Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 12 Issue 6 (2021), Pages 1-8.
DOI:10.54379/jma-2021-6-1

P-NORMALITY

LUTFI KALANTAN AND MAI MANSOURI

Abstract. A topological space X is called *P-normal* if there exist a normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|A} : A \longrightarrow f(A)$ is a homeomorphism for each paracompact subspace $A \subseteq X$. We will investigate this property and produce some examples to illustrate the relation between P-normality and other weaker kinds of normality.

1. Introduction

We introduce a new weaker version of normality and call it *P-normality.* The purpose of this paper is to investigate this property. We present some examples to show relationships between P-normality and other weaker versions of normality such as C-normality, L-normality, and epinormality. Throughout this paper, we denote an ordered pair by $\langle x, y\rangle$, the set of positive integers by $\mathbb{N}$ and the set of real numbers by $\mathbb{R}$. A T_4 space is a T_1 normal space and a Tychonoff space is a T_1 completely regular space. We do not assume T_2 in the definition of compactness, paracompactness and countable compactness. We do not assume regularity in the definition of Lindelöfness. For a subset A of a space X, int$A$ and $\overline{A}$ denote the interior and the closure of A, respectively. An ordinal γ is the set of all ordinal α such that $\alpha < \gamma$. The first infinite ordinal is ω_0, the first uncountable ordinal is ω_1, and the successor cardinal of ω_1 is ω_2.

2. P-normality

Recall that a topological space $(X, \mathcal{T})$ is paracompact if any open cover has a locally finite open refinement. For a subspace A of X, A is paracompact if $(A, \mathcal{T}_A)$ is paracompact, i.e., any open (open in the subspace) cover of A has a locally finite open (open in the subspace) refinement. We do not assume T_2 in the definition of paracompactness.

2000 *Mathematics Subject Classification.* 54D15, 54C10.

Key words and phrases. normal; P-normal; L-normal; C-normal; paracompact; epinormal; submetrizable.

Submitted June 8, 2021. Published November 21, 2021.
Communicated by P. Das.

Definition 1. A topological space X is called P-*normal* if there exist a normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|_A} : A \longrightarrow f(A)$ is a homeomorphism for each paracompact subspace $A \subseteq X$.

The Independence of P-normality

Obviously, any normal space is P-normal, just by taking, in the definition, $Y = X$ and the identity function. It is also clear that any paracompact P-normal space has to be normal. Here is an example of a P-normal space which is not normal.

Example 2. We know that $(\mathbb{R}, \mathcal{CC})$ where $\mathcal{CC}$ is the countable complement topology is not normal because it is hyper-connected and T_1, [9, Example 20]. We will show that $(\mathbb{R}, \mathcal{CC})$ is P-normal.
Claim: $C \subseteq \mathbb{R}$ is paracompact if and only if C is countable.
Proof of Claim: If C is countable, then C as a subspace is discrete and hence paracompact. On the other hand, assume that C is paracompact and suppose C is uncountable to get a contradiction. Let $\{a_n : n \in \mathbb{N}\} \subset C$. So $\{a_n : n \in \mathbb{N}\}$ is a countably infinite subset of distinct elements of C. For each $n \in \mathbb{N}$ let $V_n = (C \setminus \{a_n : n \in \mathbb{N}\}) \bigcup \{a_n\}$. Then the collection $\{V_n : n \in \mathbb{N}\}$ is an open cover for C. Let $\{W_s : s \in S\}$ be an open refinement of $\{V_n : n \in \mathbb{N}\}$. Pick $s^* \in S$ then there exists $n_{s^*} \in \mathbb{N}$ such that $W_{s^*} \subseteq V_{n_s^*}$. We will show that the subfamily $\{W_{s_n} : n \in \mathbb{N}\}$ cannot be locally finite, which in turn shows that $\{W_s : s \in S\}$ cannot be locally finite. Now, without loss of generality, we can assume the sets in the subfamily are all non empty. Thus, by being open sets in $(C, \mathcal{CC})$ we have for every $n \in \mathbb{N}$, $C \setminus W_{s_n}$ must be countable. Therefore, $\bigcup (C \setminus W_{s_n})$ is countable. $C \setminus (\bigcap W_{s_n})$ is countable. So $\bigcap W_{s_n}$ is uncountable. Pick $y \in \bigcap W_{s_n}$, any open neighborhood of y must intersect all of the W_{s_n}. That is, any open neighborhood of y intersects infinitely many members of $\{W_s : s \in S\}$. Thus the claim is proved.

Consider $id_{\mathbb{R}} : (\mathbb{R}, \mathcal{CC}) \longrightarrow (\mathbb{R}, \mathcal{D})$, where $\mathcal{D}$ is the discrete topology on $\mathbb{R}$. Let A be any arbitrary paracompact subset of $\mathbb{R}$. Then by the above discussion A is countable. Which means A as a subspace is discrete and $id_{|_A} : (A, \mathcal{CC}_A) \longrightarrow (A, \mathcal{D})$ is a homeomorphism. ■

Now, we study the independence of P-normality with respect to paracompactness. Any paracompact non-normal space cannot be P-normal. To see this, let X be any paracompact non-normal space. Suppose X is P-normal. Pick a normal space Y and bijection $f : X \longrightarrow Y$ such that $f_{|_A} : A \longrightarrow f(A)$ is a homeomorphism for every paracompact $A \subseteq X$. But X is paracompact and $f : X \longrightarrow Y$ is a homeomorphism which makes X normal, and that is a contradiction. So $(\mathbb{R}, \mathcal{CF})$, where $\mathcal{CF}$ is the finite complement topology, [9, Example 19], cannot be P-normal. Observe that $(\mathbb{R}, \mathcal{CC})$ is an example of a P-normal space which is not paracompact.

Let us recall some definitions:

Definition 3. A topological space X is called C-*normal* if there exist a normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|_A} : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$, [6]. X is called L-*normal* if there exist a normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|_A} : A \longrightarrow f(A)$ is a homeomorphism for each Lindelöf subspace $A \subseteq X$, [7].

It is clear that P-normality implies C-normality. The Dieudonné plank is an example of a C-normal space, see [6], which is not P-normal.

Example 4. Let us recall the Dieudonné plank. The ground set is

$$X = ((\omega_1 + 1) \times (\omega_0 + 1)) \setminus \{\langle \omega_1, \omega_0 \rangle\},$$

see [9, Example 89]. Let $N = \{\langle \gamma, k \rangle : \gamma < \omega_1, k < \omega_0\}$, $A = \{\langle \omega_1, k \rangle : k < \omega_0\}$, and $B = \{\langle \gamma, \omega_0 \rangle : \gamma < \omega_1\}$. Observe that N, A, and B form a partition of X. The topology $\mathcal{T}$ on X is generated by the neighborhood system: For each $\langle \gamma, k \rangle \in N$, let $\mathcal{B}(\langle \gamma, k \rangle) = \{\{\langle \gamma, k \rangle\}\}$. For each $\langle \omega_1, k \rangle \in A$, let $\mathcal{B}(\langle \omega_1, k \rangle) = \{ V_\gamma(k) = (\gamma, \omega_1] \times \{k\} : \gamma < \omega_1\}$. For each $\langle \gamma, \omega_0 \rangle \in B$, let $\mathcal{B}(\langle \gamma, \omega_0 \rangle) = \{ V_k(\gamma) = \{\gamma\} \times (k, \omega_0] : k < \omega_0\}$.

Claim 1: $N \cup A$ and $N \cup B$ are paracompact.
Proof of Claim 1: First, we show that $N \cup A$ is paracompact. Let $\mathcal{W} = \{W_s \subseteq (N \cup A) : s \in S\}$ be any open (open in the subspace $N \cup A$) cover for $N \cup A$. For each $k \in \omega_0$ there exists $s_k \in S$ such that $\langle \omega_1, k \rangle \in W_{s_k}$. For each $k \in \omega_0$ there exists $\gamma_k < \omega_1$ such that $V_{\gamma_k}(k) \subseteq W_{s_k}$. The family $\{ V_{\gamma_k}(k), \{\langle \eta, m \rangle\} : k \in \omega_0, \langle \eta, m \rangle \in (N \cup A) \setminus (\cup_{k \in \omega_0} V_{\gamma_k}(k)) \}$ is a locally finite open refinement of $\mathcal{W}$. Similarly, $N \cup B$ is paracompact and Claim 1 is proved.

Indeed, by similar idea of the proof of Claim 1, we have that *any basic open set from the neighborhood system is paracompact . . .* $(\star)$.

Suppose that the Dieudonné plank X is P-normal. Pick a normal space Y and a bijection function $f : X \longrightarrow Y$ such that $f_{|_C} : C \longrightarrow f(C)$ is a homeomorphism for each paracompact subspace C of X. Observe that $f(N)$, $f(A)$, and $f(B)$ form a partition of Y because f is a bijection function.

Claim 2: Y is T_1.
Proof of Claim 2: Let y_1 and y_2 be any two distinct elements of Y. Consider the unique elements $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. The subspace $\{ x_1, x_2 \} \subset X$ is paracompact, being finite, and discrete, because X is T_1. Thus $f_{|_{\{x_1, x_2\}}} : \{x_1, x_2\} \longrightarrow f(\{x_1, x_2\}) = \{y_1, y_2\}$ is a homeomorphism. Since $\{x_1\}$ is open in $\{x_1, x_2\}$, then $\{y_1\}$ is open in $\{y_1, y_2\}$. Thus, there exists an open subset U_1 of Y such that $U_1 \cap \{y_1, y_2\} = \{y_1\}$ which gives that $y_1 \in U_1 \not\ni y_2$. Similarly, $\{y_2\}$ is open in $\{y_1, y_2\}$. Thus, there exists an open subset U_2 of Y such that $U_2 \cap \{y_1, y_2\} = \{y_2\}$ which gives that $y_1 \notin U_1 \ni y_2$. Therefore, Y is T_1 and Claim 2 is proved.

Claim 3: For each $k < \omega_0$ and $\gamma < \omega_1$ we have that $\{ f(\langle \gamma, k \rangle) \}$ is open in Y, i.e., $f(N)$ is consisting of isolated points in Y.
Proof of Claim 3: N consists of isolated points of X, thus N is a paracompact subspace of X. So, $f_{|_N} N \longrightarrow f(N)$ is a homeomorphism. Now, let $k \in \omega_0$ and $\gamma \in \omega_1$ be arbitrary. We have $\{ f(\langle \gamma, k \rangle) \}$ is open in $f(N) \subset Y$, thus there exists an open subset U of Y such that

$$U \cap f(N) = \{ f(\langle \gamma, k \rangle) \} \ . \ . \ . \ (\star\star).$$

Suppose that there exists $y \in U$ such that $y \neq f(\langle \gamma, k \rangle)$. Then $y \notin f(N)$, hence either $y \in f(A)$ or $y \in f(B)$. If $y \in f(A)$, then there exists a unique $m \in \omega_0$ such that $y = f(\langle\omega_1, m\rangle)$. By $(\star)$, $f_{|_{V_0(m)}} : V_0(m) \longrightarrow f(V_0(m))$ is a homeomorphism. Now, we have $\langle\omega_1, m\rangle \in V_0(m)$ and U is an open neighborhood of $y = f(\langle\omega_1, m\rangle)$ and $f_{|_{V_0(m)}}$ is continuous. Thus, there exists a basic open set $V_\beta(m)$, for some $\beta < \omega_1$, such that $\langle\omega_1, m\rangle \in V_\beta(m)$ and $f_{|_{V_0(m)}}(V_0(m)) \subseteq U$. This means that U contains elements of $f(N)$ distinct from $f(\langle\gamma, k\rangle)$ which contradicts $(\star\star)$. Similarly, if $y \in f(B)$. Claim 3 is proved.

Now, $N \cup A$ and $N \cup B$ are both open in X. Also, by Claim 1, $N \cup A$ is paracompact, hence $f_{|_{N\cup A}} : N \cup A \longrightarrow f(N \cup A) \subset Y$ is a homeomorphism, in particular $f_{|_{N\cup A}}$ is continuous. Similarly, $f_{|_{N\cup B}}$ is continuous. By gluing theorem [3, Theorem 9.4], we get $f : X \longrightarrow Y$ is continuous.

Claim 4: $f(N \cup A)$ and $f(N \cup B)$ are both open in Y.
Proof of Claim 4: First, we show that $f(N \cup B)$ is open. To get a contradiction, suppose that $f(N \cup B)$ is not open in Y. This means that there exists an element $y \in f(N \cup B)$ such that for any open neighborhood $W \subseteq Y$ of y we have that $W \not\subset f(N \cup B)$. By Claim 3, y should be in $f(B)$. So, there exists $\langle\alpha, \omega_0\rangle \in B$ such that each open neighborhood W of $f(\langle\alpha, \omega_0\rangle)$ contains elements of $f(A)$. By Claim 2, Y is T_1. Since in a T_1-space, any finite subset is closed and if W is open and Z is closed, then $W \setminus Z$ is open, thus we have
any open neighborhood W of $f(\langle\alpha, \omega_0\rangle)$ contains infinitely many elements of $f(A)$,
. . . $(\star\star\star)$
Take this element $\langle\alpha, \omega_0\rangle \in B$, and consider the subspace $\{\langle\alpha, \omega_0\rangle\} \cup N \cup A = H$, then H is a paracompact subspace of X. To see this, let $\mathcal{W} = \{W_s \subset H : s \in S\}$ be any open (open in H) cover for H. Fix $s' \in S$ such that $\langle\alpha, \omega_0\rangle \in W_{s'}$, then fix $m \in \omega_0$ such that $V_m(\alpha) \subseteq W_{s'}$. For each $k \in \omega_0$, fix $s_k \in S$ such that $\langle\omega_1, k\rangle \in W_{s_k}$. For each $k \in \omega_0$, fix $\gamma_k < \omega_1$ such that $\alpha < \gamma_k$ for each $k \in \omega_0$ and $\langle\omega_1, k\rangle \in V_{\gamma_k}(k) \subseteq W_{s_k}$. The family $\{V_m(\alpha), V_{\gamma_k}(k) : k \in \omega_0\} \bigcup \{\{\langle\beta, n\rangle\} : \langle\beta, n\rangle \in H \setminus ((V_m(\alpha)) \cup (\cup_{k\in\omega_0} V_{\gamma_k}(k)))\}$ is a locally finite open refinement of $\mathcal{W}$. Thus, $f_{|_H} : H \longrightarrow f(H) \subset Y$ is a homeomorphism. Since any basic open neighborhood of $\langle\alpha, \omega_0\rangle$ in H is the same as in X and $H \cong f(H) \subset Y$, then this contradict $(\star\star\star)$. Thus $f(N \cup B)$ is open in Y.
Observe that for any $m \in \omega_0$, similar technique as H is a paracompact subspace of X can be used to show that the subspace $K = \{\langle\omega_1, m\rangle\} \cup N \cup B$ is paracompact. So, similarly, $f(N \cup A)$ is open in Y and Claim 4 is proved.

Now, $f(N \cup A)$ and $f(N \cup B)$ are both open in Y. Also, $f_{|_{N\cup A}} : N \cup A \longrightarrow f(N \cup A) \subset Y$ is a homeomorphism, in particular $f^{-1}_{|_{f(N\cup A)}}$ is continuous. Similarly, $f^{-1}_{|_{f(N\cup B)}}$ is continuous. By gluing theorem [3, Theorem 9.4], we get $f^{-1} : Y \longrightarrow X$ is continuous. Hence f is a homeomorphism which is a contradiction as the Dieudonné plank is not normal, [9, Example 89]. Therefore, the Dieudonné plank is not P-normal. ■

So any P-normal space is C-normal but the converse is not always true as shown in the previous example. We will also give another example later that shows a C-normal space need not be P-normal. This example is $(\mathbb{R}, \mathcal{RS})$ where $\mathcal{RS}$ is the rational sequence topology [9, Example 65]. But before this, we show the independence of P-normality with other related properties. We start with this useful theorem. Recall that a space X is Fréchet if for any subset A of X and any element $a \in \overline{A}$ there exists a sequence $(a_n)_{n\in\mathbb{N}}$ of members of A, i.e., $a_n \in A$ for each $n \in \mathbb{N}$, such that $a_n \longrightarrow a$, [4].

Observe that a function $f : X \longrightarrow Y$ witnessing the P-normality of X need not be continuous. For example, $(\mathbb{R}, \mathcal{CC})$ is P-normal with witness $id_{\mathbb{R}} : (\mathbb{R}, \mathcal{CC}) \longrightarrow (\mathbb{R}, \mathcal{D})$ is not continuous. But the witness function will be continuous if X is Fréchet.

Theorem 5. *If X is Fréchet and P-normal, then any function witnesses the P-normality of X is continuous.*

Proof. Assume that X is P-normal and Fréchet. Let $f : X \longrightarrow Y$ be a witness of the P-normality of X. Let $A \subseteq X$ and pick $y \in f(\overline{A})$. Pick the unique $x \in X$ such that $f(x) = y$. Thus $x \in \overline{A}$. Since X is Fréchet, there exist a sequence $(a_n) \subseteq A$ such that $a_n \longrightarrow x$. The subspace $B = \{x, a_n : n \in \mathbb{N}\}$ of X is paracompact being compact , thus $f_{|_B} : B \longrightarrow f(B)$ is a homeomorphism. Now, let $W \subseteq Y$ be any open neighborhood of y, then $W \cap f(B)$ is open in the subspace $f(B)$ containing y. By continuity of the homeomorphism $f_{|_B}$, $f^{-1}(W \cap f(B)) = f^{-1}(W) \cap B$ is an open neighborhood of x in B. Then,$(f^{-1}(W) \cap B) \cap \{a_n : n \in \mathbb{N}\} \neq \emptyset$. So $(f^{-1}(W) \cap B) \cap A \neq \emptyset$. Therefore we have, $\emptyset \neq f(\underline{(f^{-1}(W) \cap B) \cap A}) \subseteq f(f^{-1}(W) \cap A) = W \cap f(A)$ then $W \cap f(A) \neq \emptyset$. Hence $y \in \overline{f(A)}$, thus $f(\overline{A}) \subseteq \overline{f(A)}$. Therefore, f is continuous. □

Example 6. For each $x \in \mathbb{P}$, fix a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{Q}$ such that $x_n \longrightarrow x$, where the convergency is taken in $(\mathbb{R}, \mathcal{U})$, and let $A_n(x)$ denote the n^{th}-tail of the sequence. For each $x \in \mathbb{P}$, let $\mathfrak{B}(x) = \{U_n(x) : n \in \mathbb{N}\}$, where $U_n(x) = A_n(x) \cup \{x\}$. For each $x \in \mathbb{Q}$, let $\mathfrak{B}(x) = \{\{x\}\}$. The collection $\{\mathfrak{B}(x)\}_{x\in\mathbb{R}}$ satisfies the conditions of a neighborhood system. The unique topology on $\mathbb{R}$ generated by $\{\mathfrak{B}(x)\}_{x\in\mathbb{R}}$ is called *the Rational Sequence Topology*, see [9, Example 65]. Let us denote it by $\mathcal{RS}$. It is clear that $(\mathbb{R}, \mathcal{RS})$ first countable. We will show that $(\mathbb{R}, \mathcal{RS})$ is not P-normal. Suppose it is P-normal. Then there exists a normal space Y and a bijection $f : (\mathbb{R}, \mathcal{RS}) \longrightarrow (Y, \mathcal{T})$ such that the restriction $f_{|_A} : A \longrightarrow f(A)$ is a homeomorphism for each paracompact subspace $A \subseteq \mathbb{R}$. Now, we know that $(\mathbb{R}, \mathcal{RS})$ is first countable and hence Fréchet. So by Theorem 5 f is continuous. We also know that $(\mathbb{R}, \mathcal{RS})$ is separable. The continuous image of a separable space is separable, so $(Y, \mathcal{T})$ is separable. Our aim is to show that $(Y, \mathcal{T})$ has a discrete, closed uncountable subspace which means by Jone's Lemma it cannot be normal. Both $\mathbb{Q}$ and $\mathbb{P}$ are discrete subspaces in $(\mathbb{R}, \mathcal{RS})$ which means they are both paracompact. So $f_{|_{\mathbb{Q}}} : \mathbb{Q} \longrightarrow f(\mathbb{Q})$ and $f_{|_{\mathbb{P}}} : \mathbb{P} \longrightarrow f(\mathbb{P})$ are both homeomorphisms. Our claim is that $f(\mathbb{P}) \subseteq Y$ is the uncountable, discrete closed subset of the separable space Y. It is of course uncountable since f is a bijection. To see that $f(\mathbb{P})$ is discrete we want to show that every singleton is open. let $y \in f(\mathbb{P})$ be arbitrary. Then there exists a unique $x \in \mathbb{P}$ such that $y = f(x)$. But $\mathbb{P}$ is discrete in $(\mathbb{R}, \mathcal{RS})$ so $\{x\} \in \mathcal{RS}_{\mathbb{P}}$. $f_{|\mathbb{P}}$ is a homeomorphism and hence open, so $f_{|\mathbb{P}}(\{x\}) = \{y\} \in \mathcal{T}_{f(\mathbb{P})}$. In other words $\{y\}$ is open in $f(\mathbb{P})$. Since y was arbitrary, then every singleton is open in $f(\mathbb{P})$. That means it is indeed discrete. It remains to show that $f(\mathbb{P})$ is closed in $(Y, \mathcal{T})$. We will do this by showing that$Y \setminus f(\mathbb{P})$ is open in Y. Now, $\mathbb{Q} \cup \mathbb{P} = \mathbb{R}$ and $\mathbb{Q} \cap \mathbb{P} = \emptyset$. Since f is a bijection then $f(\mathbb{Q}) \cup f(\mathbb{P}) = Y$ and $f(\mathbb{Q}) \cap f(\mathbb{P}) = \emptyset$. That is, $Y \setminus f(\mathbb{P}) = f(\mathbb{Q})$. That means our goal is to show that $f(\mathbb{Q})$ is open in $(Y,\mathcal{T})$. We will do this by contradiction. Suppose $f(\mathbb{Q})$ is not open in Y. Then there exists $q \in f(\mathbb{Q})$ such that for every open neighborhood $V \in\mathcal{T} of q, q \in V \not\subseteq f(\mathbb{Q})$, where $q = f(q^*)$ for unique $q^* \in \mathbb{Q}$. Which means $V \cap f(\mathbb{P}) \neq \emptyset \longrightarrow (1)$. Notice that in a similar fashion to what we did above, we can show that $f(\mathbb{Q})$ is also discrete in $(Y,\mathcal{T})$. Which means there exists $V_q \in\mathcal{T}$ such that $V_q \cap f(\mathbb{Q}) = \{q\} \longrightarrow (2)$. By (1): since for every $V \in\mathcal{T}$ of q, $V \cap f(\mathbb{P}) \neq \emptyset$ then $V_q \cap f(\mathbb{P}) \neq \emptyset$. This implies that $f^{-1}(V_q) \cap f(\mathbb{P}) \neq \emptyset \longrightarrow (3)$. By (2) $f^{-1}(V_q) \cap f(\mathbb{Q}) = f^{-1}(\{q\})$, which means $f^{-1}(V_q) \cap f(\mathbb{Q}) = f^{-1}(\{f(q^*)\})$ and therefore $f^{-1}(V_q) \cap f(\mathbb{Q}) = \{q^*\} \longrightarrow (4)$. So combining(3) and (4) we get that $f^{-1}(V_q)$ contains at least one irrational number p. Now, $V_q \in\mathcal{T}$ and f is continuous, since $(\mathbb{R}, \mathcal{RS})$ is Fréchet so $f^{-1}(V_q) \in \mathcal{RS}$. But if $f^{-1}(V_q)$ has an irrational number p, it must contain a basic open set of p, call it U, which is of the form $U = A_m \cup \{p\}$ where A_m is a tail of a a sequence of rational numbers converging to p. Then we must have $p \in U = \{p\} \cup A_m \subseteq f^{-1}(V_q)$ but $f^{-1}(V_q) \cap \mathbb{Q} = \{q^*\}$, that is, the intersection only consists of one rational number. So it is impossible for $f^{-1}(V_q)$ to be open, but this contradicts the continuity of f. Our assumption that $f(\mathbb{Q})$ is not open lead to a contradiction. Therefore, it must be open and $f(\mathbb{P}) = Y \setminus f(\mathbb{Q})$ is closed. Hence, $f(\mathbb{P}) \subseteq Y$ (where Y is separable) is uncountable, discrete and closed. By Jone's Lemma that means Y cannot be normal and we get a contradiction. So $(\mathbb{R}, \mathcal{RS})$ is not P-normal. This is another example of a Tychonoff space that is not P-normal just like the Dieudonné Plank. ■

While T_2 local compactness implies C-normality [6, 1.8], notice that $(\mathbb{R}, \mathcal{RS})$ is not P-normal but it is T_2, locally compact and zero-dimensional. This gives us that neither zero-dimensionality implies P-normality nor being T_2 locally compact implies P-normality.

Example 7. We modify the Dieudonné Plank [9] to define a new topological space. Let

$$X = ((\omega_2 + 1) \times (\omega_0 + 1)) \setminus \{\langle \omega_2, \omega_0 \rangle\}$$

where ω_0 is the first infinite ordinal number and ω_2 is the successor cardinal of ω_1, the first uncountable ordinal. Write $X = A \cup B \cup N$, where $A = \{\langle \omega_2, n \rangle : n < \omega_0\}$, $B = \{\langle \alpha, \omega_0 \rangle : \alpha < \omega_2\}$, and $N = \{\langle \alpha, n \rangle : \alpha < \omega_2 \text{ and } n < \omega_0\}$. The topology $\mathcal{T}$ on X is generated by the following neighborhood system: For each $\langle \alpha, n \rangle \in N$, let $\mathcal{B}(\langle \alpha, n \rangle) = \{\{\langle \alpha, n \rangle\}\}$. For each $\langle \omega_2, n \rangle \in A$, let $\mathcal{B}(\langle \omega_2, n \rangle) = \{V_\alpha(n) = (\alpha, \omega_2] \times \{n\} : \alpha < \omega_2\}$. For each $\langle \alpha, \omega_0 \rangle \in B$, let $\mathcal{B}(\langle \alpha, \omega_0 \rangle) = \{V_n(\alpha) = \{\alpha\} \times (n, \omega_0] : n < \omega_0\}$. Then X is Tychonoff non-normal space which is not locally compact. This is an L-normal space [7] which is not P-normal. It can be shown that it is not P-normal in a similar fashion to Example 4. While $(\mathbb{R}, \mathcal{CC})$ is a P-normal space as we've shown in Example 2 that is not L-normal. [7]. This shows the independence of P-normality with regards to L-normality.

We have shown so far that P-normality is independent from: normality, paracompactness, C-normality and L-normality.

3. More Results

Theorem 8. *Any regular P-normal space is L-normal.*

Proof. Let $(X, \mathcal{T})$ be a regular P-normal space. We want to show it is L-normal. Let $A \subseteq X$ be an arbitrary Lindelöf subset of X. Now, since X is P-normal then there exists a normal space Y and a bijection $f : X \longrightarrow Y$ such that for every paracompact $C \subseteq X$ the restriction $f_{|_C} : C \longrightarrow f(C)$ is a homeomorphism. X is regular and regularity is hereditary. That means A is regular as well. So A is a regular Lindelöf subspace. We know that any regular Lindelöf space is paracompact, hence, $f_{|_A} : A \longrightarrow f(A)$ is a homeomorphism by P-normality. Since A was an arbitrary Lindelöf subset, this gives us that $(X,\mathcal{T})$ is L-normal. □

We conclude that in a T_3 P-normal space, any Lindelöf subspace is paracompact. Now, we have that "*if X is L-normal and of countable tightness then any witness function is continuous*" [7, Theorem 1.2]. So, by Theorem 8, we get the following corollary:

Corollary 9. *If X is P-normal, regular and of countable tightness then any witness function is continuous.*

Theorem 10. *P-normality is a topological property.*

Proof. Let X be a P-normal space and let $X \cong Z$. Let Y be a normal space and let $f : X \longrightarrow Y$ be a bijective function such that the restriction $f_{|_C} : C \longrightarrow f(C)$ is a homeomorphism for each paracompact subspace $C \subseteq X$. Let $g : Z \longrightarrow X$ be a homeomorphism. Then Y and $f \circ g : Z \longrightarrow Y$ satisfy the requirements. □

P-normality is not hereditary. Consider the Dieuodonné Plank X. X is a Tychonoff space which means it has a compactification Y, where Y is both T_2 and compact. Then Y is T_4 and hence normal. So Y is P-normal. Viewing X as a subspace of Y, X is not P-normal. This shows there exists a space Y which is P-normal and a subspace of it X which is not P-normal. Therefore, P-normality is not hereditary in general. Now, consider $(\mathbb{R}, \mathcal{RS})$. It is not only Tychonoff but it is a non compact locally compact space [9], which means it has a 1-point compactification $Y = \mathbb{R} \cup \{p\}$ where $p \notin \mathbb{R}$. $\mathcal{T}$=$\{V \in \mathcal{P}(Y) : V \in \mathcal{RS}$ or $Y \setminus V$ is a closed compact subspace of $\mathbb{R}\}$. Since the universal set $\mathbb{R} \in \mathcal{RS}$ then $\mathbb{R} \in \mathcal{T}$. $\mathbb{R}$ is an open subspace of its compactification. which shows that P-normality is not hereditary with respect to open subsets either. Since the compactification Y is normal and hence P-normal but $(\mathbb{R}, \mathcal{RS})$ is not P-normal.

Theorem 11. *P-normality is an additive property.*

Proof. Let X_α be a P-normal space for each $\alpha \in \Lambda$. We show that their sum $\oplus_{\alpha\in\Lambda} X_\alpha$ is P-normal. For each $\alpha \in \Lambda$, pick a normal space Y_α and a bijective function $f_\alpha : X_\alpha \longrightarrow Y_\alpha$ such that $f_{\alpha_{|_{C_\alpha}}} : C_\alpha \longrightarrow f_\alpha(C_\alpha)$ is a homeomorphism for each paracompact subspace C_α of X_α. Since Y_α is normal for each $\alpha \in \Lambda$, then the sum $\oplus_{\alpha\in\Lambda} Y_\alpha$ is normal, [4, 2.2.7]. Consider the function sum [4, 2.2.E], $\oplus_{\alpha\in\Lambda} f_\alpha : \oplus_{\alpha\in\Lambda} X_\alpha \longrightarrow \oplus_{\alpha\in\Lambda} Y_\alpha$ defined by $\oplus_{\alpha\in\Lambda} f_\alpha(x) = f_\beta(x)$ if $x \in X_\beta, \beta \in \Lambda$. Now, a subspace $C \subseteq \oplus_{\alpha\in\Lambda} X_\alpha$ is paracompact if and only if and $C \cap X_\alpha$ is paracompact in X_α for each $\alpha \in \Lambda$. If $C \subseteq \oplus_{\alpha\in\Lambda} X_\alpha$ is paracompact, then $(\oplus_{\alpha\in\Lambda} f_\alpha)_{|_C}$ is a homeomorphism because $f_{\alpha_{|_{C\cap X_\alpha}}}$ is a homeomorphism for each $\alpha \in \Lambda$. □

In [7, 1.6], it was proved that *"if X is T_3, separable, L-normal, and of countable tightness, then X is normal (T_4)"*. So, the Niemtyzki plane [9, Example 82] is not L-normal, hence not P-normal, by Theorem 8. Similarly, any Mrówka space $\Psi(\mathcal{A})$, where $\mathcal{A} \subset [\omega_0]^{\omega_0}$ is mad [2]. These are examples of Tychonoff spaces which are not P-normal alongside the previously shown ones in this paper: the Dieudonné Plank and $\mathbb{R}$ with the rational sequence topology. We conclude that P-normality is not multiplicative, for example, the Sorgenfrey line is T_4 hence, P-normal. But its square is a Tychonoff, separable, and first countable space (so of countable tightness) which is not P-normal because it is not normal. The Niemtyzki plane and the Sorgenfrey line square are examples that show us that a submetrizable space need not be P-normal, recall that $(X, \mathcal{T})$ is called *submetrizable* if there is a coarser metrizable topology $\mathcal{T}'$ on X, [5]. The same two examples work to show us that an epinormal space need not be P-normal either. A topological space $(X, \mathcal{T})$ is called *epinormal* if there is a coarser topology $\mathcal{T}'$ on X such that $(X, \mathcal{T}')$ is T_4, [6].

Also, by Theorem 8 and [7, 1.6] we have the following theorem:

Theorem 12. *If X is T_3, separable, P-normal, and of countable tightness then X is normal (T_4).*

References

[1] P. S. Alexandroff and P. S. Urysohn, *Mémoire sur les espaces topologiques compacts*, Verh. Akad. Wetensch. Amsterdam, 14 (1929).
[2] E.K. van Douwen, *The integers and topology*, in: Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984, 111–167.
[3] J. Dugundji, *Topology*, Allyn Bacon Inc, 1972.
[4] R. Engelking, *General Topology*, (PWN, Warszawa, 1977).
[5] G. Gruenhage, *Generalized Metric Spaces,* in: Handbook of Set Theoretic Topology, North Holland, 1984, 428-434.
[6] L. Kalantan and S. AlZahrani, *C-normal Topological Property*, Filomat, 31(2017) 407-411.
[7] L. Kalantan and M. Saeed, *L-normal Topological Property*, Topology Proceedings, 50(2017) 141-149.
[8] S. Mrówka, *On Completely Regular Spaces*, Fundamenta Mathematicae, 41(1954) 105–106.
[9] L. Steen and J. A. Seebach, *Counterexamples in Topology*, Dover Publications, INC. 1995.

Lutfi Kalantan
King Abdulaziz University, Department of Mathematics, P.O.Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: lnkalantan@hotmail.com, lkalantan@kau.edu.sa

Mai Mansouri
King Abdulaziz University, Department of Mathematics, P.O.Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: mfmansouri1@kau.edu.sa