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APPROXIMATION BY GENERALIZED q-SZÁSZ-MIRAKJAN OPERATORS

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Abstract. In this article, we introduce generalized $q-$Szász-Mirakjan operators and study their approximation properties. Based on the Voronovskaja's theorem, we obtain quantitative estimates for these operators.

1. Introduction

Approximation theory is an interesting branch of Mathematics which deals with approximating a function with simple calculative functions. The concept of $q-$calculus has emerged as a new interest in the area of approximation theory. Applications of $q-$calculus have accelerated research in this area. In recent times, the $q-$calculus has been extensively used in approximation theory (e.g. Aral [3], Aral and Gupta [4], Gal et al. [6], Mahmudov [12], Ostrovska [20], Rao et al. [23], Singh and Gairola [24] etc.). Using $q-$calculus, more suitable and useful generalizations of many classical operators have been obtained and investigated. The q-analogues of operators have better rate of convergence than classical ones as proved by Lupas [11] and Phillips [22].

We will use the abbreviations $\mathbb{N}$, $\mathbb{R}_+$, $\mathbb{R}$ for the set of natural numbers, positive real numbers and real numbers respectively.

Let us recall rudiments of q-calculus. The q-integer, factorial and binomial coefficient are defined by

$$[n]_q = \begin{cases} \sum_{k=1}^{n} q^{k-1}, & q \neq 1, \quad n \in \mathbb{N} \\ 1, & q = 1, \\ 0, & n = 0. \end{cases}$$

$$[n]_q! = \begin{cases} (1+q)(1+q+q^2)(1+q+q^2+q^3+)\cdots(1+q+q^2+q^3\cdots+q^{n-1}), & n \in \mathbb{N} \\ 1, & n = 0. \end{cases}$$

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$$\left[\begin{array}{c} n \\ k \end{array}\right]_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

respectively. For $|q|<1$, we give two $q-$analogues of classical exponential function as

$$e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!},\ \ |x| < \frac{1}{q-1}$$

$$E_q^x = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{x^n}{[n]_q!},\ \ x\in\mathbb{R}.$$

For $a \in (0,\infty)\setminus 1$ and $|q|<1$, we introduce the following generalized $q-$analogues of the exponential function

$$e_{a,q}^x = \sum_{n=0}^{\infty} \frac{(x\log a)^n}{[n]_q!} = \prod_{n=0}^{\infty} \frac{1}{1-q^n(1-q)x\log a}, \tag{1.1}$$

$$E_{a,q}^x = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{x^n}{[n]_q!} = \prod_{n=0}^{\infty} (1+q^n(1-q)x\log a) \tag{1.2}$$

for $|x| < \frac{1}{(1-q)\log a}$. It is observed that for $a=e$, the generalized exponential functions reduce to the classical ones. Making use of (1.1), (1.2), we obtain the following exponential function

$$\mathcal{E}_{a,q}^x = e_q^{\frac{x\log a}{2}} E_q^{\frac{x\log a}{2}} = \prod_{n=0}^{\infty} \frac{1+q^n(1-q)\frac{x\log a}{2}}{1-q^n(1-q)\frac{x\log a}{2}}. \tag{1.3}$$

The classical Szász -Mirakjan operators are given by [26]

$$\mathcal{S}_n(h)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} h\left(\frac{k}{n}\right) \tag{1.4}$$

where $h \in C[0,\infty), n\in\mathbb{N}$. In recent years, many generalizations and modifications of these operators have been defined and studid (e.g. Altomare [1], Ansari et al. [2], Içöz and Çekim [7], Kajla and Agrawal [8, 9], Kajla [10], Mediha and Doğru [13], Mursaleen et al. [14, 15, 16, 17], Nasiruzzaman [18, 19], Srivastava et al. [25] and Sucu [27] etc.). In the present work, we will introduce a generalization of these operators and explore approximation properties.
The rest of the paper is organized as follows. In Section 2, a new generalization of operators in (1.4) is defined and moments are computed. Section 3 studies some uniform convergence results. Section 4 contains the proof of Voronvskaja type theorem for our operators and Section 5 gives the conclusion.

2. Construction of operators and moments

Using $q-$integers, we propose a new generalization of the operators in (1.4) as follows

$$\mathcal{S}_{n,a,q}(h)(x) = \frac{1}{\mathcal{E}_{a,q}^{[n]_q x}} \sum_{k=0}^{\infty} \frac{[n]_q^k (x\log a)^k}{[k]_q!}\, h\left(\frac{[k]_q}{[n]_q}\right), \tag{2.1}$$

where, $n \in \mathbb{N}$, $q \in (0,1)$ and $0 \leq x < \left(\frac{2}{\log a(1-q)[n]_q}\right)$, $a \in (0,\infty) \setminus 1$, and $\mathcal{E}_{a,q}^{[n]_q x}$ is the $q-$analogue of the exponential function in (1.3) given by

$$\mathcal{E}_{a,q}^{[n]_q x} = \frac{1-(1-q)\frac{x}{2}}{1+(1-q)\frac{x}{2}}\,\mathcal{E}_{a,q}^{x},$$

and call them as generalized $q-$Szász-Mirakjan operators. Observe that the operators in (2.1) are positive and linear. We compute moments of test functions $e_m(t) = t^m$, $m \in \mathbb{N}$. We obtain some recurrence formulae useful in the sequel. We have

Lemma 2.1. The following recurrence formulae hold for $n, m \in \mathbb{N}$ and $0 < q < 1$:

$$\begin{aligned}
\mathcal{S}_{n,a,q}(t^{m+1})(x) &= \frac{1}{[n]_q(1-q)}\left[\mathcal{S}_{n,a,q}(t^m)(x) - \frac{1-(1-q)\frac{[n]_q x\log a}{2}}{1+(1-q)\frac{[n]_q x\log a}{2}}\mathcal{S}_{n,a,q}(t^m)(x\log aq)\right] \\
&= \sum_{j=0}^{m}\binom{m}{j}\frac{x\log aq^j}{2[n]_q^{m-j}}\left[\mathcal{S}_{n,a,q}(t^j)(x\log a) - \frac{1-(1-q)\frac{[n]_q x}{2}}{1+(1-q)\frac{[n]_q x\log a}{2}}\mathcal{S}_{n,a,q}(t^j)(x\log aq)\right] \\
&= \frac{1}{2\mathcal{E}_{a,q}^{[n]_q x}}\sum_{j=0}^{m}\binom{m}{j}\frac{x\log a}{[n]_q^{m-j}}\Big[\mathcal{E}_{a,q}^{[n]_q x q^{m-j}}\mathcal{S}_{n,a,q}(t^j)(x\log aq^{m-j}) \\
&\qquad + \mathcal{E}_{a,q}^{[n]_q x q^{m-j+1}}\mathcal{S}_{n,a,q}(t^j)(x\log aq^{m-j+1})\Big].
\end{aligned}$$

Proof. We have

$$\begin{aligned}
\mathcal{S}_{n,a,q}(t^{m+1})(x) &= \frac{1}{\mathcal{E}_{a,q}^{[n]_q x}}\sum_{k=0}^{\infty}\frac{[n]_q^k(x\log a)^k}{[k]_q!}\left(\frac{[k]_q^{m+1}}{[n]_q^{m+1}!}\right) \\
&= \frac{1}{\mathcal{E}_{a,q}^{[n]_q x}}\sum_{k=0}^{\infty}\frac{[n]_q^k x^k}{[k]_q!}\left(\frac{[k]_q^{m+1}}{[n]_q^{m}}\right)\left(\frac{1}{[n]_q}\frac{1-q^k}{1-q}\right) \\
&= \frac{1}{\mathcal{E}_{a,q}^{[n]_q x}}\frac{1}{[n]_q(1-q)}\left(\sum_{k=0}^{\infty}\frac{[n]_q^k(x\log a)^k}{[k]_q!}\frac{[k]_q^m}{[n]_q^m} - \sum_{k=0}^{\infty}\frac{[n]_q^k(x\log aq)^k}{[k]_q!}\frac{[k]_q^m}{[n]_q^m}\right) \\
&= \left(\frac{1}{[n]_q(1-q)}\right)\left[\frac{1}{\mathcal{E}_{a,q}^{[n]_q x}}\sum_{k=0}^{\infty}\frac{[n]_q^k(x\log a)^k}{[k]_q!}\frac{[k]_q^m}{[n]_q^m} - \frac{\mathcal{E}_{a,q}^{[n]_q xq}}{\mathcal{E}_{a,q}^{[n]_q x}}\frac{1}{\mathcal{E}_{a,q}^{[n]_q xq}}\sum_{k=0}^{\infty}\frac{[n]_q^k(xq)^k}{[k]_q!}\frac{[k]_q^m}{[n]_q^m}\right] \\
&= \left(\frac{1}{[n]_q(1-q)}\right)\left[\mathcal{S}_{n,a,q}(t^m)(x) - \frac{1-(1-q)\frac{[n]_q x\log a}{2}}{1-(1-q)\frac{[n]_q x\log a}{2}}\mathcal{S}_{n,a,q}(t^m)(q\,x\log a)\right],
\end{aligned}$$

which establishes the first formula. The second expression is derived by using the relation

$$[k]_q = (1+q)[k-1]_q = (1+q)\left(\frac{1-q^{k-1}}{1-q}\right) = \left(\frac{1-q^k}{1-q}\right).$$

In fact, we have

$$\begin{aligned}
\mathcal{S}_{n,a,q}(t^{m+1})(x) &= \frac{1}{\mathcal{E}_{a,q}^{[n]_q x}}\sum_{k=0}^{\infty}\frac{[n]_q^k(x\log a)^k}{[k]_q!}\frac{[k]_q^m}{[n]_q^m} \\
&= \frac{1}{\mathcal{E}_{a,q}^{[n]_q x}}\sum_{k=0}^{\infty}\binom{m}{j}\frac{x\log a}{2[n]_q^{m-j}}\sum_{k=1}^{\infty}\frac{[n]_q^{k-1}(x\log a)^{k-1}}{[k-1]_q!}\frac{[k-1]_q^j}{[n]_q^j}q^j(1+q^{k-1})
\end{aligned}$$

$$= \frac{1}{\mathcal{E}_{a,q}^{[n]_q x\log a}} \sum_{j=0}^{\infty} \binom{m}{j} \frac{x \log aq^j}{2[n]_q^{m-j}} \left[\sum_{k=1}^{\infty} \frac{[n]_q^{k-1}(x\log a)^{k-1}}{[k-1]_q!} \frac{[k-1]_q^j}{[n]_q^j} + \sum_{k=1}^{\infty} \frac{[n]_q^{k-1}(x\log aq)^{k-1}}{[k-1]_q!} \frac{[k-1]_q^j}{[n]_q^j} \right]$$

$$= \sum_{j=0}^{m} \binom{m}{j} \frac{x \log aq^j}{2[n]_q^{m-j}} \left[\frac{1}{\mathcal{E}_{a,q}^{[n]_q x}} \sum_{k=0}^{\infty} \frac{[n]_q^{k}(x\log a)^{k}}{[k]_q!} \frac{[k]_q^j}{[n]_q^j} + \frac{\mathcal{E}_{a,q}^{[n]_q x}}{\mathcal{E}_{a,q}^{[n]_q x}} \frac{1}{\mathcal{E}_{a,q}^{[n]_q xq}} \sum_{k=0}^{\infty} \frac{[n]_q^{k}(x\log aq)^{k}}{[k]_q!} \frac{[k]_q^j}{[n]_q^j} \right],$$

giving us the second formula. The proof of the third recurrence relation is based on [12]. The following relation is useful:

$$[k]_q = [k-1]_q + q^{k-1} = \frac{1-q^{k-1}}{1-q} + q^{k-1} = \left(\frac{1-q^k}{1-q} \right).$$

One obtains $\mathcal{S}_{n,a,q}(t^{m+1})(x)$

$$\begin{aligned}
&= \frac{1}{\mathcal{E}_{a,q}^{[n]_q x}} \sum_{k=0}^{\infty} \frac{[n]_q^{k}(x\log a)^{k}}{[k]_q!} \frac{[k]_q^{m+1}}{[n]_q^{m+1}} \\
&= \frac{1}{\mathcal{E}_{a,q}^{[n]_q x}} \sum_{k=0}^{\infty} \frac{[n]_q^{k-1}[n]_q x^{k}}{[k-1]_q![k]_q} \frac{[k]_q^{m}}{[n]_q^{m}[n]_q} [k]_q \frac{1+q^{k-1}}{2} \\
&= \frac{1}{\mathcal{E}_{a,q}^{[n]_q x}} \sum_{j=0}^{\infty} \binom{m}{j} \frac{x\log a}{2[n]_q^{m-j}} \left(\sum_{k=1}^{\infty} \frac{[n]_q^{k-1}.(x\log aq^{m-j})^{k-1}}{[k-1]_q!} \frac{[k-1]_q^j}{[n]_q^j} \right. \\
&+ \left. \sum_{k=1}^{\infty} \frac{[n]_q^{k-1}(x\log aq^{m}-j+1)^{k-1}}{[k-1]_q!} \frac{[k-1]_q^j}{[n]_q^j} \right) \\
&= \frac{1}{\mathcal{E}_{a,q}^{[n]_q x}} \sum_{j=0}^{\infty} \binom{m}{j} \frac{x\log a}{2[n]_q^{m-j}} \left(\frac{\mathcal{E}_{a,q}^{[n]_q xq^{m-j}}}{\mathcal{E}_{a,q}^{[n]_q xq^{(m-j)}}} \sum_{k=1}^{\infty} \frac{[n]_q^{k-1}(x\log aq^{m-j})^{k-1}}{[k-1]_q!} \frac{[k-1]_q^j}{[n]_q^j} \right. \\
&+ \left. \sum_{k=1}^{\infty} \frac{[n]_q^{k-1}(x\log aq^{m}-j+1)^{k-1}}{[k-1]_q!} \frac{[k-1]_q^j}{[n]_q^j} \right),
\end{aligned}$$

proving the last expression and hence the lemma.

Lemma 2.2. Let $n \in \mathbb{N}, 0 < q < 1$ and $m \in \mathbb{N}$. Then the following formula holds true

$$\mathcal{S}_{n,a,q}(t^m)(x) = \frac{1}{\prod_{k=0}^{m-1} \left(1 + (1-q)q^k \frac{[n]_q x \log a}{2} \right)} \sum_{j=1}^{\infty} a_{m,j}(q) \frac{(x\, loga)^j}{[n]_q^{m-j}},$$

where

$$a_{m+1,j}(q) = [j]_q a_{m,j}(q) + \frac{q^{j-1}(1+q^{m-j+1})}{2} a_{m,j-1}(q), m \geq 0, j \geq 1$$

and $a_{0,0}(q) = 1, a_{m,0}(q) = 0, m > 0, a_{m,j}(q) = 0, j > m$.

Proof. We use the recurrence formula to facilitate the induction on m.

We have $\mathcal{S}_{n,a,q}(t^{m+1})(x)$

$$
\begin{aligned}
&= \frac{1}{[n]_q(1-q)}\left[\mathcal{S}_{n,a,q}(t^m)(x) - \frac{1-(1-q)\frac{[n]_q x\,loga}{2}}{1+(1-q)\frac{[n]_q x\,loga}{2}}.\mathcal{S}_{n,a,q}(t^m)(q\,x\,loga)\right] \\
&= \frac{1}{\prod_{k=0}^{m}[1+q^k(1-q)\frac{[n]_q x\,loga}{2}]} \\
&\times \sum_{j=1}^{m} a_{m,j}(q)\frac{(x\,loga)^j}{[n]_q^{m-j}}.\frac{1}{[n]_q(1-q)}\left(1+q^m(1-q)\frac{[n]_q x\,loga}{2} - q^j + q^j(1-q)\frac{[n]_q}{2}\right) \\
&= \frac{1}{\prod_{k=0}^{m}[1+q^k(1-q)\frac{[n]_q x\,loga}{2}]} \\
&\times \left[a_{m,1}(q)\frac{x\,loga}{[n]_q^m} + \sum_{j=2}^{m}\left(a_{m,j}(q)j_q + a_{m,j-1}(q)\frac{q^{j-1}(1+q^{m-j+1})}{2}\right)\frac{(x\,loga)^j}{[n]_q^{m+1-j}}\right. \\
&+ \left. a_{m,m}(q)q^m(x\,loga)^{m+1}\right]
\end{aligned}
$$

and after simple steps of calculations, we arrive at the desired result.

Lemma 2.3. The following formulas hold for $n \in \mathbb{N}$ and $0 < q < 1$:

$$\mathcal{S}_{n,a,q}(1)(x) = 1,$$

$$\mathcal{S}_{n,a,q}(t)(x) = \frac{x\log a}{1+(1-q)\frac{[n]_q x\log a}{2}},$$

$$\mathcal{S}_{n,a,q}(t^2)(x) = \frac{1}{\prod_{k=0}^{1}\left(1+(1-q)\frac{[n]_q x\log a}{2}x\log a\right)}\left((x\log a)^2 q + \frac{x\log a}{[n]_q}\right),$$

$$
\begin{aligned}
\mathcal{S}_{n,a,q}(t^3)(x) &= \frac{1}{\prod_{k=0}^{2}\left(1+(1-q)q^k\frac{[n]_q x\log a}{2}x\log a\right)}\left((x\log a)^3 q^3 + \frac{(x\log a)^2}{[n]_q}\frac{3q(1+q)}{2}\right. \\
&+ \left.\frac{x\log a}{[n]_q^2}\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{S}_{n,a,q}(t^4)(x) &= \frac{1}{\prod_{k=0}^{3}\left(1+(1-q)q^k\frac{[n]_q x\log a}{2}x\log a\right)}\left((x\log a)^4 q^6 + \frac{(x\log a)^3}{[n]_q}\frac{q^3}{4}(7q^2+10q\right. \\
&+ \left. 7) + \frac{(x\log a)^2}{[n]_q^2}q(2q^2+3q+2+\frac{x\log a}{[n]_q^3}\right).
\end{aligned}
$$

Proof. Making use of Lemmas 2.1, 2.2, the proof follows. □

Remark. The quantities $Q_{n,a,k} = 1 + q^k(1-q)\frac{[n]_q x\log a}{2}$, where $n, k \in \mathbb{N}$ and $0 < q < 1$, can be rewritten as $Q_{n,a,k} = 1 + q^k\frac{(1-q^n)x\log a}{(1-q^{n-1})}$. We want to study the convergence of the operators $\mathcal{S}_{n,a,q}$ to the identity operator. To do this end we have the following.

Lemma 2.4. Let $x \in \left[0, \frac{1+q_n^{n-1}}{(1-q_n^n)\, log\, a}\right)$ and $(q_n)_{n\in\mathbb{N}}$ be a sequence such that $q_n \in (0,1)\ \forall\, n \in \mathbb{N},\ q_n \to 1$ and $q_n^n \to b$ as $n \to \infty$, where $0 < b \leq 1$. Then the following formulas are obtained:

$$\lim_{n\to\infty} \{\mathcal{S}_{n,a,q_n}(t)(x)\} = \frac{1+a}{1+a+x\log a(1-a)} x \log a$$

$$\lim_{n\to\infty} \left\{\mathcal{S}_{n,a,q_n}(t^2)(x)\right\} = \frac{(1+a)^2}{[1+a+x\log a(1-a)]^2} (x\log)^2,$$

$$\lim_{n\to\infty} \left\{\mathcal{S}_{n,a,q_n}(t^3)(x)\right\} = \frac{(1+a)^3}{[1+a+x\log a(1-a)]^3} (x\log a)^3,$$

$$\lim_{n\to\infty} \left\{\mathcal{S}_{n,a,q_n}(t^4)(x)\right\} = \frac{(1+a)^4}{[1+a+x\log a(1-a)]^4} (x\log)^4.$$

Proof. On utilizing Lemma 2.3 and the fact $\lim_{n\to\infty}\left(\frac{1}{[n]_{q_n}}\right) = 0$, the proof follows.

3. Uniform Approximation Results

In present section, we discuss uniform approximation of generalized q-Szász-Mirakjan operators on compact intervals of $[0,\infty)$. Let us denote by $C[0,\infty)$, the space of all continuous functions on $[0,\infty)$ and $C_B[0,\infty)$, that of bounded and continuous functions on $[0,\infty)$. Define $C_2[0,\infty) = \{h \in C[0,\infty) : \exists\, M > 0 : \|h(x)\| \leq M(1+x)^2\, \forall\, x \geq 0\}$. Let (X,d) be a metric space and $d_x : X \longrightarrow \mathbb{R}$ be the function defined by $d_x(y) = d(x,y)$ ∀ y ∈ X and $F(X)$ the linear space of all real valued functions on X.

Theorem 3.1. Let (X,d) be a locally compact metric space. Consider a lattice subspace E of $F(X)$ containing the constant function 1 and all functions $d_x^2 (x \in X)$, where $d_x^2 = e_2 - 2xe_1 + x^2.1$. Let $(L_n)_{n\geq 1}$ be a sequence of linear operators from E into $F(X)$ and such that

$$\lim_{n\to\infty} L_n(h) = h,$$

and

$$\lim_{n\to\infty} L_n d_x^2(x) = 0$$

uniformly on compact subsets of X. Then for every $h \in E \cap C_B(X)$, we have

$$\lim_{n\to\infty} L_n(h) = h$$

uniformly on compact subsets of X.

We prove the following.

Theorem 3.2. Let $q_n \in (0,1)$ be any sequence $(q_n)_{n\in\mathbb{N}}$ such that $q_n^n \to b$ as $n \to \infty, 0 < b \leq 1$. Then the sequence $\mathcal{S}_{n,a,q_n}(h)$ converges uniformly to h on any compact interval of $[0,\infty)$, for any $h \in C_2[0,\infty) \cap C_B[0,\infty)$.

Proof. For $x \in \left[0, \frac{2}{(1-q_n)\, log\, a[n]_{qn}}\right)$, as the expression $\frac{2}{(1-q_n)\, log\, a[n]_{qn}} \to \infty$ as $n \to \infty$ there exists a rank $\mathbb{N}_0$ such that $[0,d] \subset \left[0, \frac{2}{(1-q_n)\, log\, a[n]_{qn}}\right)$ where $d \in \mathbb{R}_+$. From lemma 2.4 with $d = 1$, we have,

$$\lim_{n\to\infty} \mathcal{S}_{n,a,q_n}(1)(x) = 1,$$

$$\lim_{n\to\infty} \mathcal{S}_{n,a,q_n}(t)(x) = x\log a,$$

$$\lim_{n\to\infty} \mathcal{S}_{n,a,q_n}(t^2)(x) = (x\log a)^2.$$

We prove the uniform convergence of the moment of order one. We have

$$\left|\frac{x\log a}{\left(1+\frac{1-q_n^n x\log a}{1+q_n^{n-1}}\right)} - x\log a\right| = \left|\frac{\left(\frac{1-q_n^n}{1+q_n^{n-1}}\right)(x\log a)^2}{\left(1+\frac{1-q_n^n x\log a}{1+q_n^{n-1}}\right)}\right|$$

$$\leq |1-q_n^n|\, d \to 0, n\to\infty.$$

The uniform convergence of the moment of order 2 results in the following way:

$$\left|\frac{(x\log a)^2 q_n + \frac{x\log a}{[n]_{q_n}}}{\left(1+\frac{1-q_n^n x\log a}{1+q_n^{n-1}}\right)\left(1+q_n\frac{1-q_n^n x\log a}{1+q_n^{n-1}}\right)} - (x\log a)^2\right| \tag{3.1}$$

$$\leq \left|\frac{x\log a}{[n]_{q_n}} - (x\log a)^2(1-q_n) - \frac{1-q_n^n}{1+q_n^{n-1}}(1+q_n)(x\log a)^3 - q_n\frac{(1-q_n^n)^2}{(1+q_n^{n-1})^2}(x\log a)^4\right|$$

$$\leq \frac{d}{[n]_{q_n}} + \Big|(1-q_n)\Big| d^2 + \Big|(1-q_n^n)\Big|\frac{1+q_n}{1+q_n^{n-1}}d^3 + \Big|1-q_n^n\Big|^2\frac{q_n}{(1+q_n^{n-1})^2}d^4 \to 0,\ n\to\infty.$$

The following convergence holds for $x\in[0,d]$ as $n\to\infty$

$$\mathcal{S}_{n,q_n}(1)(x)\to 1,$$

$$\mathcal{S}_{n,q_n}(t)(x)\to x,$$

$$\mathcal{S}_{n,a,q_n}(t)(x)\to x\log a.$$

We further have,

$$\lim_{n\to\infty}\left\{\mathcal{S}_{n,a,q_n}(t^2 - 2x\log at + (x\log a)^2)(x)\right\}$$

$$= \lim_{n\to\infty}[\mathcal{S}_{n,a,q_n}t^2(x) - 2x\log a\mathcal{S}_{n,a,q_n}(t)(x) + (x\log a)^2\mathcal{S}_{n,a,q_n}(1)(x)] = 0,$$

and then in view of Theorem 3.1, our theorem is proved.

Remark. Let $q\in(0,1)$ be fixed and $x\in\left[0, \frac{1+q^{n-1}}{1-q^n log\, a}\right)$. Then we have

$$\mathcal{S}_{n,a,q}(1)(x)\to 1,$$

$$\mathcal{S}_{n,a,q}(t)(x)\to\left\{\frac{x\log a}{1+x\log a}\right\}$$

as $n\to\infty$.

Proof. The proof that $\mathcal{S}_{n,a,q}(1)(x)\to 1$ is obvious. Next, since $q\in(0,1)$, one has $\lim\limits_{n\to\infty} q^n = 0$. From the second formula of Lemma 2.3, it follows that for $n\to\infty$,

$$\lim_{n\to\infty}\mathcal{S}_{n,a,q}(t)(x)\to\frac{x\log a}{1+x\log a}.$$

Moreover, one obtains

$$\begin{aligned}\left|\frac{x\log a}{1+\frac{1-q_n^n x\log a}{1+q_n^{n-1}}}-\frac{x\log a}{1+x\log a}\right| &= \left|(x\log a)^2\right|\left|\frac{1-\left(\frac{1-q_n^n}{1+q_n^{n-1}}\right)}{(1+x\log a)\frac{1-q_n^n x\log a}{1+q_n^{n-1}}}\right| \\ &\leq |x^2|\left|\frac{q^{n-1}(1+q)}{1+q^{n-1}}\right| \\ &\leq \frac{q^{n-1}(1+q)(1+q^{n-1})}{(1-q^n)^2} \\ &\to 0 \ as \ n\to\infty\end{aligned}$$

which proves the uniform convergence.
In the following we will consider only the case when q depends on n, that is $q = q_n$ and such that $q_n^n \to b$ as $n \to \infty, 0 < b \leq 1$.

4. Voronovskaja Type Theorem

The Voronovskaja theorem estimates the rate of convergence in terms of derivatives [5]. We first prove the following.

Lemma 4.1. Let $q_n \in (0,1)\,\forall\, n \in \mathbb{N}$ be such that $q_n \to 1$, and $q_n^n \to b$ where $0 \leq b \leq 1$ as $n \to \infty$. Then following formulas hold good:

$$\lim_{n\to\infty}\{[n]_{q_n}\mathcal{S}_{n,a,q_n}(t-x\log a)(x)\}=0, \tag{4.1}$$

$$\lim_{n\to\infty}\left\{[n]_{q_n}\mathcal{S}_{n,a,q_n}(t-x\log a)^2(x)\right\}=x\log a, \tag{4.2}$$

$$\lim_{n\to\infty}\left\{[n]_{q_n}^2\mathcal{S}_{n,a,q_n}(t-x\log a)^4(x)\right\}=3(x\log a)^2. \tag{4.3}$$

Moreover, the convergence is uniform on any compact interval $[0,d],\ d>0$.

Proof. First we calculate

$$\begin{aligned}&\lim_{n\to\infty}\frac{(1-q_n^n)^2}{(1-q_n)}=\lim_{n\to\infty}\left\{(1-q_n^n)^2(1+q_n+\cdots+q_{n-1}^n)\right\} \\ &=\lim_{n\to\infty}(1+q_n+\cdots+q_{n-1}^n+q_n^n-q_{n+1}^n-\cdots-q_{2n-1}^n)=0,\end{aligned} \tag{4.4}$$

because $q_n \to 1$ and $q_n^n \to b, 0<b\leq 1$, as $n\to\infty$, and

$$\begin{aligned}&\lim_{n\to\infty}\frac{(1-q_n^n)^3}{(1-q_n)}=\lim_{n\to\infty}\left\{(1-q_n^n)^n(1+q_n+\cdots+q_{n-1}^n)\right\} \\ &=\lim_{n\to\infty}(1+q_n+\cdots+q_{n-1}^n+q_n^n-q_{n+1}^n-\cdots-q_{2n-1}^n)=0,\end{aligned}$$

so that

$$\lim_{n\to\infty}\left\{1+q_n+\cdots+q_{n-1}^n-2(q_n+\cdots+q_n^{2n-1})+q_n^{2n}+\cdots+q_n^{3n-1}\right\}=0, \tag{4.5}$$

because $q_n \to 1, q_n^n \to b\,, 0<b\leq 1$, as $n\to\infty$. The calculation holds true for $x\in\left[0,\frac{2}{(1-q_n)log\,a[n]_{q_n}}\right)$. Since the expression $\frac{2}{(1-q_n)log\,a[n]_{qn}}\to\infty$ as $n\to\infty$, there exists a rank N_0 such that if $n>N_0$, then $[0,d]\subset[0,\frac{2}{(1-q_n)log\,a[n]_{qn}})$, where $d\in\mathbb{R}_+$. Then (4.1) is proved in the following way:

$$\left|[n]_{q_n}\mathcal{S}_{n,a,q_n}(t-x\log a)\right|=\left|[n]_{q_n}(\mathcal{S}_{n,a,q_n}(t)(x)-(x\log a\,)\mathcal{S}_{n,a,q_n}(1)(x)\right|$$

$$\begin{aligned}\left|[n]_{qn}\left(\frac{x\log a}{1+(1-q_n)\frac{[n]_{q_n}x\log a}{2}}-x\log a\right)\right| &= \left|-\frac{(1-q_n[n]_{qn}^2)}{2}\frac{(x\log a)^2}{1+(1-q_n)\frac{[n]_{qn}x\log a}{2}}\right|\\ &= \left|\frac{1-q_n}{2}\left(\frac{1-q_n^n}{1-q_n.\frac{2}{1+q_n^{n-1}}}\right)^2.\frac{(x\log a)^2}{1+\frac{1-q_n^n}{1+q_n^{n-1}}x\log a}\right|\\ &= \left|\frac{2(1-q_n^n)^2(1-q_n)}{(1+q_n^{n-1})^2}\frac{(x\log a)^2}{1+\frac{1-q_n^n}{1+q_n^{n-1}}x\log a}\right|\\ &\leq 2\,d^2\left\{\frac{(1-q_n^n)^2}{(1-q_n)}\right\}\to 0,\quad as\ n\to\infty.\end{aligned}$$

The proof of the limit in (4.2) is demonstrated in the following.

$$\begin{aligned}[n]_{q_n}\mathcal{S}_{n,a,q_n}\left((t-x\log a)^2\right)(x) &= [n]_{q_n}[\mathcal{S}_{n,a,q_n}(t^2)(x)-2x\log a\,\mathcal{S}_{n,a,q_n}(t)(x)+\mathcal{S}_{n,a,q_n}(1)(x)]\\ &= [n]_{q_n}\left\{\frac{(x\log a)^2q_n+\frac{x\log a}{[n]_{q_n}}}{[1+(1-q_n)\frac{[n]_{qn}x\log a}{2}][1+q_n(1-q_n)\frac{[n]_{q_n}x\log a}{2}]}\right.\\ &- \left.2x\log a\frac{x\log a}{1+(1-q_n)\frac{[n]_{q_n}x\log a}{2}}+(x\log a)^2\right\}\\ &= \frac{[n]_{q_n}}{Q_{n,a,0}\,Q_{n,a,1}}\left[(x\log a)^4q_n(1-q_n)^2\frac{[n]_{q_n}^2}{4}\right.\\ &+ \left.(x\log a)^3q_n(1-q_n)^2\frac{[n]_{q_n}^2}{2}+(x\log a)^2q_n(1-q_n)\frac{x\log a}{[n]_{q_n}}\right]\\ &= \frac{1}{Q_{n,a,0}\,Q_{n,1}}\left[(x\log a)^4\frac{q_n(1-q_n)^2}{4}\left(\frac{1-q_n^n}{1-q_n}.\frac{2}{(1+q_n^{n-1})}\right)^3\right.\\ &+ (x\log a)^3\frac{(1-q_n)^2}{2}\left(\frac{1-q_n^n}{1-q_n}\frac{2}{(1+q_n^{n-1})}\right)^2\\ &+ \left.(x\log a)^2(q_n-1)\frac{1-q_n^n}{1-q_n}\frac{2}{1+q_n^{n-1}}+x\log a\right]\\ &= \frac{1}{Q_{n,a,0}\,Q_{n,a,1}}\left[(x\log a)^4q_n\frac{(1-q_n^n)^3}{1-q_n}\frac{2}{(1+q_n^{n-1})^3}\right.\\ &+ (x\log a)^3(1-q_n^n)^2\frac{2}{(1+q_n^{n-1})^2}\\ &- \left.(x\log a)^2(1-q_n^n)\frac{2}{(1+q_n^{n-1})}+x\log a\right].\end{aligned}$$

Then, we have $\left|[n]_{q_n}\mathcal{S}_{n,a,q_n}((t-x\log a)^2)(x)-x\log a\right|$

$$\begin{aligned}&= \left|\frac{1}{Q_{n,a,0}\,Q_{n,a,1}}\left[(x\log a)^4q_n\frac{(1-q_n^n)^3}{1-q_n}\frac{2}{(1+q_n^{n-1})^3}\right.\right.\\ &+ \left.\left.(x\log a)^3(1-q_n^n)^2\frac{2}{(1+q_n^{n-1})^2}-(x\,log\,a)^2(1-q_n^n)\frac{2}{(1+q_n^{n-1})}\right]\right|\\ &\leq 2\,d^4\frac{(1-q_n^n)^3}{1-q_n}+2\,d^3(1-q_n^n)^2+d^2(1-q_n^n),\end{aligned}$$

which tends to zero by (4.5) as $n \to \infty$, which proves the limit (4.2). The third limit is similarly proved and hence the lemma. which tends to zero by (4.5) as $n \to \infty$, which proves the limit (4.2). The third limit is similarly proved and hence the lemma.

Remark. Under the conditions of Lemma 4.1, the limit (4.2) implies that $\|\mathcal{S}_{n,a,q_n}(t-.)^2\| \to 0$ as $n \to \infty$, uniformly on compact intervals.

Recall that if $I \subset \mathbb{R}$ is an interval, and $h : I \to \mathbb{R}$ a bounded function then for $\delta > 0$, the modulus of continuity of first order is defined as [21]:

$$w(h, \delta) = sup \{|h(u) - h(v)| : u, v \in I, |u - v| \leq \delta\}. \tag{4.6}$$

Moreover, the inequality $w(h, \delta) \leq [1 + (\frac{\delta}{\eta})^2] w(h, \eta), \forall \, \delta, \, \eta > 0$ holds for h bounded. Below we prove the Voronovskaja theorem.

Theorem 4.2. Let $q_n \in (0, 1)$ be a sequence such that $q_n \to 1, q_n^n \to b, 0 < b \leq 1$ as $n \to \infty$ and h be any continuous and bounded function on $[0, \infty)$ (such that $h^{'}$ and $h^{''}$ are continuous and bounded on $[0, \infty)$). Then the convergence of the operators $\mathcal{S}_{n,a,q_n}$ is uniform on every compact interval $[0, d], \; d > 0,$

and we have

$$\lim_{n\to\infty} \{[n]_{q_n} [\mathcal{S}_{n,a,q_n}(h)(x) - h(x)]\} = \frac{x}{2} h^{''}(x).$$

Proof. Let h, h', h'' be continuous and bounded on $[0, \infty)$ and $x \in [0, \infty)$ be fixed. We use Taylor's formula with integral remainder as

$$h(t) = h(x) + h'(x)(t - x) + \int_x^t (t - u) h''(u) du.$$

Let's use the notation

$$R_2(t, x) = \int_x^t (t - u) h''(u) du = \int_x^t (t - u)(h''(u) - h''(x)) du$$
$$= h''(x) \frac{(t - x)^2}{2} + \int_x^t (t - u)(h''(u) - h''(x)) du.$$

and $r(t; x) = \frac{1}{(t-x)^2} \int_x^t (t - u)[h''(u) - h''(x)] du$. Then it follows that

$$h(t) = h(x) + h(x)(t - x) + h''(x) \frac{(t - x \log a)^2}{2} + r(t; x)(t - x \log a)^2. \tag{4.7}$$

Let's rewrite (4.7) by denoting as

$$r(t, a; x) = \frac{h(t) - h(x) - h'(x)(t - x \log a) + h''(x) \frac{(t - x \log a)^2}{2}}{(t - x \log a)^2}.$$

Then, we shall prove that $\lim_{t\to x} r(t, a; x) = 0$. Let's show that $r(t, a; x)$ is bounded and continuous for $t \in [0, d]$. We have

$$\lim_{t\to x} r(t, a; x) = \lim_{t\to x} \frac{h(t) - h(x) - h'(x)(t - x \log a) + h''(x) \frac{(t - x \log a)^2}{2}}{(t - x \log a)^2}.$$
$$= \lim_{t\to x} \frac{h''(t) - h''(x)}{2} = 0,$$

because h'' is continuous. Boundedness results from the formula

$$\begin{aligned}|r(t,a;x)| &= \frac{1}{(t-x\log a)^2}\left|\int_{x\log a}^{t}(t-u)[h''(u)-h''(x)du]\right| \\ &\leq \frac{1}{(t-x\log a)^2}\int_{x\log a}^{t}\Big|(t-u)\Big|\Big[\Big|h''(u)\Big|+\Big|h''(x)\Big|\Big]\,du \\ &\leq \frac{1}{(t-x\log a)^2}2\|h''\|\int_{x}^{t}(t-u)du=\|h''\|.\end{aligned}$$

For continuity, if $t=x$, by convention $r(x,x)=0$. If $t\neq x$, then we have

$$|r(t,a;x)|=\frac{1}{(t-x\log a)^2}\int_{x}^{t}(t-u)|[h''(u)-h''(x)]|du.$$

By means of modulus of continuity of first order, we obtain

$$|h''(u)-h''(x)|\leq w(h''|(u-x)|)\leq w(h''|t-x|).$$

It follows that $|r(t;x)|\leq\frac{1}{2}w(h''|t-x|)$ for $t\neq x$. By applying $\mathcal{S}_{n,a,q_n}$ to both sides of (4.7), we have

$$\mathcal{S}_{n,a,q_n}(h)(x)=h(x)\mathcal{S}_{n,a,q_n}(1)(x)+h'(x)\mathcal{S}_{n,a,q_n}(t-x\log a)(x)$$
$$+\frac{h''(x)}{2}\mathcal{S}_{n,a,q_n}(t-x)^2(x)+\mathcal{S}_{n,a,q_n}(r(t,a;x)(t-x\log a)^2(x).$$

Then

$$[n]_{q_n}\left[\mathcal{S}_{n,a,q_n}(h)(x)-h(x)\right]=h'(x)[n]_{q_n}\mathcal{S}_{n,a,q_n}(t-x)(x)+\frac{h''(x)[n]_{q_n}}{2}\mathcal{S}_{n,a,q_n}(t-x\log a)^2$$
$$+[n]_{q_n}\mathcal{S}_{n,a,q_n}\left(r(t,a;x)(t-x\log a)^2\right)(x). \tag{4.8}$$

By applying Cauchy-Schwartz inequality, one gets

$$|\mathcal{S}_{n,a,q_n}\left(r(t,a;x)(t-x\log a)^2\right)(x)|\leq\sqrt{\mathcal{S}_{n,a,q_n}(r^2(t,a;x))(x)}\sqrt{[n]_{q_n}^2\mathcal{S}_{n,a,q_n}(t-x\log a)^4(x)}. \tag{4.9}$$

By (4.3), we have

$$\lim_{n\to\infty}\sqrt{[n]_{q_n}^2\mathcal{S}_{n,a,q_n}(t-x)^4(x)}=\sqrt{3(x\log a)^2}. \tag{4.10}$$

Then

$$\begin{aligned}|\mathcal{S}_{n,a,q_n}(r(t,a;x))(x)| &\leq \mathcal{S}_{n,a,q_n}(|r(t,a;x)|)(x)\leq\mathcal{S}_{n,a,q_n}[w(h'')|t-x\log a|](x) \\ &\leq \mathcal{S}_{n,a,q_n}\left[\left(1+\frac{(t-x\log a)^2}{(\eta)_n^2}\right)w(h,(\eta)_n)\right] \\ &= w(h,\eta_n)\mathcal{S}_{n,a,q_n}(1)(x)+\frac{w(h,\eta_n)}{\eta_n}\mathcal{S}_{n,a,q_n}((t-x\log a)^2)(x).\end{aligned}$$

On considering $\eta_n=\|\sqrt{\mathcal{S}_{n,a,q_n}(t-x\log a)^2}(x)\|$, one obtains

$$|\mathcal{S}_{n,a,q_n}r(t,x)(x)|\leq 2w\left(h,\|\sqrt{\mathcal{S}_{n,a,q_n}(t-x\log a)^2(x)}\|\right). \tag{4.11}$$

Combining (4.8), (4.9), (4.10) and (4.11), the proof is completed.

5. Conclusion

In this manuscript we introduced a generalized q- Szász-Mirakjan operators. These operators are more flexible (depending on the selection of q) than the classical Szász-Mirakjan operators while retaining approximation properties. Also, we investigated the shape preserving and convergence properties using modulus of continuity and proved Voronovskaja theorem.

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