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# ITERATED ABSTRACT RIGHT SIDE FRACTIONAL LANDAU INEQUALITIES

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Abstract. We derive uniform and $L_p$ right Caputo-Bochner abstract iterated fractional Landau inequalities over $\mathbb{R}_-$. These estimate the size of second and third iterated right abstract fractional derivates of a Banach space valued function over $\mathbb{R}_-$. We give an application when the basic fractional order is $\frac{1}{2}$.

## 1. Introduction

Let $p \in [1, \infty]$, $I = \mathbb{R}_+$ or $I = \mathbb{R}$ and $f : I \to \mathbb{R}$ is twice differentiable with $f, f'' \in L_p(I)$, then $f' \in L_p(I)$. Moreover, there exists a constant $C_p(I) > 0$ independent of $f$, such that

$$\|f'\|_{p,I} \leq C_p(I) \|f\|_{p,I}^{\frac{1}{2}} \|f''\|_{p,I}^{\frac{1}{2}}, \tag{1}$$

where $\|\cdot\|_{p,I}$ is the $p$-norm on the interval $I$, see [1], [5].

The research on these inequalities started by E. Landau [10] in 1913. For the case of $p = \infty$ he proved that

$$C_\infty(\mathbb{R}_+) = 2 \text{ and } C_\infty(\mathbb{R}) = \sqrt{2}, \tag{2}$$

are the best constants in (1).

In 1932, G. H. Hardy and J. E. Littlewood [7] proved (1) for $p = 2$, with the best constants

$$C_2(\mathbb{R}_+) = \sqrt{2} \text{ and } C_2(\mathbb{R}) = 1. \tag{3}$$

In 1935, G. H. Hardy, E. Landau and J. E. Littlewood [8] showed that the best constants $C_p(\mathbb{R}_+)$ in (1) satisfies the estimate

$$C_p(\mathbb{R}_+) \leq 2, \text{ for } p \in [1, \infty), \tag{4}$$

which yields $C_p(\mathbb{R}) \leq 2$ for $p \in [1, \infty)$.

In fact, in [6] and [9] was shown that $C_p(\mathbb{R}) \leq \sqrt{2}$.

We need the following concept from abstract fractional calculus.

Our integral next is of Bochner type [11].

We need

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**Definition 1.1.** *([4], p. 150) Let $[a,b] \subset \mathbb{R}$, $X$ be a Banach space, $\alpha > 0$; $m = \lceil \alpha \rceil \in \mathbb{N}$, ($\lceil \cdot \rceil$ is the ceiling of the number), $f : [a,b] \to X$. We assume that $f^{(m)} \in L_1([a,b], X)$. We call the Caputo-Bochner right fractional derivative of order $\alpha$:*

$$\left(D_{b-}^{\alpha} f\right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z)\, dz, \quad \forall\, x \in [a,b], \tag{5}$$

*where $\Gamma$ is the gamma function.*

*If $\alpha = m \in \mathbb{N}$, we observe that $D_{b-}^{\alpha} f = (-1)^m f^{(m)}$ the ordinary $X$-valued derivative (defined similar to numerical one, see [12], p. 83), and also set $D_{b-}^{0} f := f$.*

By ([4], p. 34), $\left(D_{b-}^{\alpha} f\right)(x)$ exists almost everywhere in $x \in [a,b]$ and $D_{b-}^{\alpha} f \in L_1([a,b], X)$.

If $\left\| f^{(m)} \right\|_{L_\infty([a,b],X)} < \infty$, and $\alpha \notin \mathbb{N}$, then by ([4], p. 37), $D_{b-}^{\alpha} f \in C([a,b], X)$, hence $\left\| D_{b-}^{\alpha} f \right\| \in C([a,b])$.

We mention the important:

**Corollary 1.2.** *([4], p. 157) Let $f \in C^m([a,b], X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $x, x_0 \in [a,b]$. Then $D_{x_0-}^{a} f(x)$ is jointly continuous functions in $(x, x_0)$ from $[a,b]^2$ into $X$, $X$ is a Banach space.*

By convention we suppose that

$$D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0,$$

for all $x, x_0 \in [a,b]$.

The author has already done an extensive body of work on fractional Landau inequalities, see [3], and on abstract fractional Landau inequalities, see [4]. However there the proving methods came out of applications of fractional Ostrowski inequalities ([2], [4]). Usually there the domains where $[A, +\infty)$ or $(-\infty, B]$, with $A, B \in \mathbb{R}$ and in one mixed case the domain was all of $\mathbb{R}$.

In this work with less assumptions we establish uniform and $L_p$ type right Caputo-Bochner abstract iterated fractional Landau inequalities over $\mathbb{R}_-$. The method of proving is based on right Caputo-Bochner iterated fractional Taylor's formula with integral remainder, see [4], pp. 129.

We give also an application for $\alpha = \frac{1}{2}$. Regardless to say that we are also inspired by [3], [4].

## 2. Main Results

We consider ($\alpha > 0$) the composition $D_{b-}^{n\alpha} := D_{b-}^{\alpha} D_{b-}^{\alpha} ... D_{b-}^{\alpha}$ ($n$-times), $n \in \mathbb{N}$.

We mention the following right modified $X$-valued Taylor's formula, $(X, \|\cdot\|)$ is a Banach space.

**Theorem 2.1.** *([4], p. 129) Let $0 < \alpha \leq 1$, $n \in \mathbb{N}$, $f \in C^1([a,b], X)$. For $k = 1, ..., n$, we assume that $D_{b-}^{k\alpha} f \in C^1([a,b], X)$ and $D_{b-}^{(n+1)\alpha} f \in C([a,b], X)$. Then*

$$f(x) = \sum_{i=0}^{n} \frac{(b-x)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{b-}^{i\alpha} f\right)(b) + \tag{6}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_x^b (t-x)^{(n+1)\alpha-1} \left(D_{b-}^{(n+1)\alpha} f\right)(t)\, dt,$$

$\forall\ x \in [a,b]$.

*When* $0<\alpha<1$ *and* $f \in C^1([a,b],X)$, *by* [4], *p. 135, we get that* $\left(D_{b-}^{\alpha}f\right)(b) = 0$.

We present the following abstact right fractional Landau inequalities over $\mathbb{R}_-$.

**Theorem 2.2.** *Let* $0<\alpha<1$, $f \in C^1(\mathbb{R}_-,X)$ *with* $\||\|f\||\|_{\infty,\mathbb{R}_-}, \||\|f'\||\|_{\infty,\mathbb{R}_-} < \infty$. *For* $k=1,2,3$, *we assume that* $D_{b-}^{k\alpha}f \in C^1((-\infty,b],X)$ *and* $D_{b-}^{4\alpha}f \in C((-\infty,b],X)$, $\forall\ b \in \mathbb{R}_-$. *We further assume that*

$$K := \left\|\left\|D_{b-}^{4\alpha}f(t)\right\|\right\|_{\infty,\mathbb{R}_-^2} < \infty, \tag{7}$$

*where* $(b,t) \in \mathbb{R}_-^2$.

*Then*

$$\sup_{b\in\mathbb{R}_-}\left\|\left(D_{b-}^{2\alpha}f\right)(b)\right\| \le \frac{\Gamma(2\alpha+1)}{2^{2\alpha-1}(2^{\alpha}-1)}\sqrt{\frac{2^{3\alpha+1}(2^{3\alpha}+1)(2^{\alpha}+1)}{\Gamma(4\alpha+1)}\||\|f\||\|_{\infty,\mathbb{R}_-}K}, \tag{8}$$

*and*

$$\sup_{b\in\mathbb{R}_-}\left\|\left(D_{b-}^{3\alpha}f\right)(b)\right\| \le \frac{4\sqrt[4]{2}\,\Gamma(3\alpha+1)\left(\Gamma(4\alpha+1)\right)^{-\frac{3}{4}}\left(2^{2\alpha}+1\right)}{\left(\sqrt[4]{3}\right)^{3}\left(\sqrt{2}\right)^{\alpha}(2^{\alpha}-1)}\||\|f\||\|_{\infty,\mathbb{R}_-}^{\frac{1}{4}}K^{\frac{3}{4}}. \tag{9}$$

*That is* $\sup_{b\in\mathbb{R}_-}\left\|\left(D_{b-}^{2\alpha}f\right)(b)\right\|$, $\sup_{b\in\mathbb{R}_-}\left\|\left(D_{b-}^{3\alpha}f\right)(b)\right\| < \infty$.

*Proof.* We notice again here that $\left(D_{b-}^{\alpha}f\right)(b) = 0$, $\forall\ b \in \mathbb{R}_-$.

We make use of Theorem 2.1 for $0<\alpha<1$ and $n=3$, applied for any $b \in \mathbb{R}_-$ and $a=-\infty$.

Momentarily we fix $b \in \mathbb{R}_-$. Let $x_2 < x_1 < b$, then

$$f(x_1) - f(b) = \frac{(b-x_1)^{2\alpha}}{\Gamma(2\alpha+1)}\left(D_{b-}^{2\alpha}f\right)(b) + \frac{(b-x_1)^{3\alpha}}{\Gamma(3\alpha+1)}\left(D_{b-}^{3\alpha}f\right)(b) + \frac{1}{\Gamma(4\alpha)}\int_{x_1}^{b}(t-x_1)^{4\alpha-1}\left(D_{b-}^{4\alpha}f\right)(t)\,dt, \tag{10}$$

and

$$f(x_2) - f(b) = \frac{(b-x_2)^{2\alpha}}{\Gamma(2\alpha+1)}\left(D_{b-}^{2\alpha}f\right)(b) + \frac{(b-x_2)^{3\alpha}}{\Gamma(3\alpha+1)}\left(D_{b-}^{3\alpha}f\right)(b) + \frac{1}{\Gamma(4\alpha)}\int_{x_2}^{b}(t-x_2)^{4\alpha-1}\left(D_{b-}^{4\alpha}f\right)(t)\,dt. \tag{11}$$

That is

$$\frac{(b-x_1)^{2\alpha}}{\Gamma(2\alpha+1)}\left(D_{b-}^{2\alpha}f\right)(b) + \frac{(b-x_1)^{3\alpha}}{\Gamma(3\alpha+1)}\left(D_{b-}^{3\alpha}f\right)(b) = f(x_1) - f(b) - \frac{1}{\Gamma(4\alpha)}\int_{x_1}^{b}(t-x_1)^{4\alpha-1}\left(D_{b-}^{4\alpha}f\right)(t)\,dt =: A, \tag{12}$$

and

$$\frac{(b-x_2)^{2\alpha}}{\Gamma(2\alpha+1)}\left(D_{b-}^{2\alpha}f\right)(b) + \frac{(b-x_2)^{3\alpha}}{\Gamma(3\alpha+1)}\left(D_{b-}^{3\alpha}f\right)(b) = f(x_2) - f(b) - \frac{1}{\Gamma(4\alpha)}\int_{x_2}^{b}(t-x_2)^{4\alpha-1}\left(D_{b-}^{4\alpha}f\right)(t)\,dt =: B. \tag{13}$$

We are solving the above system of two equations with two unknowns $\left(D_{b-}^{2\alpha}f\right)(b)$, $\left(D_{b-}^{3\alpha}f\right)(b)$.

The main determinant of system is

$$D := \begin{vmatrix} \frac{(b-x_1)^{2\alpha}}{\Gamma(2\alpha+1)} & \frac{(b-x_1)^{3\alpha}}{\Gamma(3\alpha+1)} \\ \frac{(b-x_2)^{2\alpha}}{\Gamma(2\alpha+1)} & \frac{(b-x_2)^{3\alpha}}{\Gamma(3\alpha+1)} \end{vmatrix} =$$

$$\frac{1}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)}\left[(b-x_1)^{2\alpha}(b-x_2)^{3\alpha}-(b-x_1)^{3\alpha}(b-x_2)^{2\alpha}\right] =$$

$$\frac{(b-x_1)^{2\alpha}(b-x_2)^{2\alpha}}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)}\left[(b-x_2)^{\alpha}-(b-x_1)^{\alpha}\right] > 0.$$

I.e.

$$D = \frac{(b-x_1)^{2\alpha}(b-x_2)^{2\alpha}}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)}\left[(b-x_2)^{\alpha}-(b-x_1)^{\alpha}\right] > 0. \tag{14}$$

We obtain the unique solution

$$\left(D_{b-}^{2\alpha}f\right)(b) = \frac{\begin{vmatrix} A & \frac{(b-x_1)^{3\alpha}}{\Gamma(3\alpha+1)} \\ B & \frac{(b-x_2)^{3\alpha}}{\Gamma(3\alpha+1)} \end{vmatrix}}{D},$$

$$\left(D_{b-}^{3\alpha}f\right)(b) = \frac{\begin{vmatrix} \frac{(b-x_1)^{2\alpha}}{\Gamma(2\alpha+1)} & A \\ \frac{(b-x_2)^{2\alpha}}{\Gamma(2\alpha+1)} & B \end{vmatrix}}{D}. \tag{15}$$

Therefore we have

$$\left(D_{b-}^{2\alpha}f\right)(b) = \frac{\frac{(b-x_2)^{3\alpha}}{\Gamma(3\alpha+1)}A - \frac{(b-x_1)^{3\alpha}}{\Gamma(3\alpha+1)}B}{D},$$

and

$$\left(D_{b-}^{3\alpha}f\right)(b) = \frac{\frac{(b-x_1)^{2\alpha}}{\Gamma(2\alpha+1)}B - \frac{(b-x_2)^{2\alpha}}{\Gamma(2\alpha+1)}A}{D}. \tag{16}$$

We have the following

$$\|A\| = \left\| f(x_1) - f(b) - \frac{1}{\Gamma(4\alpha)}\int_{x_1}^{b}(t-x_1)^{4\alpha-1}\left(D_{b-}^{4\alpha}f\right)(t)\,dt \right\| \leq$$

$$2\,\|\|f\|\|_{\infty,\mathbb{R}_-} + \frac{\left\|\left\|D_{b-}^{4\alpha}f(t)\right\|\right\|_{\infty,\mathbb{R}_-^2}}{\Gamma(4\alpha+1)}(b-x_1)^{4\alpha}, \tag{17}$$

under the assumption $\|\|f\|\|_{\infty,\mathbb{R}_-} < \infty$.

That is

$$\|A\| \leq 2\,\|\|f\|\|_{\infty,\mathbb{R}_-} + \frac{K}{\Gamma(4\alpha+1)}(b-x_1)^{4\alpha}, \tag{18}$$

and similarly,

$$\|B\| \leq 2\,\|\|f\|\|_{\infty,\mathbb{R}_-} + \frac{K}{\Gamma(4\alpha+1)}(b-x_2)^{4\alpha}, \tag{19}$$

where by assumption

$$K := \left\|\left\|D_{b-}^{4\alpha}f(t)\right\|\right\|_{\infty,\mathbb{R}_-^2} < \infty, \tag{20}$$

with $(b,t) \in \mathbb{R}_-^2$.

Consequently we have

$$\left\|\left(D_{b-}^{2\alpha}f\right)(b)\right\| \leq \frac{1}{\Gamma(3\alpha+1)D}\left[(b-x_2)^{3\alpha}\left(2\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}+\frac{K}{\Gamma(4\alpha+1)}(b-x_1)^{4\alpha}\right)\right.$$

$$\left.+(b-x_1)^{3\alpha}\left(2\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}+\frac{K}{\Gamma(4\alpha+1)}(b-x_2)^{4\alpha}\right)\right], \tag{21}$$

and

$$\left\|\left(D_{b-}^{3\alpha}f\right)(b)\right\| \leq \frac{1}{\Gamma(2\alpha+1)D}\left[(b-x_1)^{2\alpha}\left(2\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}+\frac{K}{\Gamma(4\alpha+1)}(b-x_2)^{4\alpha}\right)\right.$$

$$\left.+(b-x_2)^{2\alpha}\left(2\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}+\frac{K}{\Gamma(4\alpha+1)}(b-x_1)^{4\alpha}\right)\right]. \tag{22}$$

Set now $x_1 := b-h$, $x_2 := b-2h$, where $h>0$, so that $b-x_1=h$, $b-x_2=2h$.

Hence we get

$$D=\frac{2^{2\alpha}h^{5\alpha}(2^{\alpha}-1)}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)}>0. \tag{23}$$

Therefore we derive (from (16))

$$\left\|\left(D_{b-}^{2\alpha}f\right)(b)\right\| \leq \frac{\Gamma(2\alpha+1)}{2^{2\alpha}h^{5\alpha}(2^{\alpha}-1)}\left[2^{3\alpha}h^{3\alpha}\left(2\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}+\frac{K}{\Gamma(4\alpha+1)}h^{4\alpha}\right)\right.$$

$$\left.+h^{3\alpha}\left(2\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}+\frac{K}{\Gamma(4\alpha+1)}2^{4\alpha}h^{4\alpha}\right)\right]= \tag{24}$$

$$\frac{\Gamma(2\alpha+1)}{2^{2\alpha}h^{5\alpha}(2^{\alpha}-1)}\left[2\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}\left(2^{3\alpha}+1\right)h^{3\alpha}+\frac{K}{\Gamma(4\alpha+1)}\left(2^{3\alpha}+2^{4\alpha}\right)h^{7\alpha}\right]$$

$$=\left(\frac{\Gamma(2\alpha+1)}{2^{2\alpha}(2^{\alpha}-1)}\right)\left[\frac{2\left(2^{3\alpha}+1\right)\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}}{h^{2\alpha}}+\frac{2^{3\alpha}(2^{\alpha}+1)}{\Gamma(4\alpha+1)}Kh^{2\alpha}\right]. \tag{25}$$

That is

$$\left\|\left(D_{b-}^{2\alpha}f\right)(b)\right\| \leq \left(\frac{\Gamma(2\alpha+1)}{2^{2\alpha}(2^{\alpha}-1)}\right)$$

$$\left[\frac{2\left(2^{3\alpha}+1\right)\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}}{h^{2\alpha}}+\frac{2^{3\alpha}(2^{\alpha}+1)}{\Gamma(4\alpha+1)}Kh^{2\alpha}\right], \tag{26}$$

$\forall\, b\in\mathbb{R}_-$, $\forall\, h>0$.

I.e. it holds

$$\sup_{b\in\mathbb{R}_-}\left\|\left(D_{b-}^{2\alpha}f\right)(b)\right\| \leq \left(\frac{\Gamma(2\alpha+1)}{2^{2\alpha}(2^{\alpha}-1)}\right)$$

$$\left[\frac{2\left(2^{3\alpha}+1\right)\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}}{h^{2\alpha}}+\frac{2^{3\alpha}(2^{\alpha}+1)}{\Gamma(4\alpha+1)}Kh^{2\alpha}\right]<\infty, \tag{27}$$

$\forall\, h>0$, $0<\alpha<1$.

By (16) we derive

$$\left\|\left(D_{b-}^{3\alpha}f\right)(b)\right\| \leq \frac{\Gamma(3\alpha+1)}{2^{2\alpha}h^{5\alpha}(2^{\alpha}-1)}\left[h^{2\alpha}\left(2\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}+\frac{K}{\Gamma(4\alpha+1)}2^{4\alpha}h^{4\alpha}\right)\right.$$

$$\left.+2^{2\alpha}h^{2\alpha}\left(2\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}+\frac{K}{\Gamma(4\alpha+1)}h^{4\alpha}\right)\right]=$$

$$\frac{\Gamma(3\alpha+1)}{2^{2\alpha}h^{5\alpha}(2^{\alpha}-1)}\left[2\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}\left(2^{2\alpha}+1\right)h^{2\alpha}+\frac{K}{\Gamma(4\alpha+1)}\left(2^{4\alpha}+2^{2\alpha}\right)h^{6\alpha}\right] \tag{28}$$

$$= \left(\frac{\Gamma(3\alpha+1)}{2^{2\alpha}(2^{\alpha}-1)}\right)\left[\frac{2\left(2^{2\alpha}+1\right)\|\|f\|\|_{\infty,\mathbb{R}_-}}{h^{3\alpha}} + \frac{2^{2\alpha}\left(2^{2\alpha}+1\right)}{\Gamma(4\alpha+1)}Kh^{\alpha}\right]$$

$$= \frac{\Gamma(3\alpha+1)\left(2^{2\alpha}+1\right)}{2^{2\alpha}(2^{\alpha}-1)}\left[\frac{2\|\|f\|\|_{\infty,\mathbb{R}_-}}{h^{3\alpha}} + \frac{2^{2\alpha}K}{\Gamma(4\alpha+1)}h^{\alpha}\right].$$

That is

$$\left\|\left(D_{b-}^{3\alpha}f\right)(b)\right\| \leq \frac{\Gamma(3\alpha+1)\left(2^{2\alpha}+1\right)}{2^{2\alpha}(2^{\alpha}-1)}$$

$$\left[\frac{2\|\|f\|\|_{\infty,\mathbb{R}_-}}{h^{3\alpha}} + \frac{2^{2\alpha}K}{\Gamma(4\alpha+1)}h^{\alpha}\right], \tag{29}$$

$\forall\, b \in \mathbb{R}_-$, $\forall\, h > 0$.

I.e. it holds

$$\sup_{b\in\mathbb{R}_-}\left\|\left(D_{b-}^{3\alpha}f\right)(b)\right\| \leq \frac{\Gamma(3\alpha+1)\left(2^{2\alpha}+1\right)}{2^{2\alpha}(2^{\alpha}-1)}$$

$$\left[\frac{2\|\|f\|\|_{\infty,\mathbb{R}_-}}{h^{3\alpha}} + \frac{2^{2\alpha}K}{\Gamma(4\alpha+1)}h^{\alpha}\right] < \infty, \tag{30}$$

$\forall\, h > 0$, $0 < \alpha < 1$.

Call

$$\mu := 2\left(2^{3\alpha}+1\right)\|\|f\|\|_{\infty,\mathbb{R}_-}, \tag{31}$$

$$\theta = \frac{2^{3\alpha}(2^{\alpha}+1)K}{\Gamma(4\alpha+1)}, \tag{32}$$

both are greater than zero.

Set also $\rho := 2\alpha$; $0 < \rho < 2$. We consider the function

$$y(h) := \mu h^{-\rho} + \theta h^{\rho}, \quad \forall\, h > 0. \tag{33}$$

We have

$$y'(h) = -\rho\mu h^{-\rho-1} + \rho\theta h^{\rho-1} = 0, \tag{34}$$

then

$$\theta h^{\rho-1} = \mu h^{-\rho-1}$$

and

$$\theta h^{2\rho} = \mu,$$

with a unique solution

$$h_0 := h_{crit.no.} = \left(\frac{\mu}{\theta}\right)^{\frac{1}{2\rho}}. \tag{35}$$

We have that

$$y''(h) = \rho(\rho+1)\mu h^{-\rho-2} + \rho(\rho-1)\theta h^{\rho-2}. \tag{36}$$

We see that

$$y''(h_0) = y''\left(\left(\frac{\mu}{\theta}\right)^{\frac{1}{2\rho}}\right) = \rho(\rho+1)\mu\left(\frac{\mu}{\theta}\right)^{\frac{-\rho-2}{2\rho}} + \rho(\rho-1)\theta\left(\frac{\mu}{\theta}\right)^{\frac{\rho-2}{2\rho}} =$$

$$\rho\left(\frac{\theta}{\mu}\right)^{\frac{1}{\rho}}\left(2\rho\sqrt{\mu\theta}\right) = 2\rho^2\sqrt{\mu\theta}\left(\frac{\theta}{\mu}\right)^{\frac{1}{\rho}} > 0.$$

Therefore $y$ has a global minimum at $h_0 = \left(\frac{\mu}{\theta}\right)^{\frac{1}{2\rho}}$, which is

$$y(h_0) = \mu\left(\frac{\mu}{\theta}\right)^{-\frac{1}{2}} + \theta\left(\frac{\mu}{\theta}\right)^{\frac{1}{2}} = \mu\left(\frac{\theta}{\mu}\right)^{\frac{1}{2}} + \sqrt{\theta\mu} = 2\sqrt{\theta\mu}.$$

We have proved that (see (27))

$$\sup_{b\in\mathbb{R}_-}\left\|\left(D_{b-}^{2\alpha}f\right)(b)\right\| \leq \frac{\Gamma(2\alpha+1)}{2^{2\alpha-1}(2^{\alpha}-1)}$$
$$\sqrt{\frac{2^{3\alpha+1}(2^{3\alpha}+1)(2^{\alpha}+1)}{\Gamma(4\alpha+1)}\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}K}. \tag{37}$$

Call

$$\xi := 2\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}, \qquad \psi := \frac{2^{2\alpha}K}{\Gamma(4\alpha+1)}, \tag{38}$$

both are greater than zero.

We consider the function

$$\gamma(h) := \xi h^{-3\alpha} + \psi h^{\alpha}, \quad \forall\, h>0. \tag{39}$$

We have

$$\gamma'(h) = -3\alpha\xi h^{-3\alpha-1} + \alpha\psi h^{\alpha-1} = 0,$$

then

$$\psi h^{\alpha-1} = 3\xi h^{-3\alpha-1}$$

and

$$\psi h^{4\alpha} = 3\xi,$$

with unique solution

$$h_0 := h_{crit.no.} = \left(\frac{3\xi}{\psi}\right)^{\frac{1}{4\alpha}}. \tag{40}$$

We have that

$$\gamma''(h) = 3\alpha(3\alpha+1)\xi h^{-3\alpha-2} + \alpha(\alpha-1)\psi h^{\alpha-2}. \tag{41}$$

We see

$$\gamma''(h_0) = 3\alpha(3\alpha+1)\xi\left(\frac{3\xi}{\psi}\right)^{\frac{-3\alpha-2}{4\alpha}} + \alpha(\alpha-1)\psi\left(\frac{3\xi}{\psi}\right)^{\frac{\alpha-2}{4\alpha}} =$$
$$\alpha\left(\frac{3\xi}{\psi}\right)^{\frac{\alpha-2}{4\alpha}}(4\alpha\psi) = 4\alpha^2\psi\left(\frac{3\xi}{\psi}\right)^{\frac{\alpha-2}{4\alpha}} > 0. \tag{42}$$

Therefore $\gamma$ has a global minimum at $h_0 = \left(\frac{3\xi}{\psi}\right)^{\frac{1}{4\alpha}}$, which is

$$\gamma(h_0) = \xi\left(\frac{3\xi}{\psi}\right)^{-\frac{3}{4}} + \psi\left(\frac{3\xi}{\psi}\right)^{\frac{1}{4}} =$$
$$\left(\frac{3\xi}{\psi}\right)^{\frac{1}{4}}\left(\xi\frac{\psi}{3\xi} + \psi\right) = \frac{4}{3}\psi\left(\frac{3\xi}{\psi}\right)^{\frac{1}{4}}. \tag{43}$$

Consequently,

$$\gamma(h_0) = \frac{4}{\left(\sqrt[4]{3}\right)^3}\psi^{\frac{3}{4}}\xi^{\frac{1}{4}}. \tag{44}$$

We have proved that (see (30))

$$\sup_{b\in\mathbb{R}_-}\left\|\left(D_{b-}^{3\alpha}f\right)(b)\right\| \leq \frac{4\Gamma(3\alpha+1)\left(2^{2\alpha}+1\right)}{\left(\sqrt[4]{3}\right)^3 2^{2\alpha}\left(2^{\alpha}-1\right)}$$

$$\left(2\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}\right)^{\frac{1}{4}}\left(\frac{2^{2\alpha}K}{\Gamma(4\alpha+1)}\right)^{\frac{3}{4}} =$$

$$\frac{4\sqrt[4]{2}\Gamma(3\alpha+1)\Gamma(4\alpha+1)^{-\frac{3}{4}}\left(2^{2\alpha}+1\right)}{\left(\sqrt[4]{3}\right)^3 2^{\frac{\alpha}{2}}\left(2^{\alpha}-1\right)}\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}^{\frac{1}{4}} K^{\frac{3}{4}}. \tag{45}$$

The theorem is proved. ☐

We continue with abstract $L_p$ right fractional Landau inequalities over $\mathbb{R}_-$.

**Theorem 2.3.** *Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $0<\alpha<1$. Let $f\in C^1(\mathbb{R}_-,X)$ with $\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}$, $\left\|\|f'\|\right\|_{\infty,\mathbb{R}_-}<\infty$. For $k=1,2,3$, we assume that $D_{b-}^{k\alpha}f\in C^1((-\infty,b],X)$ and $D_{b-}^{4\alpha}f\in C((-\infty,b],X)$, $\forall\, b\in\mathbb{R}_-$. We further assume that*

$$\left(\sup_{b\in\mathbb{R}_-}\left\|\left\|D_{b-}^{4\alpha}f\right\|\right\|_{p,\mathbb{R}_-}\right)<\infty. \tag{46}$$

*Then*

*1) under $\frac{1}{2p}<\alpha<1$, we get*

$$\sup_{b\in\mathbb{R}_-}\left\|\left(D_{b-}^{2\alpha}f\right)(b)\right\| \leq \left[\left(\frac{2^{\alpha}\Gamma(2\alpha)\left(4\alpha-\frac{1}{p}\right)}{2^{\alpha}-1}\right)\left(\frac{4\alpha\left(1+2^{-3\alpha}\right)}{2\alpha-\frac{1}{p}}\right)^{\left(\frac{2\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)}\right.$$

$$\left.\left(\frac{1+2^{\alpha-\frac{1}{p}}}{\Gamma(4\alpha)\left(q(4\alpha-1)+1\right)^{\frac{1}{q}}}\right)^{\left(\frac{2\alpha}{4\alpha-\frac{1}{p}}\right)}\right]\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}^{\left(\frac{2\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)}$$

$$\left(\sup_{b\in\mathbb{R}_-}\left\|\left\|D_{b-}^{4\alpha}f\right\|\right\|_{p,\mathbb{R}_-}\right)^{\left(\frac{2\alpha}{4\alpha-\frac{1}{p}}\right)}<\infty. \tag{47}$$

*2) under $\frac{1}{p}<\alpha<1$, we get*

$$\sup_{b\in\mathbb{R}_-}\left\|\left(D_{b-}^{3\alpha}f\right)(b)\right\| \leq \left[\left(\frac{\Gamma(3\alpha)\left(4\alpha-\frac{1}{p}\right)}{2^{\alpha}-1}\right)\left(\frac{6\alpha\left(1+2^{-2\alpha}\right)}{\alpha-\frac{1}{p}}\right)^{\left(\frac{\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)}\right.$$

$$\left.\left(\frac{1+2^{2\alpha-\frac{1}{p}}}{\Gamma(4\alpha)\left(q(4\alpha-1)+1\right)^{\frac{1}{q}}}\right)^{\left(\frac{3\alpha}{4\alpha-\frac{1}{p}}\right)}\right]\left\|\|f\|\right\|_{\infty,\mathbb{R}_-}^{\left(\frac{\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)}$$

$$\left(\sup_{b\in\mathbb{R}_-}\left\|\left\|D_{b-}^{4\alpha}f\right\|\right\|_{p,\mathbb{R}_-}\right)^{\left(\frac{3\alpha}{4\alpha-\frac{1}{p}}\right)}<\infty. \tag{48}$$

*That is* $\sup_{b\in\mathbb{R}_-}\left\|\left(D_{b-}^{2\alpha}f\right)(b)\right\|$, $\sup_{b\in\mathbb{R}_-}\left\|\left(D_{b-}^{3\alpha}f\right)(b)\right\|<\infty$.

*Proof.* As in the proof of Theorem 2.2 we have that

$$\|A\| \overset{(12)}{=} \left\| f(x_1) - f(b) - \frac{1}{\Gamma(4\alpha)} \int_{x_1}^{b} (t - x_1)^{4\alpha - 1} \left( D_{b-}^{4\alpha} f \right)(t)\, dt \right\| \leq$$

$$2 \left\| \|f\| \right\|_{\infty, \mathbb{R}_-} + \frac{1}{\Gamma(4\alpha)} \int_{x_1}^{b} (t - x_1)^{4\alpha - 1} \left\| \left( D_{b-}^{4\alpha} f \right)(t) \right\| dt \leq$$

$$2 \left\| \|f\| \right\|_{\infty, \mathbb{R}_-} + \frac{1}{\Gamma(4\alpha)} \frac{(b - x_1)^{\frac{(q(4\alpha-1)+1)}{q}}}{(q(4\alpha - 1) + 1)^{\frac{1}{q}}} \left( \sup_{b \in \mathbb{R}_-} \left\| \left\| D_{b-}^{4\alpha} f \right\| \right\|_{p, \mathbb{R}_-} \right) \overset{(b - x_1 =: h > 0)}{=} \tag{49}$$

$$2 \left\| \|f\| \right\|_{\infty, \mathbb{R}_-} + \frac{1}{\Gamma(4\alpha)} \frac{h^{\left(4\alpha - \frac{1}{p}\right)}}{(q(4\alpha - 1) + 1)^{\frac{1}{q}}} \left( \sup_{b \in \mathbb{R}_-} \left\| \left\| D_{b-}^{4\alpha} f \right\| \right\|_{p, \mathbb{R}_-} \right),$$

with $\frac{1}{4p} < \alpha < 1$.

That is

$$\|A\| \leq 2 \left\| \|f\| \right\|_{\infty, \mathbb{R}_-} + \frac{h^{4\alpha - \frac{1}{p}}}{\Gamma(4\alpha)(q(4\alpha - 1) + 1)^{\frac{1}{q}}} \left( \sup_{b \in \mathbb{R}_-} \left\| \left\| D_{b-}^{4\alpha} f \right\| \right\|_{p, \mathbb{R}_-} \right), \tag{50}$$

where $\frac{1}{4p} < \alpha < 1$.

We also have

$$\|B\| \overset{(13)}{=} \left\| f(x_2) - f(b) - \frac{1}{\Gamma(4\alpha)} \int_{x_2}^{b} (t - x_2)^{4\alpha - 1} \left( D_{b-}^{4\alpha} f \right)(t)\, dt \right\| \leq$$

$$2 \left\| \|f\| \right\|_{\infty, \mathbb{R}_-} + \frac{(b - x_2)^{\frac{(q(4\alpha-1)+1)}{q}}}{\Gamma(4\alpha)(q(4\alpha - 1) + 1)^{\frac{1}{q}}} \left( \sup_{b \in \mathbb{R}_-} \left\| \left\| D_{b-}^{4\alpha} f \right\| \right\|_{p, \mathbb{R}_-} \right) \overset{(b - x_2 =: 2h)}{=} \tag{51}$$

$$2 \left\| \|f\| \right\|_{\infty, \mathbb{R}_-} + \frac{2^{4\alpha - \frac{1}{p}} h^{4\alpha - \frac{1}{p}}}{\Gamma(4\alpha)(q(4\alpha - 1) + 1)^{\frac{1}{q}}} \left( \sup_{b \in \mathbb{R}_-} \left\| \left\| D_{b-}^{4\alpha} f \right\| \right\|_{p, \mathbb{R}_-} \right).$$

That is

$$\|B\| \leq 2 \left\| \|f\| \right\|_{\infty, \mathbb{R}_-} + \frac{2^{4\alpha - \frac{1}{p}} h^{4\alpha - \frac{1}{p}}}{\Gamma(4\alpha)(q(4\alpha - 1) + 1)^{\frac{1}{q}}} \left( \sup_{b \in \mathbb{R}_-} \left\| \left\| D_{b-}^{4\alpha} f \right\| \right\|_{p, \mathbb{R}_-} \right), \tag{52}$$

where $\frac{1}{4p} < \alpha < 1$.

We have assumed that

$$M := \left( \sup_{b \in \mathbb{R}_-} \left\| \left\| D_{b-}^{4\alpha} f \right\| \right\|_{p, \mathbb{R}_-} \right) < \infty. \tag{53}$$

For convenience we call

$$c := \Gamma(4\alpha)(q(4\alpha - 1) + 1)^{\frac{1}{q}} > 0. \tag{54}$$

So we have

$$\begin{gathered} \|A\| \leq 2 \left\| \|f\| \right\|_{\infty, \mathbb{R}_-} + \frac{h^{4\alpha - \frac{1}{p}}}{c} M, \\ \text{and} \\ \|B\| \leq 2 \left\| \|f\| \right\|_{\infty, \mathbb{R}_-} + \frac{2^{4\alpha - \frac{1}{p}} h^{4\alpha - \frac{1}{p}}}{c} M, \end{gathered} \tag{55}$$

where $\frac{1}{4p} < \alpha < 1$.

Next we estimate the (16)-quantities and we have:

$$\left\| \left( D_{b-}^{2\alpha} f \right)(b) \right\| \leq \frac{1}{D\Gamma(3\alpha+1)} \left[ 2^{3\alpha} h^{3\alpha} \|A\| + h^{3\alpha} \|B\| \right] \overset{(23)}{=}$$

$$\frac{h^{3\alpha}\Gamma(2\alpha+1)}{2^{2\alpha}h^{5\alpha}(2^{\alpha}-1)} \left[ 2^{3\alpha} \|A\| + \|B\| \right] \overset{(55)}{\leq} \tag{56}$$

$$\frac{\Gamma(2\alpha+1)}{2^{2\alpha}(2^{\alpha}-1)h^{2\alpha}} \left[ 2^{3\alpha+1} \|\|f\|\|_{\infty,\mathbb{R}_-} + \frac{2^{3\alpha}h^{4\alpha-\frac{1}{p}}}{c} M + \right.$$

$$\left. 2\|\|f\|\|_{\infty,\mathbb{R}_-} + \frac{2^{4\alpha-\frac{1}{p}}h^{4\alpha-\frac{1}{p}}}{c} M \right] =$$

$$\frac{\Gamma(2\alpha+1)}{2^{2\alpha}(2^{\alpha}-1)h^{2\alpha}} \left[ \left(2^{3\alpha+1}+2\right) \|\|f\|\|_{\infty,\mathbb{R}_-} + \frac{\left(2^{3\alpha}+2^{4\alpha-\frac{1}{p}}\right)}{c} M h^{4\alpha-\frac{1}{p}} \right] =$$

$$\frac{\Gamma(2\alpha+1)}{2^{2\alpha}(2^{\alpha}-1)} \left[ \frac{\left(2^{3\alpha+1}+2\right) \|\|f\|\|_{\infty,\mathbb{R}_-}}{h^{2\alpha}} + \frac{\left(2^{3\alpha}+2^{4\alpha-\frac{1}{p}}\right)}{c} M h^{2\alpha-\frac{1}{p}} \right] =$$

$$\frac{2^{\alpha}\Gamma(2\alpha+1)}{(2^{\alpha}-1)} \left[ \frac{2\left(1+2^{-3\alpha}\right) \|\|f\|\|_{\infty,\mathbb{R}_-}}{h^{2\alpha}} + \frac{\left(1+2^{\alpha-\frac{1}{p}}\right) M}{c} h^{2\alpha-\frac{1}{p}} \right]. \tag{57}$$

That is

$$\left\| \left( D_{b-}^{2\alpha} f \right)(b) \right\| \leq$$

$$\left( \frac{2^{\alpha}\Gamma(2\alpha+1)}{2^{\alpha}-1} \right) \left[ \frac{2\left(1+2^{-3\alpha}\right) \|\|f\|\|_{\infty,\mathbb{R}_-}}{h^{2\alpha}} + \frac{\left(1+2^{\alpha-\frac{1}{p}}\right) M}{c} h^{2\alpha-\frac{1}{p}} \right], \tag{58}$$

$\forall\, b \in \mathbb{R}_-$, $\forall\, h > 0$.

I.e. it holds

$$\sup_{b\in\mathbb{R}_-} \left\| \left( D_{b-}^{2\alpha} f \right)(b) \right\| \leq \left( \frac{2^{\alpha}\Gamma(2\alpha+1)}{2^{\alpha}-1} \right)$$

$$\left[ \frac{2\left(1+2^{-3\alpha}\right) \|\|f\|\|_{\infty,\mathbb{R}_-}}{h^{2\alpha}} + \frac{\left(1+2^{\alpha-\frac{1}{p}}\right) M}{c} h^{2\alpha-\frac{1}{p}} \right], \tag{59}$$

$\forall\, h > 0$, under $\frac{1}{4p} < \alpha < 1$.

Again from (16) we get

$$\left\| \left( D_{b-}^{3\alpha} f \right)(b) \right\| \leq \frac{1}{\Gamma(2\alpha+1) D} \left[ h^{2\alpha} \|B\| + 2^{2\alpha} h^{2\alpha} \|A\| \right] =$$

$$\frac{h^{2\alpha}\Gamma(3\alpha+1)}{2^{2\alpha}h^{5\alpha}(2^{\alpha}-1)} \left[ \|B\| + 2^{2\alpha} \|A\| \right] \overset{(55)}{\leq} \tag{60}$$

$$\left( \frac{h^{2\alpha}\Gamma(3\alpha+1)}{2^{2\alpha}h^{5\alpha}(2^{\alpha}-1)} \right) \left[ 2\|\|f\|\|_{\infty,\mathbb{R}_-} + \frac{2^{4\alpha-\frac{1}{p}}h^{4\alpha-\frac{1}{p}}}{c} M + \right.$$

$$\left. 2^{2\alpha+1} \|\|f\|\|_{\infty,\mathbb{R}_-} + \frac{2^{2\alpha}h^{4\alpha-\frac{1}{p}}}{c} M \right] =$$

$$\left(\frac{\Gamma(3\alpha+1)}{2^{2\alpha}(2^{\alpha}-1)h^{3\alpha}}\right)\left[\left(2+2^{2\alpha+1}\right)\|\|f\|\|_{\infty,\mathbb{R}_-}+\frac{\left(2^{2\alpha}+2^{4\alpha-\frac{1}{p}}\right)}{c}Mh^{4\alpha-\frac{1}{p}}\right]=$$

$$\frac{\Gamma(3\alpha+1)}{2^{2\alpha}(2^{\alpha}-1)}\left[\frac{\left(2+2^{2\alpha+1}\right)\|\|f\|\|_{\infty,\mathbb{R}_-}}{h^{3\alpha}}+\frac{\left(2^{2\alpha}+2^{4\alpha-\frac{1}{p}}\right)}{c}Mh^{\alpha-\frac{1}{p}}\right]= \tag{61}$$

$$\frac{\Gamma(3\alpha+1)}{(2^{\alpha}-1)}\left[\frac{2\left(1+2^{-2\alpha}\right)\|\|f\|\|_{\infty,\mathbb{R}_-}}{h^{3\alpha}}+\frac{\left(1+2^{2\alpha-\frac{1}{p}}\right)}{c}Mh^{\alpha-\frac{1}{p}}\right].$$

That is

$$\left\|\left(D_{b-}^{3\alpha}f\right)(b)\right\|\leq\left(\frac{\Gamma(3\alpha+1)}{2^{\alpha}-1}\right)\left[\frac{2\left(1+2^{-2\alpha}\right)\|\|f\|\|_{\infty,\mathbb{R}_-}}{h^{3\alpha}}+\frac{\left(1+2^{2\alpha-\frac{1}{p}}\right)}{c}Mh^{\alpha-\frac{1}{p}}\right], \tag{62}$$

$\forall\ b\in\mathbb{R}_-$, $\forall\ h>0$.

I.e. it holds

$$\sup_{b\in\mathbb{R}_-}\left\|\left(D_{b-}^{3\alpha}f\right)(b)\right\|\leq\left(\frac{\Gamma(3\alpha+1)}{2^{\alpha}-1}\right)$$

$$\left[\frac{2\left(1+2^{-2\alpha}\right)\|\|f\|\|_{\infty,\mathbb{R}_-}}{h^{3\alpha}}+\frac{\left(1+2^{2\alpha-\frac{1}{p}}\right)}{c}Mh^{\alpha-\frac{1}{p}}\right], \tag{63}$$

$\forall\ h>0$, $\frac{1}{4p}<\alpha<1$.

Call

$$\mu:=2\left(1+2^{-3\alpha}\right)\|\|f\|\|_{\infty,\mathbb{R}_-},$$

$$\theta:=\frac{\left(1+2^{\alpha-\frac{1}{p}}\right)M}{c}, \tag{64}$$

both are greater than zero.

We consider the function

$$y(h)=\mu h^{-2\alpha}+\theta h^{2\alpha-\frac{1}{p}},\quad\forall\ h>0. \tag{65}$$

We have

$$y'(h)=-2\alpha\mu h^{-2\alpha-1}+\left(2\alpha-\frac{1}{p}\right)\theta h^{2\alpha-\frac{1}{p}-1}=0, \tag{66}$$

then

$$\left(2\alpha-\frac{1}{p}\right)\theta h^{2\alpha-\frac{1}{p}-1}=2\alpha\mu h^{-2\alpha-1},$$

i.e.

$$\left(2\alpha-\frac{1}{p}\right)\theta h^{4\alpha-\frac{1}{p}}=2\alpha\mu,$$

with a unique solution

$$h_0:=h_{crit.no.}=\left(\frac{2\alpha\mu}{\left(2\alpha-\frac{1}{p}\right)\theta}\right)^{\frac{1}{4\alpha-\frac{1}{p}}} \tag{67}$$

(assuming $\frac{1}{2p}<\alpha<1$).

We have that

$$y''(h) = 2\alpha(2\alpha+1)\mu h^{-2\alpha-2} + \left(2\alpha - \frac{1}{p}\right)\left(2\alpha - \frac{1}{p} - 1\right)\theta h^{2\alpha - \frac{1}{p} - 2}. \tag{68}$$

We see that

$$y''(h_0) = 2\alpha(2\alpha+1)\mu\left(\frac{2\alpha\mu}{\left(2\alpha-\frac{1}{p}\right)\theta}\right)^{\frac{-2\alpha-2}{4\alpha-\frac{1}{p}}} +$$

$$\left(2\alpha - \frac{1}{p}\right)\left(2\alpha - \frac{1}{p} - 1\right)\theta\left(\frac{2\alpha\mu}{\left(2\alpha-\frac{1}{p}\right)\theta}\right)^{\frac{2\alpha-\frac{1}{p}-2}{4\alpha-\frac{1}{p}}}$$

$$= 2\alpha\mu\left(\frac{\left(2\alpha-\frac{1}{p}\right)\theta}{2\alpha\mu}\right)^{\left(\frac{2(\alpha+1)}{4\alpha-\frac{1}{p}}\right)}\left(4\alpha - \frac{1}{p}\right) > 0. \tag{69}$$

Therefore $y$ has a global minimum at

$$h_0 = \left(\frac{2\alpha\mu}{\left(2\alpha-\frac{1}{p}\right)\theta}\right)^{\frac{1}{4\alpha-\frac{1}{p}}},$$

which is

$$y(h_0) = \mu\left(\frac{2\alpha\mu}{\left(2\alpha-\frac{1}{p}\right)\theta}\right)^{\frac{-2\alpha}{4\alpha-\frac{1}{p}}} + \theta\left(\frac{2\alpha\mu}{\left(2\alpha-\frac{1}{p}\right)\theta}\right)^{\frac{2\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}} = \tag{70}$$

$$\frac{\left(4\alpha-\frac{1}{p}\right)}{(2\alpha)^{\left(\frac{2\alpha}{4\alpha-\frac{1}{p}}\right)}}\left(2\alpha - \frac{1}{p}\right)^{\frac{-2\alpha+\frac{1}{p}}{4\alpha-\frac{1}{p}}}\mu^{\left(\frac{2\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)}\theta^{\left(\frac{2\alpha}{4\alpha-\frac{1}{p}}\right)}. \tag{71}$$

That is

$$y(h_0) = \frac{\left(4\alpha-\frac{1}{p}\right)\left(2\alpha-\frac{1}{p}\right)^{\left(\frac{-2\alpha+\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)}}{(2\alpha)^{\left(\frac{2\alpha}{4\alpha-\frac{1}{p}}\right)}}\mu^{\left(\frac{2\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)}\theta^{\left(\frac{2\alpha}{4\alpha-\frac{1}{p}}\right)}. \tag{72}$$

Therefore we derive (see (59))

$$\sup_{b\in\mathbb{R}_-}\left\|\left(D_{b-}^{2\alpha}f\right)(b)\right\| \le \left(\frac{2^{\alpha}\Gamma(2\alpha)}{2^{\alpha}-1}\right)\left(\frac{2\alpha}{2\alpha-\frac{1}{p}}\right)^{\frac{\left(2\alpha-\frac{1}{p}\right)}{\left(4\alpha-\frac{1}{p}\right)}}\left(4\alpha - \frac{1}{p}\right)$$

$$\left(2\left(1+2^{-3\alpha}\right)\right)^{\left(\frac{2\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)}\left(\frac{\left(1+2^{\alpha-\frac{1}{p}}\right)}{\Gamma(4\alpha)\left(q(4\alpha-1)+1\right)^{\frac{1}{q}}}\right)^{\left(\frac{2\alpha}{4\alpha-\frac{1}{p}}\right)}\||f\||_{\infty,\mathbb{R}_-}^{\left(\frac{2\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)}$$

$$\left(\sup_{b\in\mathbb{R}_-}\left\|\left\|D_{b-}^{4\alpha}f\right\|\right\|_{p,\mathbb{R}_-}\right)^{\left(\frac{2\alpha}{4\alpha-\frac{1}{p}}\right)} < \infty, \tag{73}$$

where $\frac{1}{2p} < \alpha < 1$.

Call

$$\xi := 2\left(1+2^{-2\alpha}\right)\||f\||_{\infty,\mathbb{R}_-},$$

$$\psi := \frac{\left(1+2^{2\alpha-\frac{1}{p}}\right)}{c}M, \tag{74}$$

both are greater than zero.

We consider the function

$$\gamma(h) := \xi h^{-3\alpha} + \psi h^{\alpha-\frac{1}{p}}, \quad \forall\ h>0. \tag{75}$$

We have

$$\gamma'(h) = -3\alpha\xi h^{-3\alpha-1} + \left(\alpha-\frac{1}{p}\right)\psi h^{\alpha-\frac{1}{p}-1} = 0,$$

then

$$\left(\alpha-\frac{1}{p}\right)\psi h^{\alpha-\frac{1}{p}-1} = 3\alpha\xi h^{-3\alpha-1}$$

and

$$\left(\alpha-\frac{1}{p}\right)\psi h^{4\alpha-\frac{1}{p}} = 3\alpha\xi,$$

with unique solution

$$h_0 := h_{crit.no.} = \left(\frac{3\alpha\xi}{\left(\alpha-\frac{1}{p}\right)\psi}\right)^{\frac{1}{4\alpha-\frac{1}{p}}} \tag{76}$$

(assuming $\frac{1}{p}<\alpha<1$).

We have that

$$\gamma''(h) = 3\alpha(3\alpha+1)\xi h^{-3\alpha-2} + \left(\alpha-\frac{1}{p}\right)\left(\alpha-\frac{1}{p}-1\right)\psi h^{\alpha-\frac{1}{p}-2}. \tag{77}$$

We observe that

$$\gamma''(h_0) = 3\alpha(3\alpha+1)\xi\left(\frac{3\alpha\xi}{\left(\alpha-\frac{1}{p}\right)\psi}\right)^{\left(\frac{-3\alpha-2}{4\alpha-\frac{1}{p}}\right)} +$$

$$\left(\alpha-\frac{1}{p}\right)\left(\alpha-\frac{1}{p}-1\right)\psi\left(\frac{3\alpha\xi}{\left(\alpha-\frac{1}{p}\right)\psi}\right)^{\left(\frac{\alpha-\frac{1}{p}-2}{4\alpha-\frac{1}{p}}\right)} =$$

$$3\alpha\xi\left(\frac{3\alpha\xi}{\left(\alpha-\frac{1}{p}\right)\psi}\right)^{\left(\frac{-3\alpha-2}{4\alpha-\frac{1}{p}}\right)}\left(4\alpha-\frac{1}{p}\right) > 0. \tag{78}$$

Therefore $y$ has a global minimum at

$$h_0 = \left(\frac{3\alpha\xi}{\left(\alpha-\frac{1}{p}\right)\psi}\right)^{\frac{1}{4\alpha-\frac{1}{p}}},$$

which is

$$\gamma(h_0) = \xi h_0^{-3\alpha} + \psi h_0^{\alpha-\frac{1}{p}} = h_0^{-3\alpha}\left(\xi+\psi h_0^{4\alpha-\frac{1}{p}}\right) = \tag{79}$$

$$\left(\frac{3\alpha\xi}{\left(\alpha-\frac{1}{p}\right)\psi}\right)^{\left(\frac{-3\alpha}{4\alpha-\frac{1}{p}}\right)}\left(\xi+\psi\left(\frac{3\alpha\xi}{\left(\alpha-\frac{1}{p}\right)\psi}\right)\right)=$$

$$\xi\left(\frac{3\alpha\xi}{\left(\alpha-\frac{1}{p}\right)\psi}\right)^{\left(\frac{-3\alpha}{4\alpha-\frac{1}{p}}\right)}\left(\frac{4\alpha-\frac{1}{p}}{\alpha-\frac{1}{p}}\right).$$

That is

$$\gamma\left(h_0\right)=\frac{\left(4\alpha-\frac{1}{p}\right)}{\left(\alpha-\frac{1}{p}\right)^{\left(\frac{\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)}(3\alpha)^{\left(\frac{3\alpha}{4\alpha-\frac{1}{p}}\right)}}\xi^{\left(\frac{\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)}\psi^{\left(\frac{3\alpha}{4\alpha-\frac{1}{p}}\right)}. \tag{80}$$

We have proved that (see (63))

$$\sup_{b\in\mathbb{R}_-}\left\|\left(D_{b-}^{3\alpha}f\right)(b)\right\|\leq\left(\frac{\left(4\alpha-\frac{1}{p}\right)\Gamma(3\alpha)}{2^{\alpha}-1}\right)\left(\frac{3\alpha}{\alpha-\frac{1}{p}}\right)^{\left(\frac{\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)} \tag{81}$$

$$\left(2\left(1+2^{-2\alpha}\right)\right)^{\left(\frac{\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)}\left(\frac{\left(1+2^{2\alpha-\frac{1}{p}}\right)}{\Gamma(4\alpha)\left(q(4\alpha-1)+1\right)^{\frac{1}{q}}}\right)^{\left(\frac{3\alpha}{4\alpha-\frac{1}{p}}\right)}\|\|f\|\|_{\infty,\mathbb{R}_-}^{\left(\frac{\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)}$$

$$\left(\sup_{b\in\mathbb{R}_-}\left\|\left\|D_{b-}^{4\alpha}f\right\|\right\|_{p,\mathbb{R}_-}\right)^{\left(\frac{3\alpha}{4\alpha-\frac{1}{p}}\right)}<\infty,$$

where $\frac{1}{p}<\alpha<1$.

The theorem is proved. □

We give an application when $\alpha=\frac{1}{2}$.

**Corollary 2.4.** *Let $f\in C^1(\mathbb{R}_-,X)$ with $\|\|f\|\|_{\infty,\mathbb{R}_-}$, $\|\|f'\|\|_{\infty,\mathbb{R}_-}<\infty$, where $(X,\|\cdot\|)$ is a Banach space. For $k=1,2,3$, we assume that $D_{b-}^{k\frac{1}{2}}f\in C^1((-\infty,b],X)$ and $D_{b-}^{4\frac{1}{2}}f\in C((-\infty,b],X)$, $\forall\ b\in\mathbb{R}_-$. We further assume that*

$$K:=\left\|\left\|D_{b-}^{4\frac{1}{2}}f(t)\right\|\right\|_{\infty,\mathbb{R}_-^2}<\infty, \tag{82}$$

*where $(b,t)\in\mathbb{R}_-^2$.*

*Then*

$$\sup_{b\in\mathbb{R}_-}\left\|\left(D_{b-}^{2\frac{1}{2}}f\right)(b)\right\|\leq\left(\frac{\sqrt{12+6\sqrt{2}}}{\sqrt{2}-1}\right)\|\|f\|\|_{\infty,\mathbb{R}_-}^{\frac{1}{2}}\left(\left\|\left\|D_{b-}^{4\frac{1}{2}}f(t)\right\|\right\|_{\infty,\mathbb{R}_-^2}\right)^{\frac{1}{2}}<\infty, \tag{83}$$

*and*

$$\sup_{b\in\mathbb{R}_-}\left\|\left(D_{b-}^{3\frac{1}{2}}f\right)(b)\right\|\leq$$

$$\left(\frac{9\sqrt{\pi}}{\left(2-\sqrt{2}\right)\sqrt[4]{2}\left(\sqrt[4]{3}\right)^3}\right)\|\|f\|\|_{\infty,\mathbb{R}_-}^{\frac{1}{4}}\left(\left\|\left\|D_{b-}^{4\frac{1}{2}}f(t)\right\|\right\|_{\infty,\mathbb{R}_-^2}\right)^{\frac{3}{4}}<\infty. \tag{84}$$

*That is* $\sup_{b\in\mathbb{R}_-}\left\|\left(D_{b-}^{2\frac{1}{2}}f\right)(b)\right\|, \sup_{b\in\mathbb{R}_-}\left\|\left(D_{b-}^{3\frac{1}{2}}f\right)(b)\right\| < \infty.$

*Proof.* By Theorem 2.2. □

**Remark.** *All of the above results are also true for any half line* $(-\infty, A]$*,* $A \in \mathbb{R}$*, instead of only* $\mathbb{R}_-$*, just replace all* $\mathbb{R}_-$ *by* $(-\infty, A]$.

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