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NEW APPROXIMATION OF CAUCHY PROBLEM FOR ELLIPTIC EQUATIONS WITH INTEGRAL CONDITION

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Abstract. In this paper, we deal with a Cauchy problem for homogeneous elliptic equation with nonlocal integral condition. We examine and study both the well-posedness and the non-wellposedness of the mild solutions. In part 2, we show that the problem is ill-posed in the sense of Hadamard. We propose a nonlocal regularized problem for approximating the problem. Under some priori assumptions on the exact solution, we establish some stability estimates of Hölder type.

1. Introduction

Let H be a Hilbert space. Let $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ be a positive-definite, self-adjoint operator with compact inverse on H. Let us assume that A admits an orthonormal eigenbasis $\{\varphi_k\}_{k\geq 1}$ in H, associated with the eigenvalues of the operator $\mathcal{A}$ and

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \lambda_j \leq ...,$$

and $\lim_{j\to\infty} \lambda_j = \infty$. For some studies on partial differential equations considered in a Hilbert space and more details about the eigenpair of the operator A, the readers can refer to [2, 9, 11, 13] and references therein.

Let $Z > 0$ be a given real number. In this paper, we consider the fractional Sobolev equation

$$\begin{cases} \dfrac{\partial^2 w}{\partial z^2} = \mathcal{A}w, \ z \in (0, Z), \\ w_z(0) = 0, \end{cases} \tag{1.1}$$

with the following integral condition

$$\int_0^Z \theta(z)w(z)dz = f. \tag{1.2}$$

The Cauchy problem for the elliptic equation has been extensively investigated in many practical areas. For example, some problems relating to geophysics [10], plasma physics [3], bioelectric field problems [7] are equivalent to solving the Cauchy

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problem for the elliptic equation. Before going to any further on the problem (1.1)-(1.2), let us state the history of the problem (1.1) with Cauchy initial data

$$w(0) = f,\ f \in H. \tag{1.3}$$

The Cauchy problem for the elliptic equation (1.1)–(1.3) is known as the non-wellposed problem i.e., a small change in the given input data will lead to a huge deviation about the solution. It is therefore difficult for classical numerical methods to give an exact approximation. In order to overcome this difficulty, the regularization methods are required [5, 6, 14].

Some other regularized method can be found in [12, 8].

The appearance of integral term (1.3) has practical significance in natural phenomena. The system with such non-local condition play the role of describing several models for meteorology, the time-averaged data, resulting in obtaining more effective long-term weather forecast, and has been used when studying radionuclides propagation in Stokes fluid, diffusion. We can refer the reader to [15] to see more details of the integral condition. Despite of the importance of the aforementioned condition, as far as we know, there hasn't been any work considering the problem (1.1)–(1.3) until now.

The main contributions of the paper are presented as follows

- The first goal is to prove the well-posedness of the problem, namely to find the functional spaces for the solutions corresponding to the given assumptions of the Cauchy input data.
- The second goal is to prove the well-posedness of the our problem and give a regularized method.

This paper is organized as follows. In section 2, we deal with the well-posedness of our problem. Section 3 give the non-wellposed of the problem (1.1)–(1.3). We also give a regularized problem. Error estimate between the regularized solution and the exact solution is also derived.

2. Well-posedness of our problem

For positive number $r \geq 0$, we also define the Hilber scale space

$$D(A^s) = \left\{ w \in H : \sum_{j=1}^{\infty} \lambda_j^{2s} \langle w, \varphi_j \rangle^2 < +\infty \right\}, \tag{2.1}$$

with the following norm $\|u\|_{D(A^s)} = \left(\sum_{j=1}^{\infty} \lambda_j^{2s} |\langle u, \varphi_j \rangle|^2 \right)^{\frac{1}{2}}$.

Theorem 2.1. *Let $\theta \in L^\infty(0,T)$ such that $M_0 \leq \|\theta\|_{L^\infty(0,T)} \leq M_1$.*

- *Let $f \in H$ then we have the following estimate for any $0 \leq z < M$*
$$\|w(z)\|_H \leq \frac{2}{M_0}(Z-z)^{-1}\|f\|_H. \tag{2.2}$$
- *Let $f \in D(A^s)$, $0 < s \leq \frac{1}{2}$. Then $w \in L^\infty(0,Z;H)$ and*
$$\|w\|_{L^\infty(0,T;H)} \leq Z^{1-2s}\|f\|_{D(A^s)}. \tag{2.3}$$
- *Let $f \in D(A^s)$, $s > \frac{1}{2}$. Then $w \in L^p(0,Z;H)$ and $1 < p < \frac{1}{2s-1}$ and*
$$\|w\|_{L^p(0,Z;H)} \leq C(M_0,s,p)\|f\|_{D(A^s)}. \tag{2.4}$$

where $C(M_0, s, p)$ depends only on M_0, s, p.

Proof. Assume that the mild solution of Problem (1.1)-(1.2) is given by Fourier series $w(z) = \sum_{k=1}^{\infty} w_k(z)\varphi_k$, where $w_k(z) = \big\langle w(z), \varphi_k \big\rangle$. Due to the main equation, we obtain the following second differential equation

$$\frac{d^2 w_k(z)}{dz^2} + \lambda_k w_k(z) = 0.$$

Then we get the following equality

$$\big\langle w(z), \varphi_k \big\rangle = \frac{e^{\sqrt{\lambda_k} z} + e^{-\sqrt{\lambda_k} z}}{2} \big\langle w(0), \varphi_k \big\rangle. \tag{2.5}$$

The integral condition $\int_0^Z \theta(z) w(z) dz = f$ gives that

$$\int_0^Z \theta(z) \left(\sum_{k=1}^{\infty} \big\langle w(z), \varphi_k \big\rangle \varphi_k \right) dz = \sum_{k=1}^{\infty} \big\langle f, \varphi_k \big\rangle \varphi_k.$$

Hence

$$\big\langle w(0), \varphi_k \big\rangle \int_0^Z \theta(z) \left(\frac{e^{\sqrt{\lambda_k} z} + e^{-\sqrt{\lambda_k} z}}{2} \right) dz = \big\langle f, \varphi_k \big\rangle. \tag{2.6}$$

Due to the uniqueness property of Fourier expansion, we find that

$$\big\langle w(0), \varphi_k \big\rangle = \frac{\big\langle f, \varphi_k \big\rangle}{\int_0^Z \theta(z) \left(\frac{e^{\sqrt{\lambda_k} z} + e^{-\sqrt{\lambda_k} z}}{2} \right) dz}. \tag{2.7}$$

Combining (2.5) and (2.7), we arrive at

$$\big\langle w(z), \varphi_k \big\rangle = \frac{e^{\sqrt{\lambda_k} z} + e^{-\sqrt{\lambda_k} z}}{\int_0^Z \theta(z) \left(e^{\sqrt{\lambda_k} z} + e^{-\sqrt{\lambda_k} z} \right) dz} \big\langle f, \varphi_k \big\rangle, \tag{2.8}$$

which allows us to obtain the explicit fomula

$$\big\langle w(z), \varphi_k \big\rangle = \frac{e^{\sqrt{\lambda_k} z} + e^{-\sqrt{\lambda_k} z}}{\int_0^Z \theta(z) \left(e^{\sqrt{\lambda_k} z} + e^{-\sqrt{\lambda_k} z} \right) dz} \big\langle f, \varphi_k \big\rangle. \tag{2.9}$$

Using Fourier series, the formula of the mild solution of Problem (1.1)-(1.2) is given by

$$w(z) = \sum_{k=1}^{\infty} \frac{e^{\sqrt{\lambda_k} z} + e^{-\sqrt{\lambda_k} z}}{\int_0^Z \theta(z) \left(e^{\sqrt{\lambda_k} z} + e^{-\sqrt{\lambda_k} z} \right) dz} \big\langle f, \varphi_k \big\rangle \varphi_k. \tag{2.10}$$

Case 1. $f \in H$.
Since $\theta(z) \geq M_0$, we know that

$$\begin{aligned} \int_0^Z \theta(z) \left(e^{\sqrt{\lambda_k} z} + e^{-\sqrt{\lambda_k} z} \right) dz &\geq M_0 \int_0^Z \left(e^{\sqrt{\lambda_k} z} + e^{-\sqrt{\lambda_k} z} \right) dz \\ &= \frac{M_0}{\sqrt{\lambda_k}} \left(e^{\sqrt{\lambda_k} Z} - e^{-\sqrt{\lambda_k} Z} \right). \end{aligned} \tag{2.11}$$

If $0 \leq z < Z$, then

$$\frac{e^{\sqrt{\lambda_k} z} + e^{-\sqrt{\lambda_k} z}}{\int_0^Z \theta(z) \left(e^{\sqrt{\lambda_k} z} + e^{-\sqrt{\lambda_k} z} \right) dz} \leq \frac{\sqrt{\lambda_k}}{M_0} \frac{e^{\sqrt{\lambda_k} z} + e^{-\sqrt{\lambda_k} z}}{e^{\sqrt{\lambda_k} Z} + e^{-\sqrt{\lambda_k} Z}} \leq \frac{2\sqrt{\lambda_k}}{M_0} e^{\sqrt{\lambda_k}(z-Z)}. \tag{2.12}$$

Using the inequality $e^{-y} \le y^{-1}$, we find that

$$e^{\sqrt{\lambda_k}(z-Z)} \le \frac{1}{\sqrt{\lambda_k}}(Z-z)^{-1}. \tag{2.13}$$

Combining (2.12) and (2.13), we get that

$$\frac{e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}}{\int_0^Z \theta(z)\left(e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}\right)dz} \le \frac{2}{M_0}(Z-z)^{-1}. \tag{2.14}$$

This follows from (2.10) that

$$\|w(z)\|_H^2 = \sum_{k=1}^{\infty}\left(\frac{e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}}{\int_0^Z \theta(z)\left(e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}\right)dz}\right)^2 \left\langle f, \varphi_k\right\rangle^2 \le \frac{4}{M_0^2}(Z-z)^{-2}\sum_{k=1}^{\infty}\left\langle f, \varphi_k\right\rangle^2. \tag{2.15}$$

Since $f \in H$ then Parseval equality gives that $\sum\limits_{k=1}^{\infty}\left\langle f, \varphi_k\right\rangle^2 = \|f\|_H^2$. Therefore, we get immediately that

$$\|w(z)\|_H \le \frac{2}{M_0}(Z-z)^{-1}\|f\|_H. \tag{2.16}$$

Case 2. $f \in D(A^s)$. Using the inequality $e^{-y} \le C_\nu y^{-\nu}$ for any $0 < \nu < 1$, we find that

$$e^{\sqrt{\lambda_k}(z-Z)} \le C_\nu \lambda_k^{-\nu/2}(Z-z)^{-\nu}. \tag{2.17}$$

Combining (2.12) and (2.17), we get that

$$\frac{e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}}{\int_0^Z \theta(z)\left(e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}\right)dz} \le \frac{2}{M_0}\lambda_k^{\frac{1-\nu}{2}}(Z-z)^{-\nu}. \tag{2.18}$$

This inequality leads to

$$\begin{aligned}\|w(z)\|_H^2 &= \sum_{k=1}^{\infty}\left(\frac{e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}}{\int_0^Z \theta(z)\left(e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}\right)dz}\right)^2 \left\langle f, \varphi_k\right\rangle^2 \\ &\le \frac{4}{M_0^2}(Z-z)^{-2\nu}\sum_{k=1}^{\infty}\lambda_k^{1-\nu}\left\langle f, \varphi_k\right\rangle^2. \end{aligned}\tag{2.19}$$

Choose $1-\nu = 2s$ and note that

$$\sum_{k=1}^{\infty}\lambda_k^{1-\nu}\left\langle f, \varphi_k\right\rangle^2 = \sum_{k=1}^{\infty}\lambda_k^{2s}\left\langle f, \varphi_k\right\rangle^2 = \|f\|_{D(A^s)}^2,$$

we can see that

$$\|w(z)\|_H \le \frac{2}{M_0}(Z-z)^{1-2s}\|f\|_{D(A^s)}. \tag{2.20}$$

Now, we continue to divide into two cases.
If $0 < s \le \frac{1}{2}$ then we know that $w \in L^\infty(0, Z; H)$ and

$$\|w\|_{L^\infty(0,T;H)} \le Z^{1-2s}\|f\|_{D(A^s)}. \tag{2.21}$$

If $s > 1/2$ then $0 < 2s - 1 < 1$, we arrive at the following bound

$$\|w\|_{L^p(0,Z;H)} = \left(\int_0^Z \|w(z)\|_H^p dz\right)^{1/p} \leq \frac{2}{M_0}\left(\int_0^Z (Z-z)^{(1-2s)p} dz\right)^{1/p} \|f\|_{D(A^s)}. \tag{2.22}$$

Since the condition $1 \leq p \leq \frac{1}{1-2s}$ and $0 < s < 1/2$, we know that the proper integral $\int_0^Z (Z-z)^{(1-2s)p} dz$ is convergent. It follows from (2.22) that $w \in L^p(0, Z; H)$ and the following regularity holds

$$\|w\|_{L^p(0,Z;H)} \leq C(M_0, s, p)\|f\|_{D(A^s)}. \tag{2.23}$$

Here $C(M_0, s, p)$ depends only on M_0, s, p. □

Theorem 2.2. *Suppose that $M_2 z^{-\delta} \leq |\theta(z)| \leq M_3 z^{-\beta}$ for any $0 < \delta, \beta < 1$. Then we have the following lower and upper bound*

$$\frac{1-\beta}{M_3 Z^{1-\beta}}\|f\|_H \leq \|w(Z)\|_H \leq \frac{1-\delta}{M_2 Z^{1-\delta}}\|f\|_H. \tag{2.24}$$

Proof. Due to (2.10), we find that

$$w(Z) = \sum_{k=1}^{\infty} \frac{e^{\sqrt{\lambda_k}Z} + e^{-\sqrt{\lambda_k}Z}}{\int_0^Z \theta(z)\left(e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}\right) dz}\langle f, \varphi_k\rangle \varphi_k. \tag{2.25}$$

Consider the following function

$$\psi(z) = e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z},\ 0 \leq z \leq Z.$$

By taking the derivative of ψ, we find that

$$\psi'(z) = e^{\sqrt{\lambda_k}z} - e^{-\sqrt{\lambda_k}z} \geq 0,\ 0 \leq z \leq Z. \tag{2.26}$$

Hence, the function ψ is increasing function on $[0, Z]$. Therefore, we get that

$$e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z} \leq \psi(Z) = e^{\sqrt{\lambda_k}Z} + e^{-\sqrt{\lambda_k}Z}. \tag{2.27}$$

Combining (2.25) and (2.27), we find that

$$\begin{aligned}\|w(Z)\|_H^2 &= \sum_{k=1}^{\infty}\left(\frac{e^{\sqrt{\lambda_k}Z} + e^{-\sqrt{\lambda_k}Z}}{\int_0^Z \theta(z)\left(e^{\sqrt{\lambda_k}Z} + e^{-\sqrt{\lambda_k}Z}\right) dz}\right)^2 \langle f, \varphi_k\rangle^2 \\ &= \sum_{k=1}^{\infty}\left(\int_0^Z \theta(z)dz\right)^{-2}\langle f, \varphi_k\rangle^2 = \left(\int_0^Z \theta(z)dz\right)^{-2}\left(\sum_{k=1}^{\infty}\langle f, \varphi_k\rangle^2\right) \\ &= \left(\int_0^Z \theta(z)dz\right)^{-2}\|f\|_H^2. \end{aligned} \tag{2.28}$$

Since the condition $\theta(z) \geq M_2 z^{-\delta}$, we know that

$$\int_0^Z \theta(z)dz \geq M_2\int_0^Z z^{-\delta}dz = \frac{M_2 Z^{1-\delta}}{1-\delta}. \tag{2.29}$$

From two above observations, we find that

$$\|w(Z)\|_H^2 \leq \left(\frac{1-\delta}{M_2 Z^{1-\delta}}\right)^2 \|f\|_H^2. \tag{2.30}$$

Combining (2.25) and (2.27) and noting that $\theta(z) \le M_3 z^{-\beta}$, we find that

$$\begin{aligned}\|w(Z)\|_H^2 &= \sum_{k=1}^{\infty} \left(\frac{e^{\sqrt{\lambda_k}Z} + e^{-\sqrt{\lambda_k}Z}}{\int_0^Z \theta(z) \left(e^{\sqrt{\lambda_k}Z} + e^{-\sqrt{\lambda_k}Z}\right) dz} \right)^2 \langle f, \varphi_k \rangle^2 \\ &= \left(\int_0^Z \theta(z) dz \right)^{-2} \left(\sum_{k=1}^{\infty} \langle f, \varphi_k \rangle^2 \right) \ge M_3^2 \left(\int_0^Z z^{-\beta} dz \right)^{-2} \|f\|_H^2,\end{aligned} \tag{2.31}$$

which allows us to get that

$$\|w(Z)\|_H \ge M_3 \left(\int_0^Z z^{-\beta} dz \right)^{-1} \|f\|_H = \frac{1-\beta}{M_3 Z^{1-\beta}} \|f\|_H. \tag{2.32}$$

□

3. Regularization and error estimate by nonlocal problem

3.1. **Ill-posedness.** To illustrate the problem easily, we need to give an example that if two inputs move towards each other, the two outputs can differ greatly. Indeed, if we take

$$\overline{f}_p = f + \frac{1}{\lambda_p^{1/4}}. \tag{3.1}$$

then we get immediately that

$$\|\overline{f}_p - f\|_H = \frac{1}{\lambda_p^{1/4}} \to +\infty, \ p \to +\infty, \tag{3.2}$$

Let $\overline{w}_p$ be the solution of

$$\begin{cases} \dfrac{\partial^2 w}{\partial z^2} = \mathcal{A} w, z \in (0, Z), \\ w_z(0) = 0, \ \in (0, Z), \\ \displaystyle\int_0^Z \theta(z) w(z) dz = \overline{f}_p \end{cases}. \tag{3.3}$$

It is easy to see that

$$\begin{aligned}\overline{w}_p(z) &= \sum_{k=1}^{\infty} \frac{e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}}{\int_0^Z \theta(z) \left(e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}\right) dz} \langle \overline{f}_p, \varphi_k \rangle \varphi_k \\ &= \sum_{k=1}^{\infty} \frac{e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}}{\int_0^Z \theta(z) \left(e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}\right) dz} \langle f, \varphi_k \rangle \varphi_k + \frac{1}{\lambda_p^{1/4}} \frac{e^{\sqrt{\lambda_p}z} + e^{-\sqrt{\lambda_p}z}}{\int_0^Z \theta(z) \left(e^{\sqrt{\lambda_p}z} + e^{-\sqrt{\lambda_p}z}\right) dz} \varphi_p \\ &= w(z) + \frac{1}{\lambda_p^{1/4}} \frac{e^{\sqrt{\lambda_p}z} + e^{-\sqrt{\lambda_p}z}}{\int_0^Z \theta(z) \left(e^{\sqrt{\lambda_p}z} + e^{-\sqrt{\lambda_p}z}\right) dz} \varphi_p.\end{aligned} \tag{3.4}$$

Therefore, it is easy to see that

$$\|\overline{w}_p(z) - w(z)\|_H = \frac{1}{\lambda_p^{1/4}} \frac{e^{\sqrt{\lambda_p}z} + e^{-\sqrt{\lambda_p}z}}{\int_0^Z \theta(z) \left(e^{\sqrt{\lambda_p}z} + e^{-\sqrt{\lambda_p}z}\right) dz}. \tag{3.5}$$

Since $\theta(z) \geq M_1$, we find that

$$\int_0^Z \theta(z)\left(e^{\sqrt{\lambda_p}z} + e^{-\sqrt{\lambda_p}z}\right)dz \leq M_1 \int_0^Z \left(e^{\sqrt{\lambda_p}z} + e^{-\sqrt{\lambda_p}z}\right)dz$$
$$\leq \frac{M_1}{\sqrt{\lambda_p}}\left(e^{\sqrt{\lambda_k}Z} - e^{-\sqrt{\lambda_k}Z}\right). \tag{3.6}$$

Combining (3.5) and (3.6), we obtain that

$$\sup_{0\leq z\leq Z} \|\overline{w}_p(z) - w(z)\|_H \geq \frac{\lambda_p^{1/4}}{M_1} \sup_{0\leq z\leq Z} \frac{e^{\sqrt{\lambda_p}z} + e^{-\sqrt{\lambda_p}z}}{e^{\sqrt{\lambda_p}Z} - e^{-\sqrt{\lambda_p}Z}} = \frac{\lambda_p^{1/4}}{M_1}\frac{e^{\sqrt{\lambda_p}Z} + e^{-\sqrt{\lambda_p}Z}}{e^{\sqrt{\lambda_p}Z} - e^{-\sqrt{\lambda_p}Z}}, \tag{3.7}$$

where we note that

$$e^{\sqrt{\lambda_p}Z} + e^{-\sqrt{\lambda_p}Z} \geq e^{\sqrt{\lambda_p}z} + e^{-\sqrt{\lambda_p}z},\ 0 \leq z \leq Z.$$

It is easy to observe that

$$\frac{e^{\sqrt{\lambda_p}Z} + e^{-\sqrt{\lambda_p}Z}}{e^{\sqrt{\lambda_p}Z} - e^{-\sqrt{\lambda_p}Z}} = 1 + \frac{2e^{-\sqrt{\lambda_p}Z}}{e^{\sqrt{\lambda_p}Z} - e^{-\sqrt{\lambda_p}Z}} \geq 1. \tag{3.8}$$

It follows from (3.7) that

$$\|\overline{w}_p - w\|_{L^\infty(0,Z;H)} \geq \frac{\lambda_p^{1/4}}{M_1}, \tag{3.9}$$

which allows us to get that

$$\lim_{p\to+\infty} \|\overline{w}_p - w\|_{L^\infty(0,Z;H)} \geq \lim_{p\to+\infty} \frac{\lambda_p^{1/4}}{M_1} = +\infty. \tag{3.10}$$

By looking closely at two observations (3.2) and (3.10), we conclude that Problem (1.1)–(1.3) is ill-posed in the sense of Hadamard.

3.2. **Regularization and error estimate.** In this section, we present a regularized problem

$$\begin{cases} \dfrac{\partial^2 w^\epsilon}{\partial z^2} = \mathcal{A}w^\epsilon, z \in (0, Z), \\ w_z^\epsilon(0) = 0,\ \in (0, Z), \\ \displaystyle\int_0^Z \theta(z)w^\epsilon(z)dz + \beta(\epsilon)w^\epsilon(Z) = f^\epsilon. \end{cases} \tag{3.11}$$

where $\beta(\epsilon)$ is a regularization parameter and such that

$$\lim_{\epsilon\to 0} \beta(\epsilon) = +\infty. \tag{3.12}$$

Theorem 3.1. *Let $f^\epsilon \in H$ such that*

$$\|f^\epsilon - f\|_H \leq \epsilon. \tag{3.13}$$

Then problem (3.11) *has a mild solution $w^\epsilon \in L^\infty(0, Z; H)$ and*

$$\|w^\epsilon(z)\|_H \leq \frac{1}{\beta(\epsilon)}\|f^\epsilon\|_H. \tag{3.14}$$

Let us choose $\beta = \epsilon^m$ for any $0 < m < 1$ and assume that

$$\|w\|_{L^\infty(0,Z;D(\mathcal{A}^{1/2}))} \leq \mathcal{M}.$$

Then, we have the following estimate

$$\|w^\epsilon(z) - w(z)\|_H \le \epsilon^{1-m} + \sqrt{\frac{1}{2M_0}\left(1 + \frac{2}{e^{\sqrt{\lambda_1}Z} - e^{-\sqrt{\lambda_1}Z}}\right)}\epsilon^{m/2}\mathcal{M}, \tag{3.15}$$

where M_0 is given in Theorem 2.1.

Proof. Assume that $w^\epsilon(z) = \sum_{k=1}^{\infty} \langle w^\epsilon(z), \varphi_k\rangle\varphi_k$, the following equality

$$\langle w^\epsilon(z), \varphi_k\rangle = \frac{e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}}{2}\langle w^\epsilon(0), \varphi_k\rangle. \tag{3.16}$$

The integral condition $\int_0^Z \theta(z)w^\epsilon(z)dz + \epsilon w^\epsilon(0) = f^\epsilon$ gives that

$$\int_0^Z \theta(z)\left(\sum_{k=1}^{\infty}\langle w^\epsilon(z), \varphi_k\rangle\varphi_k\right)dz + \beta(\epsilon)\sum_{k=1}^{\infty}\frac{e^{\sqrt{\lambda_k}Z} + e^{-\sqrt{\lambda_k}Z}}{2}\langle w^\epsilon(0), \varphi_k\rangle = \sum_{k=1}^{\infty}\langle f^\epsilon, \varphi_k\rangle\varphi_k.$$

From two latter equality as above, we find that

$$\langle w^\epsilon(0), \varphi_k\rangle \int_0^Z \theta(z)\left(\frac{e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}}{2}\right)dz + \beta(\epsilon)\frac{e^{\sqrt{\lambda_k}Z} + e^{-\sqrt{\lambda_k}Z}}{2}\langle w^\epsilon(0), \varphi_k\rangle = \langle f^\epsilon, \varphi_k\rangle. \tag{3.17}$$

Due to the uniqueness property of Fourier expansion, we find that

$$\langle w^\epsilon(0), \varphi_k\rangle = \frac{\langle f^\epsilon, \varphi_k\rangle}{\int_0^Z \theta(z)\left(\frac{e^{\sqrt{\lambda_k}z}+e^{-\sqrt{\lambda_k}z}}{2}\right)dz + \beta(\epsilon)\left(\frac{e^{\sqrt{\lambda_k}Z}+e^{-\sqrt{\lambda_k}Z}}{2}\right)}. \tag{3.18}$$

Combining (3.16) and (3.18), we get that

$$\langle w^\epsilon(z), \varphi_k\rangle = \frac{\left(e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}\right)\langle f^\epsilon, \varphi_k\rangle}{\int_0^Z \theta(z)\left(e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}\right)dz + \beta(\epsilon)\left(e^{\sqrt{\lambda_k}Z} + e^{-\sqrt{\lambda_k}Z}\right)}, \tag{3.19}$$

which allows us to get that

$$w^\epsilon(z) = \sum_{k=1}^{\infty}\frac{\left(e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}\right)\langle f^\epsilon, \varphi_k\rangle\varphi_k}{\int_0^Z \theta(z)\left(e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}\right)dz + \beta(\epsilon)\left(e^{\sqrt{\lambda_k}Z} + e^{-\sqrt{\lambda_k}Z}\right)}. \tag{3.20}$$

Using Parseval's equality, we find that

$$\begin{aligned}\|w^\epsilon(z)\|_H^2 &= \sum_{k=1}^{\infty}\left(\frac{\left(e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}\right)\langle f^\epsilon, \varphi_k\rangle}{\int_0^Z \theta(z)\left(e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}\right)dz + \beta(\epsilon)\left(e^{\sqrt{\lambda_k}Z} + e^{-\sqrt{\lambda_k}Z}\right)}\right)^2 \\ &\le \sum_{k=1}^{\infty}\left(\frac{\left(e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}\right)\langle f^\epsilon, \varphi_k\rangle}{\beta(\epsilon)\left(e^{\sqrt{\lambda_k}Z} + e^{-\sqrt{\lambda_k}Z}\right)}\right)^2 \le \frac{1}{(\beta(\epsilon))^2}\sum_{k=1}^{\infty}\langle f^\epsilon, \varphi_k\rangle^2 = \frac{1}{(\beta(\epsilon))^2}\|f^\epsilon\|_H^2.\end{aligned} \tag{3.21}$$

Hence, we arrive at the following estimate

$$\|w^\epsilon(z)\|_H \le \frac{1}{\beta(\epsilon)}\|f^\epsilon\|_H. \tag{3.22}$$

In order to estimate $\|w^\epsilon(z) - w(z)\|_H$, we set the following function

$$v^\epsilon(z) = \sum_{k=1}^{\infty} \frac{\left(e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}\right)\langle f, \varphi_k\rangle\varphi_k}{\int_0^Z \theta(z)\left(e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}\right)dz + \beta(\epsilon)\left(e^{\sqrt{\lambda_k}Z} + e^{-\sqrt{\lambda_k}Z}\right)}. \tag{3.23}$$

We treat two following steps.
Step 1. Estimate the term $\|w^\epsilon(z) - v^\epsilon(z)\|$.
Indeed, using Parseval's equality, we get

$$\begin{aligned}\|w^\epsilon(z) - v^\epsilon(z)\|_H^2 &= \sum_{k=1}^{\infty}\left(\frac{\left(e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}\right)\langle f^\epsilon - f, \varphi_k\rangle}{\int_0^Z \theta(z)\left(e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}\right)dz + \beta(\epsilon)\left(e^{\sqrt{\lambda_k}Z} + e^{-\sqrt{\lambda_k}Z}\right)}\right)^2\\ &\le \sum_{k=1}^{\infty}\left(\frac{\left(e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}\right)\langle f^\epsilon - f, \varphi_k\rangle}{\beta(\epsilon)\left(e^{\sqrt{\lambda_k}Z} + e^{-\sqrt{\lambda_k}Z}\right)}\right)^2\\ &\le \frac{1}{(\beta(\epsilon))^2}\sum_{k=1}^{\infty}\langle f^\epsilon - f, \varphi_k\rangle^2 = \frac{1}{(\beta(\epsilon))^2}\|f^\epsilon - f\|_H^2 \le \frac{\epsilon^2}{(\beta(\epsilon))^2}.\end{aligned} \tag{3.24}$$

Therefore, we obtain

$$\|w^\epsilon(z) - v^\epsilon(z)\|_H \le \frac{\epsilon}{\beta(\epsilon)}. \tag{3.25}$$

Step 2. Estimate the term $\|v^\epsilon(z) - w(z)\|$.
By applying Parseval's equality, we arrive at

$$\begin{aligned}&\|v^\epsilon(z) - w(z)\|_H^2\\ &= \sum_{k=1}^{\infty}\Bigg(\frac{\left(e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}\right)\langle f, \varphi_k\rangle}{\int_0^Z \theta(z)\left(e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}\right)dz + \beta(\epsilon)\left(e^{\sqrt{\lambda_k}Z} + e^{-\sqrt{\lambda_k}Z}\right)}\\ &\qquad - \frac{e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}}{\int_0^Z \theta(z)\left(e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}\right)dz}\langle f, \varphi_k\rangle\Bigg)^2\\ &= |\beta(\epsilon)|^2\sum_{k=1}^{\infty}\frac{\left(e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}\right)^2\left(e^{\sqrt{\lambda_k}Z} + e^{-\sqrt{\lambda_k}Z}\right)^2}{\left(\int_0^Z \theta(z)\left(e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}\right)dz + \beta(\epsilon)\left(e^{\sqrt{\lambda_k}Z} + e^{-\sqrt{\lambda_k}Z}\right)\right)^2}\\ &\qquad \left(\int_0^Z \theta(z)\left(e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}\right)dz\right)^{-2}\langle f, \varphi_k\rangle^2\\ &= |\beta(\epsilon)|^2\sum_{k=1}^{\infty}\frac{\left(e^{\sqrt{\lambda_k}Z} + e^{-\sqrt{\lambda_k}Z}\right)^2\langle w(z), \varphi_k\rangle^2}{\left(\int_0^Z \theta(z)\left(e^{\sqrt{\lambda_k}z} + e^{-\sqrt{\lambda_k}z}\right)dz + \beta(\epsilon)\left(e^{\sqrt{\lambda_k}Z} + e^{-\sqrt{\lambda_k}Z}\right)\right)^2}.\end{aligned} \tag{3.26}$$

Applying the inequality $(c+d)^2 \geq 2cd$, we find that

$$\left(\int_0^Z \theta(z)\left(e^{\sqrt{\lambda_k}z}+e^{-\sqrt{\lambda_k}z}\right)dz+\beta(\epsilon)\left(e^{\sqrt{\lambda_k}Z}+e^{-\sqrt{\lambda_k}Z}\right)\right)^2$$
$$\geq 2\left(\int_0^Z \theta(z)\left(e^{\sqrt{\lambda_k}z}+e^{-\sqrt{\lambda_k}z}\right)dz\right)\beta(\epsilon)\left(e^{\sqrt{\lambda_k}Z}+e^{-\sqrt{\lambda_k}Z}\right). \tag{3.27}$$

It follows from (3.26) that

$$\|v^\epsilon(z)-w(z)\|_H^2 \leq \frac{\beta(\epsilon)}{2}\sum_{k=1}^{\infty}\frac{\left(e^{\sqrt{\lambda_k}Z}+e^{-\sqrt{\lambda_k}Z}\right)}{\int_0^Z \theta(z)\left(e^{\sqrt{\lambda_k}z}+e^{-\sqrt{\lambda_k}z}\right)dz}\big\langle w(z),\varphi_k\big\rangle^2. \tag{3.28}$$

Noting that

$$\int_0^Z \theta(z)\left(e^{\sqrt{\lambda_k}z}+e^{-\sqrt{\lambda_k}z}\right)dz \geq M_0\int_0^Z\left(e^{\sqrt{\lambda_k}z}+e^{-\sqrt{\lambda_k}z}\right)dz=\frac{M_0}{\sqrt{\lambda_k}}\left(e^{\sqrt{\lambda_k}Z}-e^{-\sqrt{\lambda_k}Z}\right). \tag{3.29}$$

Combining (3.28) and (3.29), we find that

$$\|v^\epsilon(z)-w(z)\|_H^2 \leq \frac{\beta(\epsilon)}{2M_0}\sum_{k=1}^{\infty}\frac{\sqrt{\lambda_k}\left(e^{\sqrt{\lambda_k}Z}+e^{-\sqrt{\lambda_k}Z}\right)}{e^{\sqrt{\lambda_k}Z}-e^{-\sqrt{\lambda_k}Z}}\big\langle w(z),\varphi_k\big\rangle^2. \tag{3.30}$$

It is easy to see that

$$\frac{\left(e^{\sqrt{\lambda_k}Z}+e^{-\sqrt{\lambda_k}Z}\right)}{e^{\sqrt{\lambda_k}Z}-e^{-\sqrt{\lambda_k}Z}}=1+\frac{2e^{\sqrt{-\lambda_k}Z}}{e^{\sqrt{\lambda_k}Z}-e^{-\sqrt{\lambda_k}Z}}\leq 1+\frac{2}{e^{\sqrt{\lambda_1}Z}-e^{-\sqrt{\lambda_1}Z}}. \tag{3.31}$$

Therefore, we get that

$$\|v^\epsilon(z)-w(z)\|_H \leq \sqrt{\frac{\beta(\epsilon)}{2M_0}\left(1+\frac{2}{e^{\sqrt{\lambda_1}Z}-e^{-\sqrt{\lambda_1}Z}}\right)\sum_{k=1}^{\infty}\sqrt{\lambda_k}\big\langle w(z),\varphi_k\big\rangle^2}$$
$$=\sqrt{\frac{\beta(\epsilon)}{2M_0}\left(1+\frac{2}{e^{\sqrt{\lambda_1}Z}-e^{-\sqrt{\lambda_1}Z}}\right)}\|w(z)\|_{D(\mathcal{A}^{1/2})}. \tag{3.32}$$

This inequality together with (3.25) leads to the following estimate

$$\|w^\epsilon(z)-w(z)\|_H \leq \|v^\epsilon(z)-w(z)\|_H+\|w^\epsilon(z)-v^\epsilon(z)\|_H$$
$$\leq \frac{\epsilon}{\beta(\epsilon)}+\sqrt{\frac{\beta(\epsilon)}{2M_0}\left(1+\frac{2}{e^{\sqrt{\lambda_1}Z}-e^{-\sqrt{\lambda_1}Z}}\right)}\|w(z)\|_{D(\mathcal{A}^{1/2})}. \tag{3.33}$$

Let us choose $\beta=\epsilon^m$ for any $0<m<1$, we find that

$$\|w^\epsilon(z)-w(z)\|_H \leq \epsilon^{1-m}+\sqrt{\frac{1}{2M_0}\left(1+\frac{2}{e^{\sqrt{\lambda_1}Z}-e^{-\sqrt{\lambda_1}Z}}\right)}\epsilon^{m/2}\|w(z)\|_{D(\mathcal{A}^{1/2})}. \tag{3.34}$$

□

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