

KANNAN AND KANNAN-PREŠIĆ TYPE CONTRACTIONS IN TRIPLED CONTROLLED V -METRIC SPACES

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ABSTRACT. In this article, we introduce the concept of tripled controlled V -metric spaces and study Kannan-type contractions in this new structure. Also, we extend the concept of Kannan type contractions to Kannan-Prešić type contractions. Our results modify some known ones in the literature. To support our main results, an example and an application in integral equations are presented.

1. INTRODUCTION

According to Banach's famous fixed point theorem, the fixed point theory has attracted many researchers since 1922. There is a lot of material in this field and it is currently a very active research field in mathematical analysis. Banach's fixed point theorem is also widely used in many disciplines and branches of mathematics.

The concept of S -metric space [33], D -metric space [13], D^* -metric space [34], 2-metric space [14] and G -metric space [23] are some known generalized metric spaces with three variables.

The above structures have been generalized to more extended structures based on the definition of b -metric spaces [8]-[10], partial metric spaces, partial b -metric spaces [24] and [35], extended b -metric space [15] and [21], etc. For more details on new fixed point theorems we refer the reader to [1]-[6], [17]-[20] and [25]-[30].

The purpose of this article is to introduce the concept of tripled controlled V -metric space which we use it on Kannan type contractive mappings. Also, in the next section we combine the ideas of Kannan and Prešić to get a new extension of the Kannan contractive mapping.

2. MAIN RESULTS

Before starting the main theorem, we first introduce the following definitions.

Definition 2.1. Let X be a nonempty set and $\alpha, \beta, \gamma : X^3 \rightarrow [1, \infty)$ be continuous functions. Suppose that the mapping $V : X^3 \rightarrow \mathbb{R}$ be a function satisfying the following properties:

- (V1) $V(\ell, \ell', \ell'') \geq 0$ and $V(\ell, \ell', \ell'') = 0$ iff $\ell = \ell' = \ell''$.
- (V2) $V(\ell, \ell', \ell'') \leq \alpha(\ell, a, \ell)V(\ell, a, \ell) + \beta(\ell', a, \ell')V(\ell', a, \ell') + \gamma(\ell'', a, \ell'')V(\ell'', a, \ell'')$ for all $\ell, \ell', \ell'', a \in X$ (rectangle inequality).

Then, the function V is called a tripled controlled V -metric on X and (X, V) is called a tripled controlled V -metric space.

Remark. Note that in a V -metric space we will not have the symmetry property. Otherwise, $\alpha(\ell, a, \ell)$, $\beta(\ell', a, \ell')$, and $\gamma(\ell'', a, \ell'')$ must be equal to 1.

Example 2.2. Let $X = C([a, b], (-\infty, +\infty))$ be the set of all continuous real valued functions on $[a, b]$. Define $V : X^3 \rightarrow \mathbb{R}$ by

$$V(\ell(t), \ell'(t), \ell''(t)) = \sup_{t \in [a, b]} |\max\{\ell(t), \ell'(t), \ell''(t)\}|^2$$

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and

$$\begin{aligned}\alpha(\ell(t), \ell'(t), \ell''(t)) &= \sup_{t \in [a,b]} \max\{|\ell(t)|, |\ell'(t)|\} + |\ell''(t)| + 1, \\ \beta(\ell(t), \ell'(t), \ell''(t)) &= \sup_{t \in [a,b]} \max\{|\ell(t)|, |\ell'(t)|\} + |\ell''(t)| + 1, \\ \gamma(\ell(t), \ell'(t), \ell''(t)) &= \sup_{t \in [a,b]} \max\{|\ell(t)|, |\ell'(t)|\} + |\ell''(t)| + 1.\end{aligned}$$

For all $\ell, \ell', \ell'', a \in X$, we have

$$\begin{aligned}V(\ell(t), \ell'(t), \ell''(t)) &= \sup_{t \in [a,b]} |\max\{\ell(t), \ell'(t), \ell''(t)\}|^2 \\ &\leq \sup_{t \in [a,b]} |\max\{\ell(t), a(t), \ell(t)\}|^2 + \sup_{t \in [a,b]} |\max\{\ell'(t), a(t), \ell'(t)\}|^2 \\ &\quad + \sup_{t \in [a,b]} |\max\{\ell''(t), a(t), \ell''(t)\}|^2 \\ &\leq V(\ell(t), a(t), \ell(t)) + V(\ell'(t), a(t), \ell'(t)) + V(\ell''(t), a(t), \ell''(t)) \\ &\leq \alpha(\ell(t), a(t), \ell(t))V(\ell(t), a(t), \ell(t)) + \beta(\ell'(t), a(t), \ell'(t)) \\ &\quad V(\ell'(t), a(t), \ell'(t)) + \gamma(\ell''(t), a(t), \ell''(t))V(\ell''(t), a(t), \ell''(t)).\end{aligned}$$

In the last part of the above inequality, we use the fact that for all $\ell, \ell', a \in X$, we have $\alpha(\ell(t), a(t), \ell(t)) \geq 1$, $\beta(\ell'(t), a(t), \ell'(t)) \geq 1$ and $\gamma(\ell''(t), a(t), \ell''(t)) \geq 1$. Hence, V is a tripled controlled V -metric space.

Example 2.3. Let (X, d) be a double controlled metric space with control functions α' and β' , and $V(\ell, \ell', \ell'') = d(\ell, \ell') + d(\ell', \ell'')$. Note that V is a V -metric with $\alpha = 2\alpha'$, $\beta = \alpha' + \beta'$ and $\gamma = 2\beta'$.

Now we present some definitions and propositions in a V -metric space.

Definition 2.4. A V -metric V is said to be symmetric if $V(\ell, \ell', \ell) = V(\ell', \ell, \ell')$, for all $\ell, \ell' \in X$.

By some straight forward calculations, we can establish the following.

Proposition 2.5. Let X be a V -metric space. Then for each $\ell, \ell', \ell'', a \in X$ it follows that:

- (1) $V(\ell, \ell', \ell'') \leq \beta(\ell', \ell, \ell')V(\ell', \ell, \ell') + \gamma(\ell'', \ell, \ell'')V(\ell'', \ell, \ell'')$,
- (2) $V(\ell, \ell', \ell'') \leq \alpha(\ell, \ell', \ell)V(\ell, \ell', \ell) + \gamma(\ell'', \ell', \ell'')V(\ell'', \ell', \ell'')$,
- (3) $V(\ell, \ell', \ell'') \leq \alpha(\ell, \ell'', \ell)V(\ell, \ell'', \ell) + \beta(\ell', \ell'', \ell')V(\ell', \ell'', \ell')$.

Definition 2.6. Let (X, V) be a V -metric space. Then for $\ell_0 \in X$ and $r > 0$, the V -ball with center ℓ_0 and radius r is

$$B_V(\ell_0, r) = \{\ell' \in X \mid V(\ell', \ell_0, \ell') < r\}.$$

Based on Proposition 1.7 of [3], we have the following:

Proposition 2.7. Let X be a V -metric space. Then for any $\ell_0 \in X$ and $r > 0$, if $\ell' \in B_V(\ell_0, r)$, then there exists a $\delta > 0$ such that $B_V(\ell', \delta) \subseteq B_V(\ell_0, r)$.

Proof. Suppose that $\ell' \in B_V(\ell_0, r)$. If $\ell' = \ell_0$, then we choose $\delta = r$. If $\ell' \neq \ell_0$, then $0 < V(\ell', \ell_0, \ell') < r$. Let $A = \{n \in \mathbb{N} \mid \frac{r}{4(\max\{[\alpha + \beta](\ell_1, \ell_2, \ell_3), \beta\gamma(\ell_1, \ell_2, \ell_3)\})^{n+2}} < V(\ell', \ell_0, \ell')\}$. Suppose that $\max\{[\alpha + \beta](\ell_1, \ell_2, \ell_3), \beta\gamma(\ell_1, \ell_2, \ell_3)\} > 1$ and let

$$M = \max_{\ell_1, \ell_2, \ell_3 \in X^3} \{[\alpha + \gamma](\ell_1, \ell_2, \ell_3), [\beta(\ell_1, \ell_2, \ell_3)]^2\}.$$

Since $\lim_{n \rightarrow \infty} \frac{1}{4M^{n+2}} = 0$, hence, for $0 < \epsilon = \frac{V(\ell', \ell_0, \ell')}{r} < 1$ there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{4M^{n_0+2}} < \frac{V(\ell', \ell_0, \ell')}{r}$ or $\frac{r}{4M^{n_0+2}} < V(\ell', \ell_0, \ell')$. Hence, $n_0 \in A$ and A is a nonempty set, then by the well ordering principle, A has a least element m . Since $m - 1 \notin A$, we have $V(\ell', \ell_0, \ell') \leq \frac{r}{4M^{m+1}}$. Now, if for a $t \in B_V(\ell', \delta)$ we have

$$\begin{aligned}V(t, \ell_0, t) &\leq [\alpha(t, \ell', t) + \gamma(t, \ell', t)]V(t, \ell', t) + \beta(\ell_0, \ell', \ell_0)V(\ell_0, \ell', \ell_0) \\ &\leq [\alpha(t, \ell', t) + \gamma(t, \ell', t)]V(t, \ell', t) + \beta(\ell_0, \ell', \ell_0)\beta(\ell', \ell_0, \ell')V(\ell', \ell_0, \ell') \\ &\leq [\alpha(t, \ell', t) + \gamma(t, \ell', t)]\delta + \beta(\ell_0, \ell', \ell_0)\beta(\ell', \ell_0, \ell')\frac{r}{4M^{m+1}} \\ &\leq [\alpha(t, \ell', t) + \gamma(t, \ell', t)]\delta + \frac{r}{4}.\end{aligned}$$

and if $V(\ell', \ell_0, \ell') < \frac{r}{4M^{m+1}}$, we choose $\delta = \frac{r-4}{M}$. \square

From the above proposition the family of all V -balls

$$\mathcal{F} = \{B_V(\ell, r) \mid \ell \in X, r > 0\}$$

is a base of a topology $\tau(V)$ on X , which we call it V -metric topology.

Definition 2.8. Let (X, V) be a tripled controlled V -metric space. Then

- (i) a sequence $\{\ell_n\}$ is called convergent if and only if there exists $\ell \in X$ such that $V(\ell_n, \ell, \ell_n) \rightarrow 0$ or $V(\ell, \ell_n, \ell) \rightarrow 0$ as $n \rightarrow \infty$. In this case, we write

$$\lim_{n \rightarrow +\infty} \ell_n = \ell.$$

- (ii) a sequence $\{\ell_n\}$ is called V -Cauchy if, for each $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that $V(\ell_n, \ell_m, \ell_n) \rightarrow 0$ or $V(\ell_m, \ell_n, \ell_m) \rightarrow 0$ as $m, n \rightarrow \infty$.

Remark. According to the rectangular inequality and the boundedness of α, β and γ , we can conclude that

1. $V(\ell_n, \ell, \ell_n) \rightarrow 0 \Rightarrow V(\ell, \ell_n, \ell_n) \rightarrow 0$,
2. $V(\ell_n, \ell, \ell_n) \rightarrow 0 \Rightarrow V(\ell_n, \ell_n, \ell) \rightarrow 0$,
3. $V(\ell_n, \ell, \ell_n) \rightarrow 0 \Rightarrow V(\ell, \ell, \ell_n) \rightarrow 0$,
4. $V(\ell_n, \ell, \ell_n) \rightarrow 0 \Leftrightarrow V(\ell, \ell_n, \ell) \rightarrow 0$,
5. $V(\ell_n, \ell, \ell_n) \rightarrow 0 \Rightarrow V(\ell_n, \ell, \ell) \rightarrow 0$,
6. $V(\ell, \ell_n, \ell) \rightarrow 0 \Rightarrow V(\ell, \ell_n, \ell_n) \rightarrow 0$,
7. $V(\ell, \ell_n, \ell) \rightarrow 0 \Leftrightarrow V(\ell_n, \ell, \ell_n) \rightarrow 0$,
8. $V(\ell, \ell_n, \ell) \rightarrow 0 \Rightarrow V(\ell_n, \ell_n, \ell) \rightarrow 0$,
9. $V(\ell, \ell_n, \ell) \rightarrow 0 \Rightarrow V(\ell, \ell, \ell_n) \rightarrow 0$,
10. $V(\ell, \ell_n, \ell) \rightarrow 0 \Rightarrow V(\ell_n, \ell, \ell) \rightarrow 0$.

Remark. According to the rectangular inequality and the boundedness of α, β and γ , we can conclude that

1. $V(\ell_n, \ell_m, \ell_n) \rightarrow 0 \Rightarrow V(\ell_m, \ell_n, \ell_n) \rightarrow 0$,
2. $V(\ell_n, \ell_m, \ell_n) \rightarrow 0 \Rightarrow V(\ell_n, \ell_n, \ell_m) \rightarrow 0$,
3. $V(\ell_n, \ell_m, \ell_n) \rightarrow 0 \Rightarrow V(\ell_m, \ell_m, \ell_n) \rightarrow 0$,
4. $V(\ell_n, \ell_m, \ell_n) \rightarrow 0 \Leftrightarrow V(\ell_m, \ell_n, \ell_m) \rightarrow 0$,
5. $V(\ell_n, \ell_m, \ell_n) \rightarrow 0 \Rightarrow V(\ell_n, \ell_m, \ell_m) \rightarrow 0$,
6. $V(\ell_m, \ell_n, \ell_m) \rightarrow 0 \Rightarrow V(\ell_m, \ell_n, \ell_n) \rightarrow 0$,
7. $V(\ell_m, \ell_n, \ell_m) \rightarrow 0 \Leftrightarrow V(\ell_n, \ell_m, \ell_n) \rightarrow 0$,
8. $V(\ell_m, \ell_n, \ell_m) \rightarrow 0 \Rightarrow V(\ell_n, \ell_n, \ell_m) \rightarrow 0$,
9. $V(\ell_m, \ell_n, \ell_m) \rightarrow 0 \Rightarrow V(\ell_m, \ell_m, \ell_n) \rightarrow 0$,
10. $V(\ell_m, \ell_n, \ell_m) \rightarrow 0 \Rightarrow V(\ell_n, \ell_m, \ell_m) \rightarrow 0$.

Definition 2.9. The tripled controlled V -metric space (X, V) is called complete if, for each V -Cauchy sequence $\{\ell_n\}$, there exists $\ell \in X$ such that $\lim_{n \rightarrow +\infty} V(\ell_n, \ell, \ell_n) = 0$.

Lemma 2.10. Let (X, V) be a tripled controlled V -metric space and $\alpha, \beta, \gamma : X^3 \rightarrow [1, \infty)$. If there exist sequences $\{\ell_n\}$ and $\{\ell'_n\}$ such that $\lim_{n \rightarrow \infty} \ell_n = \ell$ and $\lim_{n \rightarrow \infty} \ell'_n = \ell'$, then

$$\begin{aligned} \beta^{-1}(\ell', \ell, \ell') \beta^{-1}(\ell, \ell', \ell) V(\ell, \ell', \ell) &\leq \liminf_{n \rightarrow \infty} V(\ell_n, \ell'_n, \ell_n) \\ &\leq \limsup_{n \rightarrow \infty} V(\ell_n, \ell'_n, \ell_n) \leq \beta(\ell', \ell, \ell') \beta(\ell, \ell', \ell) V(\ell, \ell', \ell). \end{aligned}$$

In particular, if $\ell = \ell'$, then we have $\limsup_{n \rightarrow \infty} V(\ell_n, \ell'_n, \ell_n) = 0$. Moreover, suppose that $\{\ell_n\}$ is convergent to ℓ and $\ell' \in X$ is arbitrary. Then, we have

$$\begin{aligned} [\alpha(\ell', \ell, \ell') + \gamma(\ell', \ell, \ell')]^{-1} V(\ell, \ell', \ell) &\leq \liminf_{n \rightarrow \infty} V(\ell', \ell_n, \ell') \\ &\leq \limsup_{n \rightarrow \infty} V(\ell', \ell_n, \ell') \leq [\alpha(\ell', \ell, \ell') + \gamma(\ell', \ell, \ell')] V(\ell', \ell, \ell'). \end{aligned}$$

Proof. a) Using the rectangular inequality, one obtain

$$\begin{aligned} V(\ell, \ell', \ell) &\leq \alpha(\ell, \ell_n, \ell) V(\ell, \ell_n, \ell) + \beta(\ell', \ell_n, \ell') V(\ell', \ell_n, \ell') + \gamma(\ell, \ell_n, \ell) V(\ell, \ell_n, \ell) \\ &\leq (\alpha(\ell, \ell_n, \ell) + \gamma(\ell, \ell_n, \ell)) V(\ell, \ell_n, \ell) + \beta(\ell', \ell_n, \ell') V(\ell', \ell_n, \ell') \\ &\leq (\alpha(\ell, \ell_n, \ell) + \gamma(\ell, \ell_n, \ell)) V(\ell, \ell_n, \ell) + \beta(\ell', \ell_n, \ell') \\ &\quad [(\alpha(\ell', \ell'_n, \ell') + \gamma(\ell', \ell'_n, \ell')) V(\ell', \ell'_n, \ell') + \beta(\ell_n, \ell'_n, \ell_n) V(\ell_n, \ell'_n, \ell_n)] \end{aligned}$$

and

$$\begin{aligned}
V(\ell_n, \ell'_n, \ell_n) &\leq \alpha(\ell_n, \ell, \ell_n)V(\ell_n, \ell, \ell_n) + \beta(\ell'_n, \ell, \ell'_n)V(\ell'_n, \ell, \ell'_n) + \gamma(\ell_n, \ell, \ell_n)V(\ell_n, \ell, \ell_n) \\
&\leq (\alpha(\ell_n, \ell, \ell_n) + \gamma(\ell_n, \ell, \ell_n))V(\ell_n, \ell, \ell_n) + \beta(\ell'_n, \ell, \ell'_n)V(\ell'_n, \ell, \ell'_n) \\
&\leq (\alpha(\ell_n, \ell, \ell_n) + \gamma(\ell_n, \ell, \ell_n))V(\ell_n, \ell, \ell_n) + \beta(\ell'_n, \ell, \ell'_n) \\
&\quad [(\alpha(\ell'_n, \ell', \ell'_n) + \gamma(\ell'_n, \ell', \ell'_n))V(\ell'_n, \ell', \ell'_n) + \beta(\ell, \ell', \ell)V(\ell, \ell', \ell)].
\end{aligned}$$

Taking the lower limit as $n \rightarrow \infty$ in the first inequality and the upper limit as $n \rightarrow \infty$ in the second inequality, the result is obtained.

b) Using the triangular inequality, it is proved in a similar way.

$$V(\ell', \ell, \ell') \leq \alpha(\ell', \ell_n, \ell')V(\ell', \ell_n, \ell') + \beta(\ell, \ell_n, \ell)V(\ell, \ell_n, \ell) + \gamma(\ell', \ell_n, \ell')V(\ell', \ell_n, \ell')$$

and

$$V(\ell', \ell_n, \ell') \leq \alpha(\ell', \ell, \ell')V(\ell', \ell, \ell') + \beta(\ell_n, \ell, \ell_n)V(\ell_n, \ell, \ell_n) + \gamma(\ell', \ell, \ell')V(\ell', \ell, \ell').$$

□

The concept of Kannan contractive mapping was introduced by Kannan [16] as follows.

Definition 2.11. Let (X, d) be a metric space and $\Gamma : X \rightarrow X$. The mapping Γ is said to be a Kannan contractive mapping if there exists $\alpha \in (0, \frac{1}{2})$ such that for all $\ell, \ell' \in X$ the following inequality holds:

$$d(\Gamma\ell, \Gamma\ell') \leq \alpha(d(\ell, \Gamma\ell) + d(\ell', \Gamma\ell')).$$

In 1968, Kannan [16] proved that if (X, d) is a complete metric space, then every Kannan contractive mapping on X has a unique fixed point.

Definition 2.12. Suppose that (X, V) be a tripled controlled V -metric space. We call the mapping $\Gamma : X \rightarrow X$ a Kannan-type contractive mapping on X , if there are constants η_i ($i = 1, \dots, 6$) such that

$$\begin{aligned}
&[\alpha(\Gamma(\ell), \Gamma(\ell'), \Gamma(\ell'')) + \gamma(\Gamma(\ell), \Gamma(\ell'), \Gamma(\ell''))]V(\Gamma(\ell), \Gamma(\ell'), \Gamma(\ell'')) \\
&\leq \eta_1[V(\ell, \Gamma\ell, \Gamma\ell) + V(\ell', \Gamma\ell', \Gamma\ell') + V(\ell'', \Gamma\ell'', \Gamma\ell'')] + \eta_2[V(\Gamma\ell, \ell, \Gamma\ell) + V(\Gamma\ell', \ell', \Gamma\ell') + V(\Gamma\ell'', \ell'', \Gamma\ell'')] \\
&+ \eta_3[V(\Gamma\ell, \Gamma\ell, \ell) + V(\Gamma\ell', \Gamma\ell', \ell') + V(\Gamma\ell'', \Gamma\ell'', \ell'')] + \eta_4[V(\Gamma\ell, \ell, \ell) + V(\Gamma\ell', \ell', \ell') + V(\Gamma\ell'', \ell'', \ell'')] \\
&+ \eta_5[V(\ell, \Gamma\ell, \ell) + V(\ell', \Gamma\ell', \ell') + V(\ell'', \Gamma\ell'', \ell'')] + \eta_6[V(\ell, \ell, \Gamma\ell) + V(\ell', \ell', \Gamma\ell') + V(\ell'', \ell'', \Gamma\ell'')],
\end{aligned} \tag{2.1}$$

for all $\ell, \ell', \ell'' \in X$, where

$$(3\eta_3 + 3\eta_4)\alpha(\Gamma\ell, \ell, \Gamma\ell) + (3\eta_1 + 3\eta_3 + 3\eta_5)\beta(\Gamma\ell, \ell, \Gamma\ell) + (3\eta_1 + 3\eta_6)\gamma(\Gamma\ell, \ell, \Gamma\ell) + 3\eta_2 < 1$$

for all ℓ in X .

Now, we establish the main result of this manuscript as follows.

Theorem 2.13. Let (X, V) be a complete tripled controlled V -metric space and Γ be a Kannan-type contractive mapping on X . Then Γ has a unique fixed point in X .

Proof. Choose an $\ell_0 \in X$, and set

$$\ell_n = \Gamma(\ell_{n-1}), \quad n = 1, 2, \dots$$

By condition (2.6), we get

$$\begin{aligned}
V(\ell_{n+1}, \ell_n, \ell_{n+1}) &= V(\Gamma(\ell_n), \Gamma(\ell_{n-1}), \Gamma(\ell_n)) \\
&\leq \eta_1[V(\ell_n, \Gamma\ell_n, \Gamma\ell_n) + V(\ell_{n-1}, \Gamma\ell_{n-1}, \Gamma\ell_{n-1}) + V(\ell_n, \Gamma\ell_n, \Gamma\ell_n)] \\
&\quad + \eta_2[V(\Gamma\ell_n, \ell_n, \Gamma\ell_n) + V(\Gamma\ell_{n-1}, \ell_{n-1}, \Gamma\ell_{n-1}) + V(\Gamma\ell_n, \ell_n, \Gamma\ell_n)] \\
&\quad + \eta_3[V(\Gamma\ell_n, \Gamma\ell_n, \ell_n) + V(\Gamma\ell_{n-1}, \Gamma\ell_{n-1}, \ell_{n-1}) + V(\Gamma\ell_n, \Gamma\ell_n, \ell_n)] \\
&\quad + \eta_4[V(\Gamma\ell_n, \ell_n, \ell_n) + V(\Gamma\ell_{n-1}, \ell_{n-1}, \ell_{n-1}) + V(\Gamma\ell_n, \ell_n, \ell_n)] \\
&\quad + \eta_5[V(\ell_n, \Gamma\ell_n, \ell_n) + V(\ell_{n-1}, \Gamma\ell_{n-1}, \ell_{n-1}) + V(\ell_n, \Gamma\ell_n, \ell_n)] \\
&\quad + \eta_6[V(\ell_n, \ell_n, \Gamma\ell_n) + V(\ell_{n-1}, \ell_{n-1}, \Gamma\ell_{n-1}) + V(\ell_n, \ell_n, \Gamma\ell_n)] \\
&= \eta_1[V(\ell_n, \ell_{n+1}, \ell_{n+1}) + V(\ell_{n-1}, \ell_n, \ell_n) + V(\ell_n, \ell_{n+1}, \ell_{n+1})] \\
&\quad + \eta_2[V(\ell_{n+1}, \ell_n, \ell_{n+1}) + V(\ell_n, \ell_{n-1}, \ell_n) + V(\ell_{n+1}, \ell_n, \ell_{n+1})] \\
&\quad + \eta_3[V(\ell_{n+1}, \ell_{n+1}, \ell_n) + V(\ell_n, \ell_n, \ell_{n-1}) + V(\ell_{n+1}, \ell_{n+1}, \ell_n)] \\
&\quad + \eta_4[V(\ell_{n+1}, \ell_n, \ell_n) + V(\ell_n, \ell_{n-1}, \ell_{n-1}) + V(\ell_{n+1}, \ell_n, \ell_n)] \\
&\quad + \eta_5[V(\ell_n, \ell_{n+1}, \ell_n) + V(\ell_{n-1}, \ell_n, \ell_{n-1}) + V(\ell_n, \ell_{n+1}, \ell_n)] \\
&\quad + \eta_6[V(\ell_n, \ell_n, \ell_{n+1}) + V(\ell_{n-1}, \ell_{n-1}, \ell_n) + V(\ell_n, \ell_n, \ell_{n+1})] \\
&\leq \eta_1 \left[2[\beta(\ell_{n+1}, \ell_n, \ell_{n+1}) + \gamma(\ell_{n+1}, \ell_n, \ell_{n+1})]V(\ell_{n+1}, \ell_n, \ell_{n+1}) \right. \\
&\quad \left. + [\beta(\ell_n, \ell_{n-1}, \ell_n) + \gamma(\ell_n, \ell_{n-1}, \ell_n)]V(\ell_n, \ell_{n-1}, \ell_n) \right] \\
&\quad + \eta_2[V(\ell_{n+1}, \ell_n, \ell_{n+1}) + V(\ell_n, \ell_{n-1}, \ell_n) + V(\ell_{n+1}, \ell_n, \ell_{n+1})] \\
&\quad + \eta_3 \left[2[\alpha(\ell_{n+1}, \ell_n, \ell_{n+1}) + \beta(\ell_{n+1}, \ell_n, \ell_{n+1})]V(\ell_{n+1}, \ell_n, \ell_{n+1}) \right. \\
&\quad \left. + [\alpha(\ell_n, \ell_{n-1}, \ell_n) + \beta(\ell_n, \ell_{n-1}, \ell_n)]V(\ell_n, \ell_{n-1}, \ell_n) \right] \\
&\quad + \eta_4 \left[2\alpha(\ell_{n+1}, \ell_n, \ell_{n+1})V(\ell_{n+1}, \ell_n, \ell_{n+1}) + \alpha(\ell_n, \ell_{n-1}, \ell_n)V(\ell_n, \ell_{n-1}, \ell_n) \right] \\
&\quad + \eta_5 \left[2\beta(\ell_{n+1}, \ell_n, \ell_{n+1})V(\ell_{n+1}, \ell_n, \ell_{n+1}) + \beta(\ell_n, \ell_{n-1}, \ell_n)V(\ell_n, \ell_{n-1}, \ell_n) \right] \\
&\quad + \eta_6 \left[2\gamma(\ell_{n+1}, \ell_n, \ell_{n+1})V(\ell_{n+1}, \ell_n, \ell_{n+1}) + \gamma(\ell_n, \ell_{n-1}, \ell_n)V(\ell_n, \ell_{n-1}, \ell_n) \right],
\end{aligned}$$

then

$$\begin{aligned}
V(\ell_{n+1}, \ell_n, \ell_{n+1}) &= V(\Gamma(\ell_n), \Gamma(\ell_{n-1}), \Gamma(\ell_n)) \\
&\leq \frac{(\eta_1 + \eta_3 + \eta_5)\beta(\ell_{n+1}, \ell_n, \ell_{n+1}) + (\eta_1 + \eta_6)\gamma(\ell_{n+1}, \ell_n, \ell_{n+1}) + \eta_2 + (\eta_3 + \eta_4)\alpha(\ell_{n+1}, \ell_n, \ell_{n+1})}{1 - 2(\eta_1 + \eta_3 + \eta_5)\beta(\ell_{n+1}, \ell_n, \ell_{n+1}) - 2(\eta_1 + \eta_6)\gamma(\ell_{n+1}, \ell_n, \ell_{n+1}) - 2\eta_2 - 2(\eta_3 + \eta_4)\alpha(\ell_{n+1}, \ell_n, \ell_{n+1})} V(\ell_n, \ell_{n-1}, \ell_n) \\
&\dots \\
&\leq \left(\frac{(\eta_1 + \eta_3 + \eta_5)\beta(\ell_{n+1}, \ell_n, \ell_{n+1}) + (\eta_1 + \eta_6)\gamma(\ell_{n+1}, \ell_n, \ell_{n+1}) + \eta_2 + (\eta_3 + \eta_4)\alpha(\ell_{n+1}, \ell_n, \ell_{n+1})}{1 - 2(\eta_1 + \eta_3 + \eta_5)\beta(\ell_{n+1}, \ell_n, \ell_{n+1}) - 2(\eta_1 + \eta_6)\gamma(\ell_{n+1}, \ell_n, \ell_{n+1}) - 2\eta_2 - 2(\eta_3 + \eta_4)\alpha(\ell_{n+1}, \ell_n, \ell_{n+1})} \right)^n V(\ell_1, \ell_0, \ell_1) \\
&= C^n V(\ell_1, \ell_0, \ell_1).
\end{aligned}$$

Now, if $n \rightarrow \infty$, then

$$\lim_{n \rightarrow \infty} V(\ell_{n+1}, \ell_n, \ell_{n+1}) = 0. \quad (2.2)$$

Now, we show that $\{\ell_n\}$ is a Cauchy sequence in (X, V) . Via contradiction, let there is $\varepsilon > 0$ so that for all $i \in \mathbb{N}$ there are m_i, n_i with $i < m_i < n_i$ so that

$$V(\ell_{m_i}, \ell_{n_i}, \ell_{m_i}) \geq \varepsilon, \quad (2.3)$$

and

$$V(\ell_{m_i}, \ell_{n_{i-1}}, \ell_{m_i}) < \varepsilon. \quad (2.4)$$

from (2.3), we have

$$\begin{aligned}
V(\ell_{m_i}, \ell_{n_i}, \ell_{m_i}) &= V(\Gamma\ell_{m_i-1}, \Gamma\ell_{n_i-1}, \Gamma\ell_{m_i-1}) \\
&\leq \eta_1[V(\ell_{m_i-1}, \Gamma\ell_{m_i-1}, \Gamma\ell_{m_i-1}) + V(\ell_{n_i-1}, \Gamma\ell_{n_i-1}, \Gamma\ell_{n_i-1}) + V(\ell_{m_i-1}, \Gamma\ell_{m_i-1}, \Gamma\ell_{m_i-1})] \\
&\quad + \eta_2[V(\Gamma\ell_{m_i-1}, \ell_{m_i-1}, \Gamma\ell_{m_i-1}) + V(\Gamma\ell_{n_i-1}, \ell_{n_i-1}, \Gamma\ell_{n_i-1}) + V(\Gamma\ell_{m_i-1}, \ell_{m_i-1}, \Gamma\ell_{m_i-1})] \\
&\quad + \eta_3[V(\Gamma\ell_{m_i-1}, \Gamma\ell_{m_i-1}, \ell_{m_i-1}) + V(\Gamma\ell_{n_i-1}, \Gamma\ell_{n_i-1}, \ell_{n_i-1}) + V(\Gamma\ell_{m_i-1}, \Gamma\ell_{m_i-1}, \ell_{m_i-1})] \\
&\quad + \eta_4[V(\Gamma\ell_{m_i-1}, \ell_{m_i-1}, \ell_{m_i-1}) + V(\Gamma\ell_{n_i-1}, \ell_{n_i-1}, \ell_{n_i-1}) + V(\Gamma\ell_{m_i-1}, \ell_{m_i-1}, \ell_{m_i-1})] \\
&\quad + \eta_5[V(\ell_{m_i-1}, \Gamma\ell_{m_i-1}, \ell_{m_i-1}) + V(\ell_{n_i-1}, \Gamma\ell_{n_i-1}, \ell_{n_i-1}) + V(\ell_{m_i-1}, \Gamma\ell_{m_i-1}, \ell_{m_i-1})] \\
&\quad + \eta_6[V(\ell_{m_i-1}, \ell_{m_i-1}, \Gamma\ell_{m_i-1}) + V(\ell_{n_i-1}, \ell_{n_i-1}, \Gamma\ell_{n_i-1}) + V(\ell_{m_i-1}, \ell_{m_i-1}, \Gamma\ell_{m_i-1})] \\
&\leq \eta_1[V(\ell_{m_i-1}, \ell_{m_i}, \ell_{m_i}) + V(\ell_{n_i-1}, \ell_{n_i}, \ell_{n_i}) + V(\ell_{m_i-1}, \ell_{m_i}, \ell_{m_i})] \\
&\quad + \eta_2[V(\ell_{m_i}, \ell_{m_i-1}, \ell_{m_i}) + V(\ell_{n_i}, \ell_{n_i-1}, \ell_{n_i}) + V(\ell_{m_i}, \ell_{m_i-1}, \ell_{m_i})] \\
&\quad + \eta_3[V(\ell_{m_i}, \ell_{m_i}, \ell_{m_i-1}) + V(\ell_{n_i}, \ell_{n_i}, \ell_{n_i-1}) + V(\ell_{m_i}, \ell_{m_i}, \ell_{m_i-1})] \\
&\quad + \eta_4[V(\ell_{m_i}, \ell_{m_i-1}, \ell_{m_i-1}) + V(\ell_{n_i}, \ell_{n_i-1}, \ell_{n_i-1}) + V(\ell_{m_i}, \ell_{m_i-1}, \ell_{m_i-1})] \\
&\quad + \eta_5[V(\ell_{m_i-1}, \ell_{m_i}, \ell_{m_i-1}) + V(\ell_{n_i-1}, \ell_{n_i}, \ell_{n_i-1}) + V(\ell_{m_i-1}, \ell_{m_i}, \ell_{m_i-1})] \\
&\quad + \eta_6[V(\ell_{m_i-1}, \ell_{m_i-1}, \ell_{m_i}) + V(\ell_{n_i-1}, \ell_{n_i-1}, \ell_{n_i}) + V(\ell_{m_i-1}, \ell_{m_i-1}, \ell_{m_i})].
\end{aligned}$$

Letting $i \rightarrow \infty$ and using (2.2), we get

$$\varepsilon \leq \limsup_{i \rightarrow \infty} V(\ell_{m_i}, \ell_{n_i}, \ell_{m_i}) \leq 0, \quad (2.5)$$

which is a contradiction.

So, $\{\ell_n\}$ is a Cauchy sequence in X . Since X is complete, then there exists $v \in X$ such that $\ell_n \rightarrow v$, that is,

$$\lim_{n \rightarrow \infty} V(v, \ell_n, v) = 0.$$

Now, we show that v is a fixed point of Γ .

First, let Γ be continuous. Then we have

$$v = \lim_{n \rightarrow \infty} \ell_{n+1} = \lim_{n \rightarrow \infty} \Gamma\ell_n = \Gamma v.$$

Let Γ is not continuous. Now, by Lemma 2.10,

$$\begin{aligned}
V(\Gamma v, v, \Gamma v) &\leq \frac{[\alpha(\Gamma v, v, \Gamma v) + \gamma(\Gamma v, v, \Gamma v)]}{[\alpha(\Gamma v, v, \Gamma v) + \gamma(\Gamma v, v, \Gamma v)]} V(\Gamma v, v, \Gamma v) \\
&\leq \liminf_{n \rightarrow \infty} [\alpha(\Gamma v, \Gamma\ell_n, \Gamma v) + \gamma(\Gamma v, \Gamma\ell_n, \Gamma v)] V(\Gamma v, \Gamma\ell_n, \Gamma v) \\
&\leq \limsup_{n \rightarrow \infty} \eta_1[V(v, \Gamma v, \Gamma v) + V(\ell_n, \Gamma\ell_n, \Gamma\ell_n) + V(v, \Gamma v, \Gamma v)] \\
&\quad + \limsup_{n \rightarrow \infty} \eta_2[V(\Gamma v, v, \Gamma v) + V(\Gamma\ell_n, \ell_n, \Gamma\ell_n) + V(\Gamma v, v, \Gamma v)] \\
&\quad + \limsup_{n \rightarrow \infty} \eta_3[V(\Gamma v, \Gamma v, v) + V(\Gamma\ell_n, \Gamma\ell_n, \ell_n) + V(\Gamma v, \Gamma v, v)] \\
&\quad + \limsup_{n \rightarrow \infty} \eta_4[V(\Gamma v, v, v) + V(\Gamma\ell_n, \ell_n, \ell_n) + V(\Gamma v, v, v)] \\
&\quad + \limsup_{n \rightarrow \infty} \eta_5[V(v, \Gamma v, v) + V(\ell_n, \Gamma\ell_n, \ell_n) + V(v, \Gamma v, v)] \\
&\quad + \limsup_{n \rightarrow \infty} \eta_6[V(v, v, \Gamma v) + V(\ell_n, \ell_n, \Gamma\ell_n) + V(v, v, \Gamma v)] \\
&= 2\eta_1[V(v, \Gamma v, \Gamma v)] + 2\eta_2[V(\Gamma v, v, \Gamma v)] + \eta_3[V(\Gamma v, \Gamma v, v)] \\
&\quad + 2\eta_4[V(\Gamma v, v, v)] + 2\eta_5[V(v, \Gamma v, v)] + 2\eta_6[V(v, v, \Gamma v)] \\
&\leq 2\eta_1 \left[[\beta(\Gamma v, v, \Gamma v) + \gamma(\Gamma v, v, \Gamma v)] V(\Gamma v, v, \Gamma v) \right] + 2\eta_2[V(\Gamma v, v, \Gamma v)] \\
&\quad + 2\eta_3 \left[[\alpha(\Gamma v, v, \Gamma v) + \beta(\Gamma v, v, \Gamma v)] V(\Gamma v, v, \Gamma v) \right] + 2\eta_4[\alpha(\Gamma v, v, \Gamma v) V(\Gamma v, v, \Gamma v)] \\
&\quad + 2\eta_5[\beta(\Gamma v, v, \Gamma v) V(\Gamma v, v, \Gamma v)] + 2\eta_6[\gamma(\Gamma v, v, \Gamma v) V(\Gamma v, v, \Gamma v)].
\end{aligned}$$

As

$$2\eta_1[\beta(\ell) + \gamma(\ell)] + 2\eta_2 + 2\eta_3[\alpha(\ell) + \beta(\ell)] + 2\eta_4[\alpha(\ell)] + 2\eta_5[\beta(\ell)] + 2\eta_6[\gamma(\ell)] < 1,$$

so, $V(\Gamma v, v, \Gamma v) = 0$ which yields that $v = \Gamma v$.

Obviously, the fixed point of Γ is unique. \square

Definition 2.14. Suppose that (X, V) be a V_b -metric space. We call the mapping $\Gamma : X \rightarrow X$ a Kannan-type contractive mapping on X , if there are constants η_i ($i = 1, \dots, 6$) such that

$$\begin{aligned} & [\alpha(\Gamma(\ell), \Gamma(\ell'), \Gamma(\ell'')) + \gamma(\Gamma(\ell), \Gamma(\ell'), \Gamma(\ell''))]V(\Gamma(\ell), \Gamma(\ell'), \Gamma(\ell'')) \\ & \leq \eta_1[V(\ell, \Gamma\ell, \Gamma\ell) + V(\ell', \Gamma\ell', \Gamma\ell') + V(\ell'', \Gamma\ell'', \Gamma\ell'')] + \eta_2[V(\Gamma\ell, \ell, \Gamma\ell) + V(\Gamma\ell', \ell', \Gamma\ell') + V(\Gamma\ell'', \ell'', \Gamma\ell'')] \\ & + \eta_3[V(\Gamma\ell, \Gamma\ell, \ell) + V(\Gamma\ell', \Gamma\ell', \ell') + V(\Gamma\ell'', \Gamma\ell'', \ell'')] + \eta_4[V(\Gamma\ell, \ell, \ell) + V(\Gamma\ell', \ell', \ell') + V(\Gamma\ell'', \ell'', \ell'')] \\ & + \eta_5[V(\ell, \Gamma\ell, \ell) + V(\ell', \Gamma\ell', \ell') + V(\ell'', \Gamma\ell'', \ell'')] + \eta_6[V(\ell, \ell, \Gamma\ell) + V(\ell', \ell', \Gamma\ell') + V(\ell'', \ell'', \Gamma\ell'')], \end{aligned} \quad (2.6)$$

for all $\ell, \ell', \ell'' \in X$, where

$$s(3\eta_3 + 3\eta_4) + s(3\eta_1 + 3\eta_3 + 3\eta_5) + s(3\eta_1 + 3\eta_6) + 3\eta_2 < 1.$$

Now, we establish the main result of this manuscript as follows.

Corollary 2.15. Let (X, V) be a complete V_b -metric space and $\Gamma : X \rightarrow X$ be a mapping so that there are constants η_i ($i = 1, \dots, 6$) and

$$\begin{aligned} & [\alpha(\Gamma(\ell), \Gamma(\ell'), \Gamma(\ell'')) + \gamma(\Gamma(\ell), \Gamma(\ell'), \Gamma(\ell''))]V(\Gamma(\ell), \Gamma(\ell'), \Gamma(\ell'')) \\ & \leq \eta_1[V(\ell, \Gamma\ell, \Gamma\ell) + V(\ell', \Gamma\ell', \Gamma\ell') + V(\ell'', \Gamma\ell'', \Gamma\ell'')] + \eta_2[V(\Gamma\ell, \ell, \Gamma\ell) + V(\Gamma\ell', \ell', \Gamma\ell') + V(\Gamma\ell'', \ell'', \Gamma\ell'')] \\ & + \eta_3[V(\Gamma\ell, \Gamma\ell, \ell) + V(\Gamma\ell', \Gamma\ell', \ell') + V(\Gamma\ell'', \Gamma\ell'', \ell'')] + \eta_4[V(\Gamma\ell, \ell, \ell) + V(\Gamma\ell', \ell', \ell') + V(\Gamma\ell'', \ell'', \ell'')] \\ & + \eta_5[V(\ell, \Gamma\ell, \ell) + V(\ell', \Gamma\ell', \ell') + V(\ell'', \Gamma\ell'', \ell'')] + \eta_6[V(\ell, \ell, \Gamma\ell) + V(\ell', \ell', \Gamma\ell') + V(\ell'', \ell'', \Gamma\ell'')], \end{aligned} \quad (2.7)$$

for all $\ell, \ell', \ell'' \in X$, where

$$s(3\eta_3 + 3\eta_4) + s(3\eta_1 + 3\eta_3 + 3\eta_5) + s(3\eta_1 + 3\eta_6) + 3\eta_2 < 1.$$

. Then Γ has a unique fixed point in X .

Example 2.16. Let $X = C([a, b], \mathbb{R})$ be the set of all continuous real valued functions on $[a, b]$. Consider the tripled controlled V -metric space and the functions α , β and γ presented in Example 2.2. Choose $\Gamma\ell = \frac{\ell}{4}$ for all $\ell \in X$. Note that,

$$V(\Gamma\ell, \Gamma\ell', \Gamma\ell'') = \sup_{t \in [a, b]} |\max\{\frac{\ell(t)}{4}, \frac{\ell'(t)}{4}, \frac{\ell''(t)}{4}\}|^2 \leq \frac{1}{4}[V(\Gamma\ell, \ell, \Gamma\ell) + V(\Gamma\ell', \ell', \Gamma\ell') + V(\Gamma\ell'', \ell'', \Gamma\ell'')].$$

In this case, we have $\eta_2 = \frac{1}{4}$ and $\eta_i = 0$ for all $i = 1, 3, 4, 5, 6$. Therefore, all hypotheses of Theorem 2.13 are satisfied and $\ell = 0$ is the unique fixed point.

3. KANNAN-PRESIĆ TYPE FIXED POINT RESULTS

In this section, we combine the ideas of Kannan and Presić.

Theorem 3.1. [7] Let (X, d) be a complete metric space and let $\Gamma : X \rightarrow X$ so that

$$d(\Gamma\ell, \Gamma\kappa) \leq \gamma d(\ell, \kappa) \text{ for all } \ell, \kappa \in X,$$

where $\gamma \in [0, 1)$. Then, there is a unique σ in X such that $\sigma = \Gamma\sigma$. Also, for each $\ell_0 \in X$, the sequence $\ell_{n+1} = \Gamma\ell_n$ converges to σ .

The BCP has been expanded and generalized in a variety of ways (see, for example, [?] and [32]). Presić [28] came up with the following outcome.

Theorem 3.2. [28] Let (X, d) be a complete metric space and let $\Gamma : X^k \rightarrow X$ (k is a positive integer). Suppose that

$$d(\Gamma(\ell_1, \dots, \ell_k), \Gamma(\ell_2, \dots, \ell_{k+1})) \leq \sum_{i=1}^k \lambda_i d(\ell_i, \ell_{i+1}) \quad (3.1)$$

for all $\ell_1, \dots, \ell_{k+1}$ in X , where $\lambda_i \geq 0$ and $\sum_{i=1}^k \lambda_i \in [0, 1)$. Then Γ has a unique fixed point ℓ^* (that is, $\Gamma(\ell^*, \dots, \ell^*) = \ell^*$). Moreover, for all arbitrary points $\ell_1, \dots, \ell_{k+1}$ in X , the sequence $\{\ell_n\}$ defined by $\ell_{n+k} = \Gamma(\ell_n, \ell_{n+1}, \dots, \ell_{n+k-1})$, converges to ℓ^* .

It is obvious that for $k = 1$, Theorem 3.2 coincides with the BCP. The above theorem generalized by Ćirić and Presić [12] as follows.

Theorem 3.3. [12] Let (X, d) be a complete metric space and $\Gamma : X^k \rightarrow X$ (k is a positive integer). Suppose that

$$d(\Gamma(\ell_1, \dots, \ell_k), \Gamma(\ell_2, \dots, \ell_{k+1})) \leq \lambda \max\{d(\ell_i, \ell_{i+1}) : 1 \leq i \leq k\}, \quad (3.2)$$

for all $\ell_1, \dots, \ell_{k+1}$ in X , where $\lambda \in [0, 1)$. Then Γ has a fixed point $\ell^* \in X$. Also, for all points $\ell_1, \dots, \ell_{k+1} \in X$, the sequence $\{\ell_n\}$ defined by $\ell_{n+k} = \Gamma(\ell_n, \ell_{n+1}, \dots, \ell_{n+k-1})$, converges to ℓ^* . The fixed point of Γ is unique if

$$d(\Gamma(\rho, \dots, \rho), \Gamma(\varrho, \dots, \varrho)) < d(\rho, \varrho),$$

for all $\rho, \varrho \in X$ with $\rho \neq \varrho$.

For more details on Presić type contractive mappings, we refer the reader to [11, 28, 26].

Motivated by Theorem 6 of [31], we prove the following lemma which will help us in Presić type results.

Theorem 3.4. Suppose that V_1, V_2, \dots, V_n be tripled controlled V -Metric spaces on nonempty sets X_1, X_2, \dots, X_n , respectively, and let $v : [0, \infty)^n \rightarrow [0, \infty)$ so that $v(\sigma_1, \dots, \sigma_n) = 0$ if and only if $\sigma_i = 0$ for all $i = 1, 2, 3, \dots, n$ and

$$\begin{aligned} & v(\alpha_1(\ell_{11}, \ell_1, \ell_{11})V_1(\ell_{11}, \ell_1, \ell_{11}) + \beta_1(\ell_{21}, \ell_1, \ell_{21})V_1(\ell_{21}, \ell_1, \ell_{21}) + \gamma_1(\ell_{31}, \ell_1, \ell_{31})V_1(\ell_{31}, \ell_1, \ell_{31})), \\ & \alpha_2(\ell_{12}, \ell_2, \ell_{12})V_2(\ell_{12}, \ell_2, \ell_{12}) + \beta_2(\ell_{22}, \ell_2, \ell_{22})V_2(\ell_{22}, \ell_2, \ell_{22}) + \gamma_2(\ell_{32}, \ell_2, \ell_{32})V_2(\ell_{32}, \ell_2, \ell_{32})) \\ & , \dots, \\ & \alpha_n(\ell_{1n}, \ell_n, \ell_{1n})V_n(\ell_{1n}, \ell_n, \ell_{1n}) + \beta_n(\ell_{2n}, \ell_n, \ell_{2n})V_n(\ell_{2n}, \ell_n, \ell_{2n}) + \gamma_n(\ell_{3n}, \ell_n, \ell_{3n})V_n(\ell_{3n}, \ell_n, \ell_{3n})) \\ & \leq \alpha((\ell_{11}, \ell_{12}, \dots, \ell_{1n}), (\ell_1, \ell_2, \dots, \ell_n), (\ell_{11}, \ell_{12}, \dots, \ell_{1n})) \\ & v(V_1(\ell_{11}, \ell_1, \ell_{11}), V_2(\ell_{12}, \ell_2, \ell_{12}), \dots, V_n(\ell_{1n}, \ell_n, \ell_{1n})) \\ & + \beta((\ell_{21}, \ell_{22}, \dots, \ell_{2n}), ((\ell_1, \ell_2, \dots, \ell_n), (\ell_{21}, \ell_{22}, \dots, \ell_{2n}))) \\ & v(V_1(\ell_{21}, \ell_1, \ell_{21}), V_2(\ell_{22}, \ell_2, \ell_{22}), \dots, V_n(\ell_{2n}, \ell_n, \ell_{2n})) \\ & + \gamma(((\ell_{31}, \ell_{23}, \dots, \ell_{3n}), ((\ell_1, \ell_2, \dots, \ell_n), (\ell_{31}, \ell_{32}, \dots, \ell_{3n}))) \\ & v(V_1(\ell_{31}, \ell_1, \ell_{31}), V_2(\ell_{32}, \ell_2, \ell_{32}), \dots, V_n(\ell_{3n}, \ell_n, \ell_{3n})) \end{aligned}$$

for all $\ell_{ij} \in [0, \infty)$ where

$$\begin{aligned} & \alpha((\ell_{11}, \ell_{12}, \dots, \ell_{1n}), (\ell_{21}, \ell_{22}, \dots, \ell_{2n}), (\ell_{31}, \ell_{32}, \dots, \ell_{3n})) \\ & = v(\alpha_1(\ell_{11}, \ell_{21}, \ell_{31}), \alpha_2(\ell_{12}, \ell_{22}, \ell_{32}), \dots, \alpha_n(\ell_{1n}, \ell_{2n}, \ell_{3n})), \end{aligned}$$

$$\begin{aligned} & \beta((\ell_{11}, \ell_{12}, \dots, \ell_{1n}), (\ell_{21}, \ell_{22}, \dots, \ell_{2n}), (\ell_{31}, \ell_{32}, \dots, \ell_{3n})) \\ & = v(\beta_1(\ell_{11}, \ell_{21}, \ell_{31}), \beta_2(\ell_{12}, \ell_{22}, \ell_{32}), \dots, \beta_n(\ell_{1n}, \ell_{2n}, \ell_{3n})), \end{aligned}$$

and .

$$\begin{aligned} & \gamma((\ell_{11}, \ell_{12}, \dots, \ell_{1n}), (\ell_{21}, \ell_{22}, \dots, \ell_{2n}), (\ell_{31}, \ell_{32}, \dots, \ell_{3n})) \\ & = v(\gamma_1(\ell_{11}, \ell_{21}, \ell_{31}), \gamma_2(\ell_{12}, \ell_{22}, \ell_{32}), \dots, \gamma_n(\ell_{1n}, \ell_{2n}, \ell_{3n})), \end{aligned}$$

Then

$$\begin{aligned} & \tilde{V}((\ell_{11}, \ell_{12}, \dots, \ell_{1n}), (\ell_{21}, \ell_{22}, \dots, \ell_{2n}), (\ell_{31}, \ell_{32}, \dots, \ell_{3n})) \\ & = v(V_1(\ell_{11}, \ell_{21}, \ell_{31}), V_2(\ell_{12}, \ell_{22}, \ell_{32}), \dots, V_n(\ell_{1n}, \ell_{2n}, \ell_{3n})), \end{aligned}$$

is a tripled controlled V -metric in $[X_1 \times X_2 \times \dots \times X_n]^3$.

Proof. We only show the triangular inequality in a tripled controlled V -metric space. Let $\ell_{ij} \in X_j$ for all $1 \leq i \leq 3$ and $1 \leq j \leq n$. So,

$$\begin{aligned}
& \tilde{V}((\ell_{11}, \ell_{12}, \dots, \ell_{1n}), (\ell_{21}, \ell_{22}, \dots, \ell_{2n}), (\ell_{31}, \ell_{32}, \dots, \ell_{3n})) \\
&= v(V_1(\ell_{11}, \ell_{21}, \ell_{31}), V_2(\ell_{12}, \ell_{22}, \ell_{32}), \dots, V_n(\ell_{1n}, \ell_{2n}, \ell_{3n})) \\
&\leq v(\alpha_1(\ell_{11}, \ell_{12}, \ell_{11})V_1(\ell_{11}, \ell_{12}, \ell_{11}) + \beta_1(\ell_{21}, \ell_{12}, \ell_{21})V_1(\ell_{21}, \ell_{12}, \ell_{21}) + \gamma_1(\ell_{31}, \ell_{12}, \ell_{31})V_1(\ell_{31}, \ell_{12}, \ell_{31})), \\
&\quad \alpha_2(\ell_{12}, \ell_{21}, \ell_{12})V_2(\ell_{12}, \ell_{21}, \ell_{12}) + \beta_2(\ell_{22}, \ell_{12}, \ell_{22})V_2(\ell_{22}, \ell_{12}, \ell_{22}) + \gamma_2(\ell_{32}, \ell_{12}, \ell_{32})V_2(\ell_{32}, \ell_{12}, \ell_{32})) \\
&\quad , \dots, \\
&\alpha_n(\ell_{1n}, \ell_{2n}, \ell_{1n})V_n(\ell_{1n}, \ell_{2n}, \ell_{1n}) + \beta_n(\ell_{2n}, \ell_{3n}, \ell_{2n})V_n(\ell_{2n}, \ell_{3n}, \ell_{2n}) + \gamma_n(\ell_{3n}, \ell_{2n}, \ell_{3n})V_n(\ell_{3n}, \ell_{2n}, \ell_{3n})) \\
&\leq \alpha((\ell_{11}, \ell_{12}, \dots, \ell_{1n}), (\ell_{12}, \ell_{21}, \dots, \ell_{1n}), (\ell_{11}, \ell_{12}, \dots, \ell_{1n})) \\
&\quad v(V_1(\ell_{11}, \ell_{12}, \ell_{11}), V_2(\ell_{12}, \ell_{21}, \ell_{12}), \dots, V_n(\ell_{1n}, \ell_{12}, \ell_{1n})) \\
&\quad + \beta((\ell_{21}, \ell_{22}, \dots, \ell_{2n}), ((\ell_{12}, \ell_{21}, \dots, \ell_{1n}), (\ell_{21}, \ell_{22}, \dots, \ell_{2n}))) \\
&\quad v(V_1(\ell_{21}, \ell_{22}, \ell_{21}), V_2(\ell_{22}, \ell_{21}, \ell_{22}), \dots, V_n(\ell_{2n}, \ell_{21}, \ell_{2n})) \\
&\quad + \gamma(((\ell_{31}, \ell_{23}, \dots, \ell_{3n}), ((\ell_{12}, \ell_{21}, \dots, \ell_{1n}), (\ell_{31}, \ell_{32}, \dots, \ell_{3n}))) \\
&\quad v(V_1(\ell_{31}, \ell_{32}, \ell_{31}), V_2(\ell_{32}, \ell_{31}, \ell_{32}), \dots, V_n(\ell_{3n}, \ell_{31}, \ell_{3n})) \\
&\leq \alpha((\ell_{11}, \ell_{12}, \dots, \ell_{1n}), (\ell_{12}, \ell_{21}, \dots, \ell_{1n}), (\ell_{11}, \ell_{12}, \dots, \ell_{1n})) \\
&\quad \tilde{V}((\ell_{11}, \ell_{12}, \dots, \ell_{1n}), (\ell_{12}, \ell_{21}, \dots, \ell_{1n}), (\ell_{11}, \ell_{12}, \dots, \ell_{1n})) \\
&\quad + \beta((\ell_{21}, \ell_{22}, \dots, \ell_{2n}), ((\ell_{12}, \ell_{21}, \dots, \ell_{1n}), (\ell_{21}, \ell_{22}, \dots, \ell_{2n}))) \\
&\quad \tilde{V}((\ell_{21}, \ell_{22}, \dots, \ell_{2n}), ((\ell_{12}, \ell_{21}, \dots, \ell_{1n}), (\ell_{21}, \ell_{22}, \dots, \ell_{2n}))) \\
&\quad + \gamma(((\ell_{31}, \ell_{23}, \dots, \ell_{3n}), ((\ell_{12}, \ell_{21}, \dots, \ell_{1n}), (\ell_{31}, \ell_{32}, \dots, \ell_{3n}))) \\
&\quad \tilde{V}((\ell_{31}, \ell_{32}, \dots, \ell_{3n}), ((\ell_{12}, \ell_{21}, \dots, \ell_{1n}), (\ell_{31}, \ell_{32}, \dots, \ell_{3n}))).
\end{aligned}$$

□

Theorem 3.5. Having (X, V) as a tripled controlled V -metric space and $\Gamma : X^n \rightarrow X$ as a continuous function such that

$$\begin{aligned}
& [\alpha(\Gamma(\ell_1, \ell_2, \dots, \ell_k), \Gamma(\ell_2, \ell_3, \dots, \ell_{k+1}), \Gamma(\ell_3, \ell_4, \dots, \ell_{k+1})) + \gamma(\Gamma(\ell_1, \ell_2, \dots, \ell_k), \Gamma(\ell_2, \ell_3, \dots, \ell_{k+1}), \Gamma(\ell_3, \ell_4, \dots, \ell_{k+1}))] \\
& \cdot V(\Gamma(\ell_1, \ell_2, \dots, \ell_k), \Gamma(\ell_2, \ell_3, \dots, \ell_{k+1}), \Gamma(\ell_3, \ell_4, \dots, \ell_{k+1})) \\
& \leq \frac{1}{k} \left[\eta_1 \sum_{i=1}^k V(\ell_i, \Gamma(\ell_1, \dots, \ell_k), \Gamma(\ell_1, \dots, \ell_k)) + \eta_1 \sum_{i=2}^{k+1} V(\ell_i, \Gamma(\ell_2, \dots, \ell_{k+1}), \Gamma(\ell_2, \dots, \ell_{k+1})) + \eta_1 \sum_{i=3}^{k+2} V(\ell_i, \Gamma(\ell_3, \dots, \ell_{k+2}), \Gamma(\ell_3, \dots, \ell_{k+2})) \right. \\
& \quad + \eta_2 \sum_{i=1}^k V(\Gamma(\ell_1, \dots, \ell_k), \ell_i, \Gamma(\ell_1, \dots, \ell_k)) + \eta_2 \sum_{i=2}^{k+1} V(\Gamma(\ell_2, \dots, \ell_{k+1}), \ell_i, \Gamma(\ell_2, \dots, \ell_{k+1})) + \eta_2 \sum_{i=3}^{k+2} V(\Gamma(\ell_3, \dots, \ell_{k+2}), \ell_i, \Gamma(\ell_3, \dots, \ell_{k+2})) \\
& \quad + \eta_3 \sum_{i=1}^k V(\Gamma(\ell_1, \dots, \ell_k), \Gamma(\ell_1, \dots, \ell_k)\ell_i) + \eta_3 \sum_{i=2}^{k+1} V(\Gamma(\ell_2, \dots, \ell_{k+1}), \Gamma(\ell_2, \dots, \ell_{k+1})\ell_i) + \eta_3 \sum_{i=3}^{k+2} V(\Gamma(\ell_3, \dots, \ell_{k+2}), \Gamma(\ell_3, \dots, \ell_{k+2})\ell_i) \\
& \quad + \eta_4 \sum_{i=1}^k V(\Gamma(\ell_1, \dots, \ell_k), \ell_i, \ell_i) + \eta_4 \sum_{i=2}^{k+1} V(\Gamma(\ell_2, \dots, \ell_{k+1}), \ell_i, \ell_i) + \eta_4 \sum_{i=3}^{k+2} V(\Gamma(\ell_3, \dots, \ell_{k+2}), \ell_i, \ell_i) \\
& \quad + \eta_5 \sum_{i=1}^k V(\ell_i, \Gamma(\ell_1, \dots, \ell_k), \ell_i) + \eta_5 \sum_{i=2}^{k+1} V(\ell_i, \Gamma(\ell_2, \dots, \ell_{k+1}), \ell_i) + \eta_5 \sum_{i=3}^{k+2} V(\ell_i, \Gamma(\ell_3, \dots, \ell_{k+2}), \ell_i) \\
& \quad \left. + \eta_6 \sum_{i=1}^k V(\ell_i, \ell_i, \Gamma(\ell_1, \dots, \ell_k)) + \eta_6 \sum_{i=2}^{k+1} V(\ell_i, \ell_i, \Gamma(\ell_2, \dots, \ell_{k+1})) + \eta_6 \sum_{i=3}^{k+2} V(\ell_i, \ell_i, \Gamma(\ell_3, \dots, \ell_{k+2})) \right]
\end{aligned} \tag{3.3}$$

for all $\ell_{ij} \subseteq X$, where

$$(3\eta_3 + 3\eta_4)\alpha(\tilde{\Gamma}\tilde{\sigma}, \tilde{\sigma}, \tilde{\Gamma}\tilde{\sigma}) + (3\eta_1 + 3\eta_3 + 3\eta_5)\beta(\tilde{\Gamma}\tilde{\sigma}, \tilde{\sigma}, \tilde{\Gamma}\tilde{\sigma}) + (3\eta_1 + 3\eta_6)\gamma(\tilde{\Gamma}\tilde{\sigma}, \tilde{\sigma}, \tilde{\Gamma}\tilde{\sigma}) + 3\eta_2 < 1$$

in which

$$\tilde{\Gamma}\tilde{\sigma} = (\Gamma(\sigma_1, \dots, \sigma_n), \dots, \Gamma(\sigma_1, \dots, \sigma_n))$$

and $\tilde{\sigma} = (\sigma_1, \dots, \sigma_n)$. Then Γ has at least a Kannan-Presić type fixed point.

Proof. We define the mapping $\tilde{\Gamma} : X^n \rightarrow X^n$ by

$$\tilde{\Gamma}(\sigma_1, \dots, \sigma_n) = (\Gamma(\sigma_1, \dots, \sigma_n), \dots, \Gamma(\sigma_1, \dots, \sigma_n)).$$

Clearly, $\tilde{\Gamma}$ is continuous. We demonstrate that $\tilde{\Gamma}$ satisfies all the conditions of Theorem 2.13. We know that

$$\tilde{V}((\ell_1, \ell_2, \dots, \ell_k), (\kappa_1, \kappa_2, \dots, \kappa_k), (\sigma_1, \sigma_2, \dots, \sigma_k)) = \frac{V(\ell_1, \kappa_1, \sigma_1) + V(\ell_2, \kappa_2, \sigma_2) + \dots + V(\ell_k, \kappa_k, \sigma_k)}{k}.$$

is a tripled controlled V -Metric Space. From (3.3) we have

$$\begin{aligned}
& \tilde{V}(\tilde{\Gamma}(\ell_1, \ell_2, \dots, \ell_k), \tilde{\Gamma}(\ell_2, \ell_3, \dots, \ell_{k+1}), \tilde{\Gamma}(\ell_2, \ell_3, \dots, \ell_{k+1})) \\
&= \tilde{V}\left((\Gamma(\ell_1, \ell_2, \dots, \ell_k), \Gamma(\ell_1, \ell_2, \dots, \ell_k), \dots, \Gamma(\ell_1, \ell_2, \dots, \ell_k)), \right. \\
&\quad (\Gamma(\ell_2, \ell_3, \dots, \ell_{k+1}), \Gamma(\ell_2, \ell_3, \dots, \ell_{k+1}), \dots, \Gamma(\ell_2, \ell_3, \dots, \ell_{k+1})), \\
&\quad \left. (\Gamma(\ell_2, \ell_3, \dots, \ell_{k+1}), \Gamma(\ell_2, \ell_3, \dots, \ell_{k+1}), \dots, \Gamma(\ell_2, \ell_3, \dots, \ell_{k+1}))\right) \\
&= V(\Gamma(\ell_1, \ell_2, \dots, \ell_k), \Gamma(\ell_2, \ell_3, \dots, \ell_{k+1}), \Gamma(\ell_2, \ell_3, \dots, \ell_{k+1})) \\
&\leq \frac{1}{k} \left[\eta_1 \sum_{i=1}^k V(\ell_i, \Gamma(\ell_1, \dots, \ell_k), \Gamma(\ell_1, \dots, \ell_k)) + \eta_1 \sum_{i=2}^{k+1} V(\ell_i, \Gamma(\ell_2, \dots, \ell_{k+1}), \Gamma(\ell_2, \dots, \ell_{k+1})) + \eta_1 \sum_{i=3}^{k+2} V(\ell_i, \Gamma(\ell_3, \dots, \ell_{k+2}), \Gamma(\ell_3, \dots, \ell_{k+2})) \right. \\
&\quad + \eta_2 \sum_{i=1}^k V(\Gamma(\ell_1, \dots, \ell_k), \ell_i, \Gamma(\ell_1, \dots, \ell_k)) + \eta_2 \sum_{i=2}^{k+1} V(\Gamma(\ell_2, \dots, \ell_{k+1}), \ell_i, \Gamma(\ell_2, \dots, \ell_{k+1})) + \eta_2 \sum_{i=3}^{k+2} V(\Gamma(\ell_3, \dots, \ell_{k+2}), \ell_i, \Gamma(\ell_3, \dots, \ell_{k+2})) \\
&\quad + \eta_3 \sum_{i=1}^k V(\Gamma(\ell_1, \dots, \ell_k), \Gamma(\ell_1, \dots, \ell_k) \ell_i,) + \eta_3 \sum_{i=2}^{k+1} V(\Gamma(\ell_2, \dots, \ell_{k+1}), \Gamma(\ell_2, \dots, \ell_{k+1}) \ell_i,) + \eta_3 \sum_{i=3}^{k+2} V(\Gamma(\ell_3, \dots, \ell_{k+2}), \Gamma(\ell_3, \dots, \ell_{k+2}) \ell_i,) \\
&\quad + \eta_4 \sum_{i=1}^k V(\Gamma(\ell_1, \dots, \ell_k), \ell_i, \ell_i) + \eta_4 \sum_{i=2}^{k+1} V(\Gamma(\ell_2, \dots, \ell_{k+1}), \ell_i, \ell_i) + \eta_4 \sum_{i=3}^{k+2} V(\Gamma(\ell_3, \dots, \ell_{k+2}), \ell_i, \ell_i) \\
&\quad + \eta_5 \sum_{i=1}^k V(\ell_i, \Gamma(\ell_1, \dots, \ell_k), \ell_i) + \eta_5 \sum_{i=2}^{k+1} V(\ell_i, \Gamma(\ell_2, \dots, \ell_{k+1}), \ell_i) + \eta_5 \sum_{i=3}^{k+2} V(\ell_i, \Gamma(\ell_3, \dots, \ell_{k+2}), \ell_i) \\
&\quad + \eta_6 \sum_{i=1}^k V(\ell_i, \ell_i, \Gamma(\ell_1, \dots, \ell_k)) + \eta_6 \sum_{i=2}^{k+1} V(\ell_i, \ell_i, \Gamma(\ell_2, \dots, \ell_{k+1})) + \eta_6 \sum_{i=3}^{k+2} V(\ell_i, \ell_i, \Gamma(\ell_3, \dots, \ell_{k+2})) \Big] \\
&\leq \eta_1 [\tilde{V}((\ell_1, \ell_2, \dots, \ell_k), \tilde{\Gamma}(\ell_1, \ell_2, \dots, \ell_k), \tilde{\Gamma}(\ell_1, \ell_2, \dots, \ell_k)) + \tilde{V}((\ell_2, \ell_3, \dots, \ell_{k+1}), \tilde{\Gamma}(\ell_2, \ell_3, \dots, \ell_{k+1}), \tilde{\Gamma}(\ell_2, \ell_3, \dots, \ell_{k+1})) \\
&\quad + \tilde{V}((\ell_3, \ell_4, \dots, \ell_{k+2}), \tilde{\Gamma}(\ell_3, \ell_4, \dots, \ell_{k+2}), \tilde{\Gamma}(\ell_3, \ell_4, \dots, \ell_{k+2}))] \\
&\quad + \eta_2 [\tilde{V}(\tilde{\Gamma}(\ell_1, \ell_2, \dots, \ell_k), (\ell_1, \ell_2, \dots, \ell_k), \tilde{\Gamma}(\ell_1, \ell_2, \dots, \ell_k)) + \tilde{V}(\tilde{\Gamma}(\ell_2, \ell_3, \dots, \ell_{k+1}), (\ell_2, \ell_3, \dots, \ell_{k+1}), \tilde{\Gamma}(\ell_2, \ell_3, \dots, \ell_{k+1})) \\
&\quad + \tilde{V}(\tilde{\Gamma}(\ell_3, \ell_4, \dots, \ell_{k+2}), (\ell_3, \ell_4, \dots, \ell_{k+2}), \tilde{\Gamma}(\ell_3, \ell_4, \dots, \ell_{k+2}))] \\
&\quad + \eta_3 [\tilde{V}(\tilde{\Gamma}(\ell_1, \ell_2, \dots, \ell_k), \tilde{\Gamma}(\ell_1, \ell_2, \dots, \ell_k), (\ell_1, \ell_2, \dots, \ell_k)) + \tilde{V}(\tilde{\Gamma}(\ell_2, \ell_3, \dots, \ell_{k+1}), \tilde{\Gamma}(\ell_2, \ell_3, \dots, \ell_{k+1}), (\ell_2, \ell_3, \dots, \ell_{k+1})) \\
&\quad + \tilde{V}(\tilde{\Gamma}(\ell_3, \ell_4, \dots, \ell_{k+2}), \tilde{\Gamma}(\ell_3, \ell_4, \dots, \ell_{k+2}), (\ell_3, \ell_4, \dots, \ell_{k+2}))] \\
&\quad + \eta_4 [\tilde{V}(\tilde{\Gamma}(\ell_1, \ell_2, \dots, \ell_k), (\ell_1, \ell_2, \dots, \ell_k), (\ell_1, \ell_2, \dots, \ell_k)) + \tilde{V}(\tilde{\Gamma}(\ell_2, \ell_3, \dots, \ell_{k+1}), (\ell_2, \ell_3, \dots, \ell_{k+1}), (\ell_2, \ell_3, \dots, \ell_{k+1})) \\
&\quad + \tilde{V}(\tilde{\Gamma}(\ell_3, \ell_4, \dots, \ell_{k+2}), (\ell_3, \ell_4, \dots, \ell_{k+2}), (\ell_3, \ell_4, \dots, \ell_{k+2}))] \\
&\quad + \eta_5 [\tilde{V}((\ell_1, \ell_2, \dots, \ell_k), \tilde{\Gamma}(\ell_1, \ell_2, \dots, \ell_k), (\ell_1, \ell_2, \dots, \ell_k)) + \tilde{V}((\ell_2, \ell_3, \dots, \ell_{k+1}), \tilde{\Gamma}(\ell_2, \ell_3, \dots, \ell_{k+1}), (\ell_2, \ell_3, \dots, \ell_{k+1})) \\
&\quad + \tilde{V}((\ell_3, \ell_4, \dots, \ell_{k+2}), \tilde{\Gamma}(\ell_3, \ell_4, \dots, \ell_{k+2}), (\ell_3, \ell_4, \dots, \ell_{k+2}))] \\
&\quad + \eta_6 [\tilde{V}((\ell_1, \ell_2, \dots, \ell_k), (\ell_1, \ell_2, \dots, \ell_k), \tilde{\Gamma}(\ell_1, \ell_2, \dots, \ell_k)) + \tilde{V}((\ell_2, \ell_3, \dots, \ell_{k+1}), (\ell_2, \ell_3, \dots, \ell_{k+1}), \tilde{\Gamma}(\ell_2, \ell_3, \dots, \ell_{k+1})) \\
&\quad + \tilde{V}((\ell_3, \ell_4, \dots, \ell_{k+2}), (\ell_3, \ell_4, \dots, \ell_{k+2}), \tilde{\Gamma}(\ell_3, \ell_4, \dots, \ell_{k+2}))],
\end{aligned}$$

Now, according to Theorem 2.13 we deduce that $\tilde{\Gamma}$ admits at least a fixed point which implies that there exists $\sigma_1, \dots, \sigma_n$ such that $\Gamma(\sigma_1, \dots, \sigma_n) = \sigma_1 = \dots = \sigma_n$, that is, Γ possesses at least a Kannan-Presić type fixed point. \square

4. APPLICATION

Let $X = C([a, b], (-\infty, +\infty))$ be the set of real continuous functions defined on $[a, b]$. Consider the following Fredholm integral Equation:

$$\ell(t) = \int_a^b M(t, s, \ell(s)) ds + g(t), \tag{4.1}$$

for all $s, t \in [a, b]$, where $M : [a, b]^2 \times (-\infty, +\infty) \rightarrow (-\infty, +\infty)$ and $g : [a, b] \rightarrow (-\infty, +\infty)$. Let

$$d(\ell, \ell') = \max_{t \in [a, b]} |\ell(t) - \ell'(t)|^p.$$

Define $V : X^3 \rightarrow \mathbb{R}$ by:

$$V(\ell, \ell', \ell'') = (d(\ell, \ell') + d(\ell', \ell''))$$

for all $\ell, \ell', \ell'' \in X$ where $p \geq 1$. Now, if we define the mappings $\alpha, \beta, \gamma : X \times X \times X \rightarrow [1, \infty)$ by

$$\begin{aligned}\alpha(\ell(t), \ell'(t), \ell''(t)) &= \beta(\ell(t), \ell'(t), \ell''(t)) = \gamma(\ell(t), \ell'(t), \ell''(t)) \\ &= 2^p,\end{aligned}$$

then (X, V) is a tripled controlled V -Metric Space. Now, we consider the following assumption:

For all $\ell, \ell' \in X$, assume that the following condition holds:

$$\begin{aligned}|M(s, t, \ell(s)) - M(s, t, \ell'(s))| \\ \leq \frac{\eta_6}{2^{2p+1}(b-a)^{1+\frac{p}{q}}} \left(\left| \left(\int_a^b M(t, s, \ell(s)) ds + g(t) - \ell(s) \right) \right| + \left| \left(\int_a^b M(t, s, \ell'(s)) ds + g(t) - \ell'(s) \right) \right| \right).\end{aligned}$$

Theorem 4.1. Suppose that above assumptions hold. Then the integral equation (4.1) has a unique solution in X .

Proof. We define $\Gamma : X \rightarrow X$ by

$$\Gamma(\ell)(t) = \int_a^b M(t, s, \ell(s)) ds + g(t), \quad \forall \ell \in X \text{ and } \forall s, t \in [a, b].$$

For every $\ell, \ell', \ell'' \in X$, we have

$$\begin{aligned}|\Gamma(\ell) - \Gamma(\ell')|^p + |\Gamma(\ell') - \Gamma(\ell'')|^p \\ = \left| \int_E [M(t, s, \ell(s)) - M(t, s, \ell'(s))] ds \right|^p + \left| \int_E [M(t, s, \ell'(s)) - M(t, s, \ell''(s))] ds \right|^p \\ \leq \left[\int_E |M(t, s, \ell(s)) - M(t, s, \ell'(s))| ds \right]^p + \left[\int_E |M(t, s, \ell'(s)) - M(t, s, \ell''(s))| ds \right]^p \\ \leq \frac{\eta_6}{2^{2p+1}(b-a)^{1+\frac{p}{q}}} \left[\int_E [|\ell(s) - \Gamma\ell(s)| + |\ell'(s) - \Gamma\ell'(s)|] ds \right]^p \\ + \frac{\eta_6}{2^{2p+1}(b-a)^{1+\frac{p}{q}}} \left[\int_E [|\ell'(s) - \Gamma\ell'(s)| + |\ell''(s) - \Gamma\ell''(s)|] ds \right]^p \\ \leq \frac{\eta_6}{2^{2p+1}(b-a)^{1+\frac{p}{q}}} \left[\left[\int_E [|\ell(s) - \Gamma\ell(s)| + |\ell'(s) - \Gamma\ell'(s)|]^p ds \right]^{\frac{1}{p}} \left[\int_E ds \right]^{\frac{1}{q}} \right]^p \\ + \frac{\eta_6}{2^{2p+1}(b-a)^{1+\frac{p}{q}}} \left[\left[\int_E [|\ell'(s) - \Gamma\ell'(s)| + |\ell''(s) - \Gamma\ell''(s)|]^p ds \right]^{\frac{1}{p}} \left[\int_E ds \right]^{\frac{1}{q}} \right]^p \\ \leq \frac{\eta_6}{2^{2p+1}(b-a)^{1+\frac{p}{q}}} \left[\left[\int_E 2^{p-1} [d(\ell, \Gamma\ell) + d(\ell', \Gamma\ell')] ds \right]^{\frac{1}{p}} \left[\int_E ds \right]^{\frac{1}{q}} \right]^p \\ + \frac{\eta_6}{2^{2p+1}(b-a)^{1+\frac{p}{q}}} \left[\left[\int_E 2^{p-1} [d(\ell', \Gamma\ell') + d(\ell'', \Gamma\ell'')] ds \right]^{\frac{1}{p}} \left[\int_E ds \right]^{\frac{1}{q}} \right]^p \\ \leq \frac{\eta_6}{2^{2p+1}(b-a)^{1+\frac{p}{q}}} \left[\left[\int_E 2^{p-1} [d(\ell, \Gamma\ell) + d(\ell', \Gamma\ell')] ds \right]^{\frac{1}{p}} \left[\int_E ds \right]^{\frac{1}{q}} \right]^p \\ + \frac{\eta_6}{2^{2p+1}(b-a)^{1+\frac{p}{q}}} \left[\left[\int_E 2^{p-1} [d(\ell', \Gamma\ell') + d(\ell'', \Gamma\ell'')] ds \right]^{\frac{1}{p}} \left[\int_E ds \right]^{\frac{1}{q}} \right]^p \\ \leq \frac{\eta_6}{2^{2p+1}(b-a)^{1+\frac{p}{q}}} \left[\left[\int_E 2^{p-1} [d(\ell, \Gamma\ell) + d(\ell', \Gamma\ell') + d(\ell'', \Gamma\ell'')] ds \right]^{\frac{1}{p}} \left[\int_E ds \right]^{\frac{1}{q}} \right]^p \\ + \frac{\eta_6}{2^{2p+1}(b-a)^{1+\frac{p}{q}}} \left[\left[\int_E 2^{p-1} [d(\ell, \Gamma\ell) + d(\ell', \Gamma\ell') + d(\ell'', \Gamma\ell'')] ds \right]^{\frac{1}{p}} \left[\int_E ds \right]^{\frac{1}{q}} \right]^p \\ \leq \frac{\eta_6}{2^{2p+1}(b-a)^{1+\frac{p}{q}}} \left[\left[\int_E 2^{p-1} [V(\ell, \ell, \Gamma\ell) + V(\ell', \ell', \Gamma\ell') + V(\ell'', \ell'', \Gamma\ell'')] ds \right]^{\frac{1}{p}} \left[\int_E ds \right]^{\frac{1}{q}} \right]^p \\ + \frac{\eta_6}{2^{2p+1}(b-a)^{1+\frac{p}{q}}} \left[\left[\int_E 2^{p-1} [V(\ell, \ell, \Gamma\ell) + V(\ell', \ell', \Gamma\ell') + V(\ell'', \ell'', \Gamma\ell'')] ds \right]^{\frac{1}{p}} \left[\int_E ds \right]^{\frac{1}{q}} \right]^p \\ \leq \frac{\eta_6}{2^{2p+1}(b-a)^{1+\frac{p}{q}}} 2^p (b-a)^{1+\frac{p}{q}} [V(\ell, \ell, \Gamma\ell) + V(\ell', \ell', \Gamma\ell') + V(\ell'', \ell'', \Gamma\ell'')].\end{aligned}$$

So,

$$\begin{aligned}
 & 2^{p+1} [|\Gamma(\ell) - \Gamma(\ell')|^p + |\Gamma(\ell') - \Gamma(\ell'')|^p] \\
 & \leq \frac{\eta_6}{2^{2p+1}(b-a)^{1+\frac{p}{q}}} 2^{2p+1} (b-a)^{1+\frac{p}{q}} [V(\ell, \ell, \Gamma\ell) + V(\ell', \ell', \Gamma\ell') + V(\ell'', \ell'', \Gamma\ell'')] \\
 & = \eta_6 [V(\ell, \ell, \Gamma\ell) + V(\ell', \ell', \Gamma\ell') + V(\ell'', \ell'', \Gamma\ell'')].
 \end{aligned}$$

Therefore, it is the mapping Γ satisfies all the conditions of Theorem 2.13. Hence, Γ has a unique fixed point, that is, the Fredholm integral Equation (4.1) has a unique solution. \square

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