic dynamic system with multiple bifurcation parameters [1]. This method gives important results on the dynamic behavior of the latter, stability, bifurcations and chaos. This method consists of two steps, a projection on the plane to obtain a dynamic system of lower dimension [2], then the choice of the appropriate parameter, to simplify, we choose a three-dimensional dynamic system with seven parameters. We will examine a subsystem of the original

2000 *Mathematics Subject Classification.* 30D05, 32H50.
Key words and phrases. Dynamic Analysis, Chaotic, 3D Quadratic System, Planar Projection.

Submitted December 17, 2021. Published May 14, 2022.
Communicated by Kais Ammari.

system by analyzing its dynamic behavior in a lower dimension. This will be useful for the final study of the dynamic behaviour of the original system.

1.1. **Dynamic analysis of a nonlinear system in three dimensions.** Let the dynamic system defined as follows

$$\begin{cases} \frac{dx_1}{dt} = a_1x_1 + a_2x_2 + a_3x_3 \\ \frac{dx_2}{dt} = x_1x_3 + b \\ \frac{dx_3}{dt} = c_1x_1 + c_2x_2x_3 + c \end{cases} \tag{1.1}$$

Where, $a_i \neq 0, (1 \leq i \leq 3)$ $c_i \neq 0, b \neq 0$ and $c \neq 0$ are real parameters. When projecting onto the plane $(x_1 - x_2)$, the following new system is obtained

$$\begin{cases} \frac{dx_1}{dt} = a_1x_1 + a_2x_2 + a_3x_3 \\ \frac{dx_2}{dt} = x_1x_3 + b \end{cases} \tag{1.2}$$

Where, x_3 is considered as a known function of the time variable t. When $t = t_0$ the system (2) becam linear and bi-dimensional with constant coefficient [1]. The Jacobian matrix of the system (2) is given by

$$J = \begin{pmatrix} a_1 & a_2 \\ x_3 & 0 \end{pmatrix}.$$

The determinant of matrix J is

$$\det(J) = -a_2x_3.$$

Notice that for $x_3 \neq 0$, $\det(J)$ is not zero.

1.1.1. *The fixed point of the system (2).* The fixed point of the system (2) obtained from

$$\frac{dx_1}{dt} = \frac{dx_2}{dt} = 0,$$

hence, we have

$$\begin{cases} a_1x_1 + a_2x_2 + a_3x_3 = 0 \\ x_1x_3 + b = 0 \end{cases}. \tag{1.3}$$

With a simple calculation, we get

$$x_1^e = -\frac{b}{x_3} \text{ and } x_2^e = \frac{a_1b - a_3x_3^2}{a_2x_3}.$$

Thus the system (2) has a single fixed point $e\left(-\frac{b}{x_3}, \frac{a_1b-a_3x_3^2}{a_2x_3}\right)$. Using the translation $(x = x_1 - x_1^e, y = x_2 - x_2^e)$ the point e can be reduced to the origin O .

1.1.2. *Fixed point classification according to eigenvalues.* To perform a clacification of the fixed points of (2), we calculate the eigenvalues of the matrix J.

$$\det(\lambda I - J) = \lambda^2 - a_1\lambda - a_2x_3,$$

we put

$$\det(\lambda I - J) = 0,$$

hence

$$\lambda^2 - a_1\lambda - a_2x_3 = 0, \tag{1.4}$$

we will study only the case $a_2 > 0$, as for the case $a_2 < 0$, it will be studied in the same method.

For $a_2 > 0$, we have the following cases

1.: For $x_3 > 0$, the equation (4) has two solutions λ_1 and λ_2 such that, $\lambda_1 < 0 < \lambda_2$ then the fixed point e is a "saddle point". The curve of the solution in the plane $(x_1 - x_2)$ is represented in (Figure 1.a) where, the directions of the orbits are represented by arrows when time t increases. When t tend to infinity only two orbits tend towards the fixed point e and the others diverge towards infinity in two different directions.

2.: For $a_1 < 0$, when $x_3 < -\frac{a_1^2}{4a_2}$: The equation (4) has two solutions λ_1 and λ_2 such that, $\lambda_1 < \lambda_2 < 0$, so the fixed point e is a "Node", which explains the tendency of solution curves on the plane $(x_1 - x_2)$ to infinity with the exception of two orbits that tend towards point e. This is shown in (Figure 1.b), where the direction of the orbits is represented by arrows.

3.: For $a_1 < 0$, when $-\frac{a_1^2}{4a_2} < x_3 < 0$: The equation (4) has two complex solutions conjugated with a negative real part, the fixed point e is a "focus". The curve of the solutions on the plane $(x_1 - x_2)$ is shown in (Figure 1.c), where the direction of the arrow is the direction of the orbit when the time t increases. When t tends to infinity, all the orbits move in spiral around to point e.

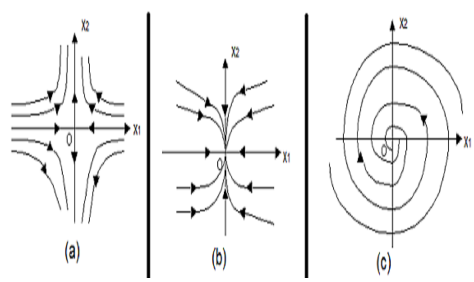

FIGURE 1. (a): the fixed point e is a saddle point. (b): The fixed point e is a "node". (c): the fixed point e is a "focus".

1.1.3. *The relationship between the time variable t and the function $x_3(t\ )$.* When t tends to infinity, The orbit $x_3(t\ )$ intersects the two straight lines $x_3 = -\frac{a_1^2}{4a_2}$ and $x_3 = 0$ alternately and several times. Hence, the division of the x_3 axis into three disjoint domains $\left(-\infty\ ,\ -\frac{a_1^2}{4a_2}\right)$, $\left(-\frac{a_1^2}{4a_2}\ ,\ 0\right)$ and $(0\ ,\ +\infty)$, Which implies the possession of the system (2) of different dynamic behaviors in the three domains above. When t tends to infinity the system (2) changes its dynamic behavior and $x_3(t\ )$ passes through these domains repeatedly, leading to complex dynamics such as the appearance of bifurcations and chaos. It is noticed that the system (2) depends on time t when $x_3(t\ )$ varies over time. The two systems (1) and (2) can be verified that are chaotic when the function $x_3(t\ )$ passes through the straight lines $x_3 = -\frac{a_1^2}{4a_2}$ and $x_3 = 0$ alternately [1], [2].

1.1.4. *The relationship between the time variable t and the function $x_3(t\ )$.* When t tends to infinity, The orbit $x_3(t\ )$ intersects the two straight lines $x_3 = -\frac{a_1^2}{4a_2}$ and $x_3 = 0$ alternately and several times. Hence, the division of the x_3 axis into three disjoint domains $\left(-\infty\ ,\ -\frac{a_1^2}{4a_2}\right)$, $\left(-\frac{a_1^2}{4a_2}\ ,\ 0\right)$ and $(0\ ,\ +\infty)$, Which implies the possession of the system (2) of different dynamic behaviors in the three domains above. When t tends to infinity the system (2) changes its dynamic behavior and $x_3(t\ )$ passes through these domains repeatedly, leading to complex dynamics such as the appearance of bifurcations and chaos. It is noticed that the system (2) depends on time t when $x_3(t\ )$ varies over time. The two systems (1) and (2) can be verified that are chaotic when the function $x_3(t\ )$ passes through the straight lines $x_3 = -\frac{a_1^2}{4a_2}$ and $x_3 = 0$ alternately [1] , [2].

1.2. **The fixed point of the system (1).** The system fixed point (1) results from the first and second equation, so we have

$$x_3 = -\frac{b}{x_1}, \tag{1.5}$$

and

$$x_2 = \frac{a_3 b - a_1 x_1^2}{a_2 x_1}, \tag{1.6}$$

by substituting (5) and (6) in the third system equation (1), we obtain the following equation (7)

$$a_2 c_1 x_1^3 + (a_1 b c_2 + a_2 c) x_1^2 - a_3 c_2 b = 0 \tag{1.7}$$

To obtain a single fixed point, you must take the case

$$a_1 b c_2 + a_2 c = 0 \text{ or } c = -\frac{a_1 c_2}{a_2} b \tag{1.8}$$

Then, under the condition (8), the equation (7) has a unique real root $x_1 = \sqrt[3]{\frac{a_3 c_2}{a_2 c_1} b}$. therefore the fixed point of system (1) is given by $E(\sqrt[3]{\frac{a_3 c_2}{a_2 c_1} b}, \frac{a_3 b - a_1 x_1^2}{a_2 x_1}, -\frac{b}{x_1})$.

1.2.1. *linearization of the system (2) at fixed point $E(x_1, x_2, x_3)$.* The stability of the equilibrium state (point E), is analyzed by linearizing the system (1) to point E under the linear transformation [2] , [3] , [5].

$$\begin{cases} x = x_1 - x_0 \\ y = x_2 - y_0 \\ z = x_3 - z_0 \end{cases}$$

Where

$$\begin{cases} x_0 = \sqrt[3]{\frac{a_3 c_2}{a_2 c_1} b} \\ y_0 = \frac{a_3 b - a_1 x_1^2}{a_2 x_1} \\ z_0 = -\frac{b}{x_1} \end{cases} \tag{1.9}$$

The system (1) becomes in the form

$$\begin{cases} \frac{dx}{dt} = a_1 x + a_2 y + a_3 z \\ \frac{dy}{dt} = z_0 x + x_0 z + xz \\ \frac{dz}{dt} = c_1 x + c_2 z_0 y + c_2 y_0 z + c_2 yz \end{cases}, \tag{1.10}$$

the Jacobian matrix $A(E)$ of the system (10) is given as follows

$$A(E) = \begin{pmatrix} a_1 & a_2 & a_3 \\ z_0 & 0 & x_0 \\ c_1 & c_2 z_0 & c_2 y_0 \end{pmatrix}, \tag{1.11}$$

its characteristic polynomial is given by

$$P(\lambda) = \lambda^3 + A\lambda^2 + B\lambda + C \tag{1.12}$$

Where

$$\begin{cases} A = -(c_2 y_0 + a_1) \\ B = bc_2 - a_3 c_1 + a_1 c_2 y_0 - a_2 z_0 \\ C = a_2 c_1 x_0 \end{cases} \tag{1.13}$$

Then, the conditions of Routh-Hurwitz lead to the condition that the real parts of the roots are $\lambda-$négative if $A > 0$, $C > 0$ and $AB - C > 0$. It is noticed that the coefficients of the polynomial (12) are all positive, so $P(\lambda) > 0$ for all $\lambda > 0$, therefore the only fixed point is unstable ($rel(\lambda) > 0$) If $P(\lambda)$ has two conjugate complex eigenvalues [1]. It is noticed that $\lambda_1 = +i\omega$ and $\lambda_2 = -i\omega$, because the sum of the three roots of the cubic $P(\lambda)$ is

$$\lambda_1 + \lambda_2 + \lambda_3 = -A \tag{1.14}$$

So, we have $\lambda_3 = -A = a_1 + c_2 y_0$ which is located on the system stability margin (10), hence

$$\lambda_3 = \frac{-c_2 a_1 x_0^2 + a_1 a_2 x_0 + c_2 a_3 b}{a_2 x_0}, \tag{1.15}$$

on the other hand, we have

$$P(\lambda_3) = -AB + C = 0, \tag{1.16}$$

where

$$\begin{cases} A = \frac{a_2 a_3 c_1 c_2 x_0^2 + a_1 a_3 b c_2^2 x_0 - a_1 a_2 a_3 c_2 b}{a_2 a_3 c_2 b} \\ B = \frac{(a_1 a_2 a_3 c_1 c_2 + a_2^3 c_1) x_0^2 - a_1^2 a_3 b c_2^2 x_0 + a_3 b c_2 (a_2 b c_2 - a_2 a_3 c_1)}{a_2 a_3 c_2 b} \\ C = a_2 c_1 x_0 \end{cases}, \tag{1.17}$$

and

$$x_0 = \sqrt[3]{\frac{a_3 c_2}{a_2 c_1} b}.$$

A substitution of (17) in (16) we obtain

$$\begin{aligned} &(a_2^2 a_3 b c_1 c_2^3 - a_2^2 a_3^2 c_1^2 c_2^2 - a_1^3 a_3 b c_1^4 - a_1^2 a_2^2 a_3 c_1 c_2 - a_1 a_2^4 c_1) x_0^2 \\ &+(-a_1 a_2 a_3^2 b c_1 c_2^3 - a_2^3 a_3 b c_1 c_2^2 + a_1 a_2 a_3 b^2 c_2^4 - a_1 a_2 a_3^2 b c_1 c_2^2 + a_1^3 a_2 a_3 b c_2^2 + a_2^3 a_3 b c_1 c_2^2) x_0 \\ &- a_1^2 a_3^2 b^2 c_2^4 + a_1 a_3^2 b^2 c_1 c_2^4 + a_1 a_2^2 a_3 b c_2^3 - a_1 a_2^2 a_3 b^2 c_2^2 + a_1 a_2^2 a_3^2 b c_1 c_2 = 0, \end{aligned} \tag{1.18}$$

or

$$\alpha a_1^3 + \beta a_1^2 + \gamma a_1 + \delta = 0, \tag{1.19}$$

where

$$\begin{cases} \alpha = -a_3 b c_1^4 x_0^2 + a_2 a_3 b c_2^2 x_0 \\ \beta = -a_2^2 a_3 c_1 c_2 x_0^2 - a_3^2 b^2 c_2^4 \\ \gamma = -a_2^4 c_1 x_0^2 + (-a_2 a_3^2 b c_1 c_2^3 + a_2 a_3 b^2 c_2^4 - a_2 a_3^2 b c_1 c_2^2) x_0 \\ +a_3^2 b^2 c_1 c_2^4 + a_2^2 a_3 b c_2^3 - a_2^2 a_3 b^2 c_2^2 + a_2^2 a_3^2 b c_1 c_2 \\ \delta = (a_2^2 a_3 b c_1 c_2^3 - a_2^2 a_3^2 c_1^2 c_2^2) x_0^2 + (-a_2^3 a_3 b c_1 c_2^2 + a_2^3 a_3 b c_1 c_2^2) x_0 \end{cases} \tag{1.20}$$

Assume that $\alpha > 0$ and the equation (19) has only one solution $a_1 = a_0$,hence for $a_1 = a_0$ the fixed point E will lose its stability, so a hopf bifurcation can occur [2] , [6] , [7]. Using the two conditions (14), (16) and $a_1 = a_0$, the polynomial $P(\lambda)$ can be written in the form

$$P(\lambda) = (\lambda - a_0 - c_2 y_0)(\lambda^2 + \widetilde{B}), \tag{1.21}$$

where

$$\widetilde{B} = \frac{(a_0a_2a_3c_1c_2 + a_2^3c_1)x_0^2 - a_0^2a_3bc_2^2x_0 + a_3bc_2(a_2bc_2 - a_2a_3c_1)}{a_2a_3c_2b}. \tag{1.22}$$

It is obvious that, the equation $P(\lambda) = 0$ has three roots, one negative, $\lambda_3 = a_0 + c_2y_0$ and a pair of conjugated purely imaginary roots

$$\lambda_{1,2} = \pm i\sqrt{\frac{(a_0a_2a_3c_1c_2 + a_2^3c_1)x_0^2 - a_0^2a_3bc_2^2x_0 + a_3bc_2(a_2bc_2 - a_2a_3c_1)}{a_2a_3c_2b}} = \pm id \tag{1.23}$$

Differentiate the two sides of equation (12) with respect to a, we obtain

$$\frac{d\lambda}{dt} = \frac{\left(1 - \frac{c_2x_0}{a_2}\right)\lambda^2 + \left(\frac{2a_1bc_2x_0 - a_2c_1x_0^2}{a_2b}\right)\lambda}{3\lambda^2 + 2A\lambda + B}, \tag{1.24}$$

hence

$$\frac{d\,\mathrm{Re}\,\lambda}{da_1}\,|_{a_1=a_0} = -\frac{1}{2}\frac{\left(\frac{c_2x_0}{a_2} - 1\right)d^2 \ + \ (a_0 + c_2y_0)\left(\frac{2a_0bc_2x_0 - a_2c_1x_0^2}{a_2b}\right)}{d^2 + (a_0 + c_2y_0)^2} < 0, \tag{1.25}$$

and

$$\frac{d\,\mathrm{Im}\,\lambda}{da_1}\,|_{a_1=a_0} = -\frac{1}{2}d\frac{\left(\frac{2a_0bc_2x_0 - a_2c_1x_0^2}{a_2b}\right) \ + \ (a_0 + c_2y_0)\left(1 - \frac{c_2x_0}{a_2}\right)}{d^2 + (a_0 + c_2y_0)^2}. \tag{1.26}$$

Conclusion:

(1) According to the hopf bifurcation theorem, it can be concluded that a_0 is the critical value.
(2) The fixed point E is stable, when $a_1 < a_0$, and there are a periodic solutions when $a_1 > a_0$.
(3) When a_1 crosses the value a_0, the system (1) undergoes a Hopf bifurcation at fixed point E.

2. Property of Hopf bifurcation

In this section we will give the explicit formulas to determine the direction, stability and period of these periodic bifurcation solutions at point E for the critical value $a_1 = a_0$, using regular form techniques [1] , [9] , [14].

2.1. **Supercritical and subcritical bifurcation.**

Let the eigenvectors corresponding to the eigenvalues $\lambda_3 = a_0 + c_2y_0$ and $\lambda_2 = id$ are v_1 and $v_2 + iv_3$.

By a direct calculations, we obtain

$$\left\{ \begin{array}{l} v_1 \begin{pmatrix} 1 \\ \frac{a_2 c_1 x_0^2 - a_0 a_2 b}{-a_0^2 c_2 x_0^2 + (a_0^2 a_2 + a_2 b c_2) x_0 + a_0 a_3 b c_2} \\ \frac{(a_2^2 c_1 + a_0 b c_2^2) x_0^2 - a_3 b^2 c_2^2}{a_0 a_2^2 x_0^2 - a_2 a_3 b c_2 x_0} \end{pmatrix} \\ v_2 \begin{pmatrix} 1 \\ \frac{-a_0 a_2 x_0^2 + a_3 d^2 x_0 - a_2 a_3 b}{a_2^2 x_0^2 + a_3^2 d^2} \\ \frac{c_2 b^2 (b c_2 - d^2) - c_1 c_2^2 x_0 y_0}{((b c_2 - d^2) x_0)^2 + (c_2 d x_0 y_0)^2} \end{pmatrix} \\ v_3 \begin{pmatrix} 0 \\ \frac{d(a_3 b^2 c_2 + a_0 a_3 c_1 x_0 + a_3^2 b c_1)}{a_2 a_3 b^2 c_2 + a_3^2 c_1 d^2 x_0} \\ \frac{(c_1 c_2 b - c_1 d^2) x_0^2 + c_2^2 b^2 y_0}{((b c_2 - d^2) x_0)^2 + (c_2 d x_0 y_0)^2} \end{pmatrix} \end{array} \right. \tag{2.1}$$

We put

$$P = (v_1, v_2, v_3) = \begin{pmatrix} 1 & 1 & 0 \\ \frac{a_2 c_1 x_0^2 - a_0 a_2 b}{-a_0^2 c_2 x_0^2 + (a_0^2 a_2 + a_2 b c_2) x_0 + a_0 a_3 b c_2} & \frac{-a_0 a_2 x_0^2 + a_3 d^2 x_0 - a_2 a_3 b}{a_2^2 x_0^2 + a_3^2 d^2} & \frac{d(a_3 b^2 c_2 + a_0 a_3 c_1 x_0 + a_3^2 b c_1)}{a_2 a_3 b^2 c_2 + a_3^2 c_1 d^2 x_0} \\ \frac{(a_2^2 c_1 + a_0 b c_2^2) x_0^2 - a_3 b^2 c_2^2}{a_0 a_2^2 x_0^2 - a_2 a_3 b c_2 x_0} & \frac{c_2 b^2 (b c_2 - d^2) - c_1 c_2^2 x_0 y_0}{((b c_2 - d^2) x_0)^2 + (c_2 d x_0 y_0)^2} & \frac{(c_1 c_2 b - c_1 d^2) x_0^2 + c_2^2 b^2 y_0}{((b c_2 - d^2) x_0)^2 + (c_2 d x_0 y_0)^2} \end{pmatrix} \tag{2.2}$$

To simplify the calculation, we will replace the matrix (28) by the matrix (29)

$$\begin{pmatrix} 1 & 1 & 0 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{pmatrix}, \tag{2.3}$$

where

$$\left\{ \begin{array}{l} \alpha_1 = \frac{a_2 c_1 x_0^2 - a_0 a_2 b}{-a_0^2 c_2 x_0^2 + (a_0^2 a_2 + a_2 b c_2) x_0 + a_0 a_3 b c_2} \\ \alpha_2 = \frac{-a_0 a_2 x_0^2 + a_3 d^2 x_0 - a_2 a_3 b}{a_2^2 x_0^2 + a_3^2 d^2} \\ \alpha_3 = \frac{d(a_3 b^2 c_2 + a_0 a_3 c_1 x_0 + a_3^2 b c_1)}{a_2 a_3 b^2 c_2 + a_3^2 c_1 d^2 x_0} \end{array} \right. . \tag{2.4}$$

Then, perform the following transformation on the system (10)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = P \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

hence

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = P^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

in order to obtain

$$\left\{ \begin{array}{c} u_1 = m_1 x + m_2 y + m_3 z \\ u_2 = n_1 x + n_2 y + n_3 z \\ u_3 = k_1 x + k_2 y + k_3 z \end{array} \right. , \tag{2.5}$$

therefore

$$\left\{\begin{array}{l} \frac{du_1}{dt} = -du_2 + F(u_1,u_2,u_3) \\ \frac{du_2}{dt} = du_1 + G(u_1,u_2,u_3) \\ \frac{du_3}{dt} = \lambda u_3 + H(u_1,u_2,u_3) \end{array}\right. , \tag{2.6}$$

where

$$\left\{\begin{array}{l} F(u_1,u_2,u_3) = m_2 f(u_1,u_2,u_3) + m_3 c_2 g(u_1,u_2,u_3) \\ G(u_1,u_2,u_3) = \quad n_2 f(u_1,u_2,u_3) + n_3 c_2 g(u_1,u_2,u_3) \\ H(u_1,u_2,u_3) = k_2 f(u_1,u_2,u_3) + k_3 c_2 g(u_1,u_2,u_3) \\ f(u_1,u_2,u_3) = (u_1+u_2)(\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3) \\ g(u_1,u_2,u_3) = (\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3)(\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3) \end{array}\right.$$

$$\left\{\begin{array}{l} m_1 = \left(1 + \frac{\alpha_1\beta_3 - \alpha_3\beta_1}{(\beta_3-\alpha_3)(\alpha_2-\alpha_1)}\right) \\ m_2 = -\frac{\beta_3}{(\beta_3-\alpha_3)(\alpha_2-\alpha_1)} \\ m_3 = \frac{\alpha_3}{(\beta_3-\alpha_3)(\alpha_2-\alpha_1)} \end{array}\right.$$

$$\left\{\begin{array}{l} n_1 = \frac{\alpha_3\beta_1 - \alpha_1\beta_3}{\beta_3(\alpha_2-\alpha_1) - \alpha_3(\beta_2-\beta_1)} \\ n_2 = \frac{\beta_3}{\beta_3(\alpha_2-\alpha_1) - \alpha_3(\beta_2-\beta_1)} \\ n_3 = -\frac{\alpha_3}{\beta_3(\alpha_2-\alpha_1) - \alpha_3(\beta_2-\beta_1)} \end{array}\right.$$

$$\left\{\begin{array}{l} k_1 = \frac{\alpha_1\beta_2 - \beta_1\alpha_2}{\beta_3(\alpha_2-\alpha_1) - \alpha_3(\beta_2-\beta_1)} \\ k_2 = \frac{\beta_1 - \beta_2}{\beta_3(\alpha_2-\alpha_1) - \alpha_3(\beta_2-\beta_1)} \\ k_3 = \frac{\alpha_2 - \alpha_1}{\beta_3(\alpha_2-\alpha_1) - \alpha_3(\beta_2-\beta_1)} \end{array}\right. .$$

By now applying the method of Auchmuty and Nicolas in [1] from the system (32), the following quantities can be calculated in $a_1 = a_0$ and $O(0,0,0)$.

$$\begin{aligned} g_{11} &= \frac{1}{4}\left[\frac{\partial^2 F}{\partial u_1^2} + \frac{\partial^2 F}{\partial u_2^2} + i\left(\frac{\partial^2 G}{\partial u_1^2} + \frac{\partial^2 G}{\partial u_2^2}\right)\right], \\ &= \frac{1}{2}\left[(m_2\beta_1 + m_2\beta_2 + m_3c_2\alpha_1\beta_1 + m_3c_2\alpha_2\beta_2)\right] + \\ &\frac{1}{2}i\left[(n_2\beta_1 + n_2\beta_2 + n_3c_2\alpha_1\beta_1 + n_3c_2\alpha_2\beta_2)\right]. \end{aligned}$$

$$\begin{aligned} g_{02} &= \frac{1}{4}\frac{\partial^2 F}{\partial u_1^2} - \frac{\partial^2 F}{\partial u_2^2} - 2\frac{\partial^2 G}{\partial u_1 \partial u_2} + i\left(\frac{\partial^2 G}{\partial u_1^2} - \frac{\partial^2 G}{\partial u_2^2} + 2\frac{\partial^2 F}{\partial u_1 \partial u_2}\right), \\ &= \frac{1}{2}\left[m_2(\beta_1-\beta_2) + m_3c_2(\alpha_1\beta_1 - \alpha_2\beta_2) - n_2(\beta_1+\beta_2) - n_3c_2(\alpha_2\beta_1 + \alpha_1\beta_2)\right] + \\ &\frac{1}{2}i\left[n_2(\beta_1-\beta_2) + n_3c_2(\alpha_1\beta_1 - \alpha_2\beta_2) + m_2(\beta_1+\beta_2) + m_3c_2(\alpha_2\beta_1 + \alpha_1\beta_2)\right]. \end{aligned}$$

$$\begin{aligned} g_{20} &= \frac{1}{4}\left[\frac{\partial^2 F}{\partial u_1^2} - \frac{\partial^2 F}{\partial u_2^2} + 2\frac{\partial^2 G}{\partial u_1 \partial u_2} + i\left(\frac{\partial^2 G}{\partial u_1^2} - \frac{\partial^2 G}{\partial u_2^2} - 2\frac{\partial^2 F}{\partial u_1 \partial u_2}\right)\right], \\ &= \frac{1}{2}\left[m_2(\beta_1-\beta_2) + m_3c_2(\alpha_1\beta_1 - \alpha_2\beta_2) + m_2(\beta_1+\beta_2) + m_3c_2(\alpha_2\beta_1 + \alpha_1\beta_2)\right] + \\ &\frac{1}{2}i\left[n_2(\beta_1-\beta_2) + n_3c_2(\alpha_1\beta_1 - \alpha_2\beta_2) - m_2(\beta_1+\beta_2) - m_3c_2(\alpha_2\beta_1 + \alpha_1\beta_2)\right]. \end{aligned}$$

$$G_{21} = \frac{1}{8}\left[\frac{\partial^3 F}{\partial u_1^3} + \frac{\partial^3 F}{\partial u_1 \partial u_2^2} + \frac{\partial^3 G}{\partial u_1^2 \partial u_2} + \frac{\partial^3 G}{\partial u_2^3} + i\left(\frac{\partial^3 G}{\partial u_1^3} + \frac{\partial^3 G}{\partial u_1 \partial u_2^2} - \frac{\partial^3 F}{\partial u_1^2 \partial u_2} - \frac{\partial^3 F}{\partial u_2^3}\right)\right] = 0.$$

$$\begin{aligned} h_{11} &= \frac{1}{4}\left(\frac{\partial^2 H}{\partial u_1^2} + \frac{\partial^2 H}{\partial u_2^2}\right), \\ &= \frac{1}{2}\left[k_2\left(\beta_1 + \beta_2\right) + k_3 c_2(\alpha_1\beta_1 + \alpha_2\beta_2)\right]. \end{aligned}$$

$$\begin{aligned} h_{20} &= \frac{1}{4}\left(\frac{\partial^2 H}{\partial u_1^2} - \frac{\partial^2 H}{\partial u_2^2} - 2i\frac{\partial^2 H}{\partial u_1 \partial u_2}\right), \\ &= \frac{1}{2}\left[k_2\left(\beta_1 - \beta_2\right) + k_3 c_2(\alpha_1\beta_1 - \alpha_2\beta_2) - i\left(k_2\left(\beta_1 + \beta_2\right) + k_3 c_2(\alpha_2\beta_1 + \alpha_1\beta_2)\right)\right]. \end{aligned}$$

: Then, we get the following system

$$\begin{cases} \lambda_1 \omega_{11} = -h_{11} \\ \left(\lambda_1 - 2id\right)\omega_{20} = -h_{20} \end{cases} \tag{2.7}$$

The solution of the system (33) is

$$\begin{cases} \omega_{11} = -\frac{h_{11}}{\lambda_1} \\ \omega_{20} = -\frac{h_{20}}{\lambda_1 - 2id} \end{cases},$$

therefore

$$\begin{cases} \omega_{11} = -\frac{K_2(\beta_1+\beta_2)+K_3c_2(\alpha_1\beta_1+\alpha_2\beta_2)}{2\lambda_1} \\ \omega_{20} = -\frac{K_2\lambda_1(\beta_1-\beta_2)+K_3c_2(\alpha_1\beta_1-\alpha_2\beta_2)+2d(K_2(\beta_1+\beta_2)+K_3c_2(\alpha_2\beta_1+\beta_2\alpha_1))}{2\left(\lambda_1^2+4d^2\right)} + \\ i\frac{2d(K_2(\beta_1-\beta_2)+K_3c_2(\alpha_1\beta_1-\alpha_2\beta_2))-\lambda_1(K_2(\beta_1+\beta_2)+K_3c_2(\alpha_2\beta_1+\alpha_1\beta_2)}{2\left(\lambda_1^2+4d^2\right)} \end{cases}$$

On the other hand, we have the following quantities

therefore

$$\begin{cases} \omega_{11} = -\frac{K_2(\beta_1+\beta_2)+K_3c_2(\alpha_1\beta_1+\alpha_2\beta_2)}{2\lambda_1} \\ \omega_{20} = -\frac{K_2\lambda_1(\beta_1-\beta_2)+K_3c_2(\alpha_1\beta_1-\alpha_2\beta_2)+2d(K_2(\beta_1+\beta_2)+K_3c_2(\alpha_2\beta_1+\beta_2\alpha_1))}{2\left(\lambda_1^2+4d^2\right)} + \\ i\frac{2d(K_2(\beta_1-\beta_2)+K_3c_2(\alpha_1\beta_1-\alpha_2\beta_2))-\lambda_1(K_2(\beta_1+\beta_2)+K_3c_2(\alpha_2\beta_1+\alpha_1\beta_2)}{2\left(\lambda_1^2+4d^2\right)} \end{cases}$$

On the other hand, we have the following quantities

$$\begin{aligned} G_{110} &= \frac{1}{2}\left[\frac{\partial^2 F}{\partial u_1 \partial_3 u} + \frac{\partial^2 G}{\partial u_2 \partial_3 u} + i\left(\frac{\partial^2 G}{\partial u_1 \partial_3 u} - \frac{\partial^2 F}{\partial u_2 \partial_3 u}\right)\right], \\ &= \frac{1}{2}\left[\beta_3\left(m_2 + n_2\right) + \alpha_3 c_2\left(m_3\beta_1 + n_3\beta_2\right) + c_2\beta_3\left(m_3\alpha_1 + n_3\alpha_2\right)\right] + \\ &\frac{1}{2} i\left[\beta_3\left(n_2 - m_2\right) + \alpha_3 c_2\left(n_3\beta_1 - m_3\beta_2\right) + c_2\beta_3\left(n_3\alpha_1 - m_3\alpha_2\right)\right]. \end{aligned}$$

$$
\begin{aligned}
g_{21} &= G_{21} + (2G_{110}\omega_{11} + G_{110}\omega_{20}) \\
&= [\beta_3 (m_2 + n_2) + c_2\alpha_3 (m_3\beta_1 + n_3\beta_2) + c_2\beta_3 (m_3\alpha_1 + n_3\alpha_2)] \\
&\times \left[-\frac{k_2 (\beta_1 + \beta_2) + k_3 c_2 (\alpha_1\beta_1 + \alpha_2\beta_2)}{2\lambda_1} \right] \\
&- \frac{1}{4} [\beta_3 (m_2 - n_2) + \alpha_3 c_2 (m_3\beta_1 - n_3\beta_2) + c_2\beta_3 (m_3\alpha_1 - n_3\alpha_2)] \\
&\times \left[\frac{k_2\lambda_1 (\beta_1 - \beta_2) + k_3 c_2 \lambda_1 (\alpha_1\beta_1 - \alpha_2\beta_2) + 2d (k_2 (\beta_1 + \beta_2) + k_3 c_2)}{\lambda_1^2 + 4d^2} \right] \\
&- \frac{1}{4} [\beta_3 (m_2 + n_2) + \alpha_3 c_2 (n_3\beta_1 + m_3\beta_2) + c_2\beta_3 (n_3\alpha_1 + m_3\alpha_2)] \\
&\times \left[\frac{2d(k_2 (\beta_1 - \beta_2) + k_3 c_2 (\alpha_1\beta_1 - \alpha_2\beta_2)) - \lambda_1 (k_2 (\beta_1 + \beta_2) + k_3 c_2 (\alpha_2\beta_1 + \alpha_1\beta_2))}{\lambda_1^2 + 4d^2} \right] \\
&+ \frac{1}{4} i [(\beta_3 (m_2 - n_2) + \alpha_3 c_2 (m_3\beta_1 - n_3\beta_2) + c_2\beta_3 (m_3\alpha_1 - n_3\alpha_2)] \\
&\times \left[\frac{2d (k_2 (\beta_1 + \beta_2) + k_3 c_2 (\alpha_1\beta_1 - \alpha_2\beta_2)) - \lambda_1 (k_2 (\beta_1 + \beta_2) + k_3 c_2 (\alpha_2\beta_1 + \alpha_1\beta_2))}{\lambda_1^2 + 4d^2} \right] \\
&+ \frac{1}{4} i [(\beta_3 (m_2 + n_2) + \alpha_3 c_2 (n_3\beta_1 + m_3\beta_2) + c_2\beta_3 (n_3\alpha_1 + m_3\alpha_2))] \\
&\times \left[-\frac{\lambda_1(k_2 (\beta_1 - \beta_2) + k_3 c_2 (\alpha_1\beta_1 - \alpha_2\beta_2)) + 2d (k_2 (\beta_1 + \beta_2) + k_3 c_2)}{\lambda_1^2 + 4d^2} \right]
\end{aligned}
$$

: also, we put

$$C_1(0) = \frac{1}{2d} \left(g_{20}g_{11} - 2 |g_{11}|^2 - \frac{1}{3} |g_{02}|^2 \right) + \frac{g_{21}}{2},$$

: then

$$\mu_2 = -\frac{\mathrm{Re}(C_1(0))}{\mathrm{Re}(\lambda'(a_0))}$$

$$\tau_2 = -\frac{\mathrm{Im}(C_1(0)) + \mu_2 \,\mathrm{Im}(\lambda'(a_0))}{d},$$

and

$$\gamma_2 = 2\,\mathrm{Re}(C_1(0)).$$

We note that [1] .

1.: μ_2 determines the direction of the Hopf bifurcation.

i.: If $\mu_2 > 0$, the hopf bifurcation is subcritical.

ii.: If $\mu_2 < 0$, the hopf bifurcation is supercritical and the bifurcated periodic solution exists for $a_1 > a_0$ and $a_1 < a_0$.

2.: γ_2 determines the bifurcated periodic solution stability.

i.: If $\gamma_2 < 0$, the bifurcated periodic solutions on the central collector are stable.

ii.: If $\gamma_2 > 0$, the bifurcated periodic solutions on the central collector are unstable.

3.: τ_2 determines the periods of the bifurcation of the periodic solutions.

i.: If $\tau_2 > 0$, the periods increase.

ii.: If $\tau_2 < 0$, the periods decrease.

2.2. **Numerical simulation.** The numerical simulation confirms the results obtained by this method, for two different values of the parameter a_1, we have two different chaotic attractors.

For $a_2 = 1,5$, $a_3 = 2$, $b = -1,3$, $c_1 = -1,5$ and $c_2 = -1$.

Case.1: $a_1 = -1,2$.

The attractor generated by the chaotic system (1) as shown in (Figure 2).

Case.2: $a_1 = -1,4$.

The attractor generated by the chaotic system (1) as shown in (Figure 3).

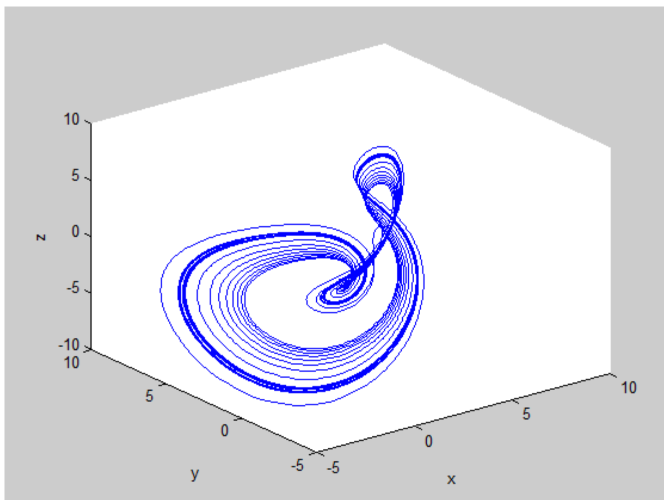

FIGURE 2. The chaotic attractor of the system (1), for $a_1 = -1,221$, $a_2 = 1,5$, $a_3 = 2$, $b = -1,3$, $c_1 = -1,5$ and $c_2 = -1$.

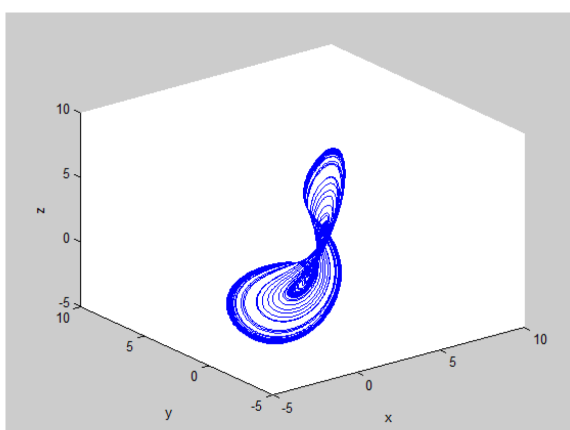

FIGURE 3. The chaotic attractor of the system (1), for $a_1 = -1,44$, $a_2 = 1,5$, $a_3 = 2$, $b = -1,3$, $c_1 = -1,5$ and $c_2 = -1$.

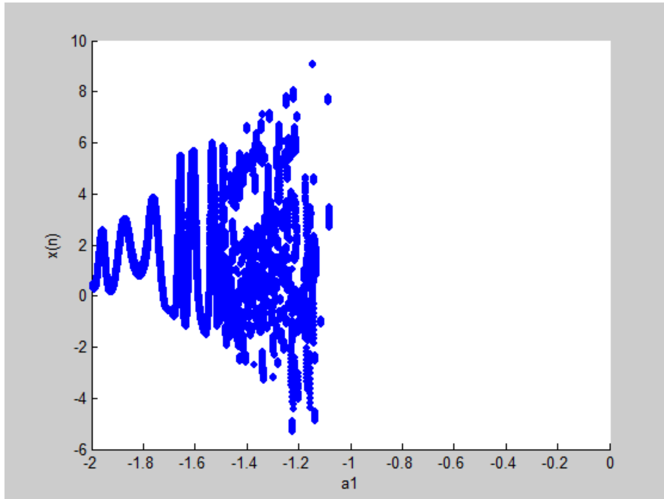

FIGURE 4. Diagramme de bifurcation du sysème (1): $(a, x(n))$ avec $-2 \leq a_1 \leq 0$.

3. CONCLUSION

In this paper, a three - dimensional quadratic system with seven bifurcation parameters has been studied. Using a projection on the plane and choosing a suitable bifurcation parameter, this method has been proved that can help us to simplify the study of bifurcations and in particular the Hopf bifurcation, which have been demonstrated that it occurs when the bifurcation parameter crosses the critical value. The direction of the Hopf bifurcation and the stability of the bifurcated periodic solutions are analyzed in detail.

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