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THE ITERATIONS OF STRONGLY QUASI ϕ-NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we study the iterations of strongly (asymptotic) quasi ϕ-nonexpansive mappings in Banach spaces. First, we prove weak convergence of the generated sequence to a common fixed point of an infinite family of strongly asymptotic quasi ϕ-nonexpansive mappings. Next we prove strong convergence of the generated sequence by an additional assumption. In the sequel, invoke of Halpern regularization method, we prove strong convergence of the generated sequence to a common fixed point of the family of mappings without any extra conditions. Finally, we give some applications of our main results in convex minimization and equilibrium problems and present numerical examples to illustrate and support them.

1. Introduction

We denote the dual of a real Banach space E with E^*, its norm with $\|.\|$ and the value of $v \in E^*$ at $x \in E$ by $\langle x, v\rangle$. The mapping $J : E \to 2^{E^*}$ defined by

$$Jx = \{v \in E^* : \langle x, v\rangle = \|x\|^2 = \|v\|^2\}$$

for all $x \in E$, is called the duality mapping.
A Banach space E for which $\|\frac{x+y}{2}\| < 1$ for any $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$, is called strictly convex. Also, it is called uniformly convex if for any $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|\frac{x+y}{2}\| < 1 - \delta$, for any $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$. We know that a uniformly convex Banach space is reflexive and strictly convex. A Banach space E is called smooth if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.1}$$

exists for all $x, y \in U = \{z \in E : \|z\| = 1\}$. If for all $x, y \in U$, the limit (1.1) is attained uniformly, then E is called the uniformly smooth Banach space. For a smooth Banach space E, we will use the following function

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \tag{1.2}$$

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for any $x, y \in E$ which was used in [1] by Alber, in [8] by Kamimura and Takahashi and in [15] by Reich. By the definition of the function ϕ, we have

$$0 \leq (\|x\| - \|y\|)^2 \leq \phi(x, y). \tag{1.3}$$

If E is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$, because the duality mapping is the identity operator in Hilbert spaces.

Proposition 1.1. [8] *Suppose that $\{x_k\}$ and $\{y_k\}$ are two sequences in a uniformly convex and smooth Banach space E. If $\phi(x_k, y_k)$ tends to zero, as $k \to \infty$, and either $\{x_k\}$ or $\{y_k\}$ is bounded, then $\lim_{k\to\infty} \|x_k - y_k\| = 0$.*

Proposition 1.2. [8] *Let C be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space E and $x \in E$. Then we can find a unique element $x_0 \in C$, such that*

$$\phi(x_0, x) = \inf\{\phi(z, x) : z \in C\}.$$

Regarding Proposition 1.2, we denote the unique element $x_0 \in C$ by $P_C(x)$, where the mapping P_C is called the generalized projection from E onto C. It is well known that the generalized projection mapping P_C is coincident with the metric projection from E onto C in Hilbert spaces. We need the following proposition to prove the strong convergence in Section 3.

Proposition 1.3. [8] *Let C be a convex subset of a smooth Banach space E, $x \in E$ and $x_0 \in C$. Then*

$$\phi(x_0, x) = \inf\{\phi(z, x) : z \in C\}$$

if and only if

$$\langle z - x_0, Jx - Jx_0\rangle \leq 0, \quad \forall z \in C.$$

Throughout this paper, the strong convergence of a sequence $\{x_k\}$ in E to $x \in E$ is denoted by $x_k \to x$ and its weak convergence by $x_k \rightharpoonup x$. Let C be a closed and convex subset of a Banach space E. We denote the set of all fixed points of a mapping $T : C \to C$ by $F(T)$, i.e. $F(T) = \{x \in C : Tx = x\}$.

Definition 1.4. *A mapping $T : C \to C$ is called nonexpansive, if and only if for any $x, y \in C$,*

$$\|Tx - Ty\| \leq \|x - y\|$$

and T is called quasi-nonexpansive, whenever $F(T) \neq \varnothing$ and for any $(q, x) \in F(T) \times C$,

$$\|Tx - q\| \leq \|x - q\|.$$

Regarding the definitions of nonexpansive and quasi-nonexpansive mappings, we define ϕ-nonexpansive and quasi ϕ-nonexpansive mappings in Banach spaces.

Definition 1.5. *A mapping $T : C \to C$ is said to be ϕ-nonexpansive if and only if*

$$\phi(Tx, Ty) \leq \phi(x, y), \qquad \forall x, y \in C$$

and T is said to be quasi ϕ-nonexpansive, whenever $F(T) \neq \varnothing$ and

$$\phi(q, Tx) \leq \phi(q, x), \qquad \forall (q, x) \in F(T) \times C.$$

Definition 1.6. *The sequence $\{T_k\}$ of quasi ϕ-nonexpansive mappings is called a strongly quasi ϕ-nonexpansive sequence if and only if $\bigcap_k F(T_k) \neq \varnothing$ and $\phi(T_k x_k, x_k)$ tends to zero, whenever $\{x_k\}$ is a bounded sequence in C and $\phi(q, x_k) - \phi(q, T_k x_k) \to 0$, for $q \in \bigcap_k F(T_k)$. When $T_k \equiv T$, T is called strongly quasi ϕ-nonexpansive mapping.*

Definition 1.7. *The sequence $\{T_k\}$ is called asymptotically quasi ϕ-nonexpansive mappings if and only if $\bigcap_k F(T_k) \neq \varnothing$ and there exists a nonnegative sequence $\{\mu_k\}$ with $\sum_{k=1}^{\infty} \mu_k < \infty$ such that for all $k \in \mathbb{N}$,*

$$\phi(q, T_k x) \leq (1 + \mu_k)\phi(q, x), \qquad \forall (q, x) \in F(T_k) \times C.$$

Definition 1.8. *The sequence $\{T_k\}$ of asymptotically quasi ϕ-nonexpansive mappings is called a strongly asymptotic quasi ϕ-nonexpansive sequence if and only if $\phi(T_k x_k, x_k) \to 0$ for any bounded sequence $\{x_k\}$ in C and $q \in \bigcap_k F(T_k)$ where $\phi(q, x_k) - \phi(q, T_k x_k) \to 0$.*

(Asymptotically) quasi ϕ-nonexpansive mappings and some related topics have been studied in recent years for example in [6, 12, 14]. Δ-convergece of the iterations for a sequence of strongly quasi-nonexpansive mappings to a common fixed point of the mappings has been studied in Hadamard spaces by Khatibzadeh and Mohebbi in [9]. They also used the Halpern regularization method to prove the strong convergence of the generated sequence. A new iterative method for solving split feasibility problems and also approximating common fixed points by a new faster iteration process has been studied by Garodia and Uddin, respectively in [2] and [3]. Convergence theorems for a hybrid pair of generalized nonexpansive mappings in Banach spaces have been investigated by Uddin, Imdad and Ali in [20]. For the iteration scheme of a family of multivalued mappings in CAT(0) spaces and Δ-convergence and strong convergence theorems, we refer the readers to [19].

In this paper, we investigate the iterations of strongly (asymptotic) quasi ϕ-nonexpansive mappings in Banach spaces. In Section 2, we consider the iterations of a sequence of strongly asymptotic quasi ϕ-nonexpansive mappings and prove weak convergence of their iterations to a common fixed point of the sequence. Then we prove strong convergence of the generated sequence by an additional assumption. To achieve strong convergence without any additional assumption, we use the Halpern regularization method which was used by Xu [21] in Hilbert spaces. In Section 3, we will use the Halpern regularization method and prove strong convergence of their iterations to a common fixed point of the sequence. Finally in Section 4, we give some applications of the main results in convex minimization and equilibrium problems and present numerical examples to illustrate and support them.

2. Weak and strong convergence

In this section, we first study the weak convergence of the sequence generated by (2.2). Next we prove the strong convergence of the generated sequence by an additional assumption. In order to study the convergence analysis of the generated sequence, we need the definition of demiclosedness for a sequence of quasi ϕ-nonexpansive mappings. Nonexpansive mappings are demiclosed, but for quasi ϕ-nonexpansive mappings, we have to assume this property even in Hilbert spaces.

Definition 2.1. *A sequence $\{T_k\}$ of quasi ϕ-nonexpansive mappings is said to be demiclosed, whenever for each subsequence $\{x_{k_n}\}$ of $\{x_k\}$ and $\{T_{k_n}\}$ of $\{T_k\}$;*

$$\text{if} \quad x_{k_n} \rightharpoonup p \quad \text{and} \quad \lim_{n\to\infty} \|T_{k_n} x_{k_n} - x_{k_n}\| = 0, \quad \text{then} \quad p \in \bigcap_k F(T_k). \tag{2.1}$$

In the following, we give an example of a sequence of quasi ϕ-nonexpansive mappings which is demiclosed.

Example 1. Assume that E is a uniformly smooth and uniformly convex Banach space and $B(0,r)$ denotes the closed ball of radius r centered at 0. We define a sequence $\{T_k\}$ from E to itself by $T_k(x) = P_{B(0,\frac{k+1}{2k}\|x\|)}(x)$ where P is the generalized projection on $B(0,\frac{k+1}{2k}\|x\|)$. It is easy to see that

$$\phi(q,T_k x) \le \phi(q,x),$$

for all $(q,x) \in F(T_k) \times E$ and for all $k \in \mathbb{N}$. Therefore $\{T_k\}$ is a sequence of quasi ϕ-nonexpansive mappings. Now, we show that $\{T_k\}$ is demiclosed. Suppose that $x_{k_n} \rightharpoonup p$ and $\lim_{n\to\infty} \|T_{k_n} x_{k_n} - x_{k_n}\| = 0$. This implies that

$$\lim_{n\to\infty} \|P_{B(0,\frac{k_n+1}{2k_n}\|x_{k_n}\|)}(x_{k_n}) - x_{k_n}\| = 0.$$

Therefore we obtain that $x_{k_n} \rightharpoonup 0$, that is $p = 0$. It is easy to see that $p \in \bigcap_k F(T_k)$, i.e. $\{T_k\}$ is demiclosed.

Throughout this paper, we assume that E is a uniformly smooth and uniformly convex Banach space. Let $T_k : C \to C$ be a sequence of strongly asymptotic quasi ϕ-nonexpansive mappings. In the following theorem, we show the weak convergence of the sequence $\{x_k\}$ generated by

$$x_{k+1} = T_k x_k \tag{2.2}$$

to an element of $\bigcap_k F(T_k) \neq \varnothing$. In order to prove uniqueness of the weak limit point of the generated sequence in the following theorem, we need the following condition on the Banach space E:
If $\{y_k\}$ and $\{z_k\}$ are sequences in C that converge weakly to y and z respectively and $y \neq z$, then

$$\liminf_{k\to\infty} |\langle y - z, Jy_k - Jz_k\rangle| > 0. \tag{2.3}$$

For example, it is known that ℓ_p spaces for $1 < p < \infty$ satisfy the above condition (see [7]). It is also valuable to mention that when we prove the strong convergence theorems in this paper we do not need the above condition.

Theorem 2.2. *Suppose that C is a nonempty, closed and convex subset of a uniformly smooth and uniformly convex Banach space E, and $T_k : C \to C$ is a demiclosed sequence of strongly asymptotic quasi ϕ-nonexpansive mappings and $x_1 \in C$. Let $x_{k+1} = T_k \cdots T_1 x_1$. Then the sequence $\{x_k\}$ converges weakly to a point of $\bigcap_k F(T_k)$.*

Proof. Take $x^* \in \bigcap_k F(T_k)$. Since the sequence $\{T_k\}$ is a sequence of asymptotically quasi ϕ-nonexpansive mappings, there is a sequence $\{\mu_k\}$ with $\sum_{k=1}^{\infty} \mu_k < \infty$ such that

$$\begin{aligned}\phi(x^*, x_{k+1}) &= \phi(x^*, T_k x_k) \leq (1+\mu_k)\phi(x^*, x_k) \\ &\leq (1+\mu_k)(1+\mu_{k-1})\phi(x^*, x_{k-1}) \\ &\leq \cdots \leq \phi(x^*, x_n)\prod_{i=n}^{i=k}(1+\mu_i) \\ &\leq \phi(x^*, x_n)\prod_{i=n}^{\infty}(1+\mu_i) \\ &\leq \phi(x^*, x_n)e^{\sum_{i=n}^{\infty}\mu_i}. \end{aligned} \tag{2.4}$$

We take limsup in (2.4) as $k \to \infty$ and then we take liminf as $n \to \infty$, since

$$\limsup_{k\to\infty}\phi(x^*, x_k) \leq \liminf_{k\to\infty}\phi(x^*, x_k),$$

$\lim_{k\to\infty}\phi(x^*, x_k)$ exists for each $x^* \in \bigcap_k F(T_k)$. In addition, $\{x_k\}$ is bounded by (1.3). Therefore, there exist $\{x_{k_n}\}$ of $\{x_k\}$ and $p \in C$ such that $x_{k_n} \rightharpoonup p \in C$. On the other hand, since the sequence $\{T_k\}$ is strongly asymptotic quasi ϕ-nonexpansive and $\lim_{k\to\infty}\phi(x^*, x_k)$ exists for all $x^* \in \bigcap_k F(T_k)$, we get $\lim_{k\to\infty}\phi(T_{k_n}x_{k_n}, x_{k_n}) = 0$. Then we have $\lim_{n\to\infty}\|T_{k_n}x_{k_n} - x_{k_n}\| = 0$ by Proposition 1.1. Now, (2.1) shows that $p \in \bigcap_k F(T_k)$.

In the sequel, we show that there exists only one weak limit point of $\{x_k\}$. If q is an other weak limit point of $\{x_k\}$, then there exists a subsequence $\{x_{k_j}\}$ such that $x_{k_j} \rightharpoonup q$. Similar to the above argument, we can prove that q is an element of $\bigcap_k F(T_k)$, also $\lim_{k\to\infty}\phi(p, x_k)$ and $\lim_{k\to\infty}\phi(q, x_k)$ exist. Note that

$$\begin{aligned}2\langle p-q, Jx_{k_n} - Jx_{k_j}\rangle &= 2\langle p, Jx_{k_n}\rangle - 2\langle q, Jx_{k_n}\rangle - 2\langle p, Jx_{k_j}\rangle + 2\langle q, Jx_{k_j}\rangle \\ &= -\phi(p, x_{k_n}) + \phi(q, x_{k_n}) + \phi(p, x_{k_j}) - \phi(q, x_{k_j}). \end{aligned} \tag{2.5}$$

Taking limit from (2.5) when $n \to \infty$ and then when $j \to \infty$, we obtain $p = q$ by (2.3). Therefore, $\{x_k\}$ converges weakly to a point of $\bigcap_k F(T_k)$. □

Theorem 2.3. *Suppose that the assumptions of Theorem 2.2 hold. If the interior of $\bigcap_k F(T_k)$ is nonempty, then the sequence $\{x_k\}$ is strongly convergent to an element of $\bigcap_k F(T_k)$.*

Proof. Note that $\{x_k\}$ is bounded by Theorem 2.2. Since $\text{int}(\bigcap_k F(T_k)) \neq \emptyset$, there exist $r > 0$ and $x^* \in \text{int}(\bigcap_k F(T_k))$ such that $\bar{B}_r(x^*) \subset \text{int}(\bigcap_k F(T_k))$. It is known that since E is smooth, the duality mapping J is single-valued and since E is uniformly convex, J is one to one and also E is reflexive. In addition, J is surjective because E is reflexive (see [11, 12]). Hence, if $\|J^{-1}(Jx_{k+1} - Jx_k)\| \neq 0$ by letting $\tilde{x_k} = x^* - r\frac{J^{-1}(Jx_{k+1}-Jx_k)}{\|J^{-1}(Jx_{k+1}-Jx_k)\|}$, we have $\tilde{x_k} \in \bigcap_k F(T_k)$ and

$$\phi(\tilde{x_k}, x_{k+1}) = \phi(\tilde{x_k}, T_k x_k) \leq (1+\mu_k)\phi(\tilde{x_k}, x_k).$$

We define $M = \sup_{k\in\mathbb{N}}\{\phi(\tilde{x_k}, x_k), \phi(x^*, x_k)\}$. Therefore we have

$$\phi(\tilde{x_k}, x_{k+1}) \leq \phi(\tilde{x_k}, x_k) + M\mu_k$$

or equivalently

$$\begin{aligned}\phi(\tilde{x_k}, x_{k+1}) &= \|\tilde{x_k}\|^2 - 2\langle x^* - r\frac{J^{-1}(Jx_{k+1} - Jx_k)}{\|J^{-1}(Jx_{k+1} - Jx_k)\|}, Jx_{k+1}\rangle + \|x_{k+1}\|^2 \\ &\leq \|\tilde{x_k}\|^2 - 2\langle x^* - r\frac{J^{-1}(Jx_{k+1} - Jx_k)}{\|J^{-1}(Jx_{k+1} - Jx_k)\|}, Jx_k\rangle \\ &+ \|x_k\|^2 + M\mu_k. \end{aligned} \tag{2.6}$$

Since,

$$2r\|Jx_{k+1} - Jx_k\| = \frac{2r}{\|J^{-1}(Jx_{k+1} - Jx_k)\|}\langle J^{-1}(Jx_{k+1} - Jx_k), Jx_{k+1} - Jx_k\rangle,$$

therefore (2.6) implies that

$$\begin{aligned} 2r\|Jx_{k+1} - Jx_k\| + \|x^*\|^2 - 2\langle x^*, Jx_{k+1}\rangle + \|x_{k+1}\|^2 \\ \leq \|x^*\|^2 - 2\langle x^*, Jx_k\rangle + \|x_k\|^2 + M\mu_k. \end{aligned}$$

Using (1.2), we can write the above inequality as

$$2r\|Jx_{k+1} - Jx_k\| \leq \phi(x^*, x_k) - \phi(x^*, x_{k+1}) + M\mu_k. \tag{2.7}$$

It is clear that if $\|J^{-1}(Jx_{k+1} - Jx_k)\| = 0$, then using (2.4), again (2.7) holds. Summing up (2.7) from $k = 1$ to $k = n$, we obtain

$$2r\sum_{k=1}^{n}\|Jx_{k+1} - Jx_k\| \leq \phi(x^*, x_1) - \phi(x^*, x_{n+1}) + M\sum_{k=1}^{n}\mu_k.$$

Now, if $n \to +\infty$, we get

$$\sum_{k=1}^{+\infty}\|Jx_{k+1} - Jx_k\| < +\infty. \tag{2.8}$$

It follows that $\{Jx_k\}$ converges strongly to an element in E^*. Since E is uniformly convex, E^* is uniformly smooth and so, the duality mapping J^{-1} is uniformly norm to norm continuous on each bounded subset of E^*. Therefore, $\{x_k\}$ converges strongly to an element of C. On the other hand by Theorem 2.2, since $\{x_k\}$ has a weakly convergence subsequence to an element of $\bigcap_k F(T_k)$, we get $\{x_k\}$ converges strongly to an element of $\bigcap_k F(T_k)$. □

3. Halpern regularization method

In this section, we study the Halpern type regularization of (2.2). Consider the sequence $\{x_k\}$ given by the following process

$$x_{k+1} = J^{-1}(\alpha_k Ju + (1 - \alpha_k)JT_k x_k), \tag{3.1}$$

where $u, x_1 \in E$, $\{\alpha_k\} \subset (0,1)$ such that $\lim_{k\to\infty}\alpha_k = 0$ and $\sum_{k=1}^{\infty}\alpha_k = \infty$. We show that the strong convergence of the sequence generated by (3.1) to the generalized projection of u on $\bigcap_k F(T_k)$. Let E be a strictly convex, smooth and reflexive Banach space and $J : E \to E^*$ be the duality mapping. Then J^{-1} is single-valued, one-to-one and surjective and it is the duality mapping from E^* into E. We make use of the following mapping V studied by Alber in [1]

$$V(x,v) = \|x\|^2 - 2\langle x, v\rangle + \|v\|^2 \tag{3.2}$$

for all $x \in E$ and $v \in E^*$. In other words, $V(x,v) = \phi(x, J^{-1}v)$ for all $x \in E$ and $v \in E^*$. In order to prove the main theorem of this section, we need the following two lemmas.

Lemma 3.1. [11] *Suppose that E is a strictly convex, smooth and reflexive Banach space and V is as defined in* (3.2). *Then*

$$V(x,v) \le V(x, v+w) - 2\langle J^{-1}(v) - x, w\rangle \tag{3.3}$$

for all $x \in E$ and $v, w \in E^*$.

Lemma 3.2. [18] *Suppose that $\{s_k\}$ is a sequence of nonnegative real numbers, $\{a_k\} \subset (0,1)$ where $\sum_{k=1}^{\infty} a_k = \infty$ and $\{t_k\} \subset \mathbb{R}$. Let*

$$s_{k+1} \le (1-a_k)s_k + a_k t_k, \qquad \forall\ k \ge 1.$$

If $\limsup_{m\to\infty} t_{k_m} \le 0$ for each subsequence $\{s_{k_m}\}$ of $\{s_k\}$ satisfying $\liminf_{m\to\infty}(s_{k_m+1} - s_{k_m}) \ge 0$, then $\lim_{k\to\infty} s_k = 0$.

Theorem 3.3. *Assume that E is a uniformly convex and uniformly smooth Banach space. Let $T_k : C \to C$ be a sequence of strongly quasi ϕ-nonexpansive mappings and $\{T_k\}$ be demiclosed. Then the sequence $\{x_k\}$ defined by (3.1), is strongly convergent to $P_{\bigcap_k F(T_k)}u$.*

Proof. Since $\bigcap_k F(T_k)$ is nonempty, closed and convex, we set $x^* := P_{\bigcap_k F(T_k)}u$. Note that

$$\begin{aligned}
\phi(x^*, x_{k+1}) &= \phi(x^*, J^{-1}(\alpha_k Ju + (1-\alpha_k)JT_k x_k)) \\
&= V(x^*, \alpha_k Ju + (1-\alpha_k)JT_k x_k) \le \alpha_k V(x^*, Ju) + (1-\alpha_k)V(x^*, JT_k x_k) \\
&\le \alpha_k \phi(x^*, u) + (1-\alpha_k)\phi(x^*, T_k x_k) \le \alpha_k \phi(x^*, u) + (1-\alpha_k)\phi(x^*, x_k) \\
&\le \max\{\phi(x^*, u), \phi(x^*, x_k)\} \le \cdots \le \max\{\phi(x^*, u), \phi(x^*, x_1)\},
\end{aligned}$$

which follows $\{\phi(x^*, x_k)\}$ is bounded. Thus, by (1.3), $\{x_k\}$ is bounded. On the other hand, by Lemma 3.1, we have

$$\begin{aligned}
\phi(x^*, x_{k+1}) &= V(x^*, \alpha_k Ju + (1-\alpha_k)JT_k x_k) \\
&\le V(x^*, \alpha_k Ju + (1-\alpha_k)JT_k x_k - \alpha_k(Ju - Jx^*)) \\
-2 &\ \langle J^{-1}(\alpha_k Ju + (1-\alpha_k)JT_k x_k) - x^*, -\alpha_k(Ju - Jx^*)\rangle \\
&= V(x^*, (1-\alpha_k)JT_k x_k + \alpha_k Jx^*) + 2\langle x_{k+1} - x^*, \alpha_k(Ju - Jx^*)\rangle \\
&\le (1-\alpha_k)V(x^*, JT_k x_k) + \alpha_k V(x^*, Jx^*) + 2\alpha_k\langle x_{k+1} - x^*, Ju - Jx^*\rangle \\
&= (1-\alpha_k)\phi(x^*, T_k x_k) + 2\alpha_k\langle x_{k+1} - x^*, Ju - Jx^*\rangle \\
&\le (1-\alpha_k)\phi(x^*, x_k) + 2\alpha_k\langle x_{k+1} - x^*, Ju - Jx^*\rangle.
\end{aligned}$$

We want to prove $\phi(x^*, x_k) \to 0$. By Lemma 3.2, it is enough to show that $\limsup_{m\to\infty}\langle x_{k_m+1} - x^*, Ju - Jx^*\rangle \le 0$ for each subsequence $\{\phi(x^*, x_{k_m})\}$ of $\{\phi(x^*, x_k)\}$ satisfying $\liminf_{m\to\infty}(\phi(x^*, x_{k_m+1}) - \phi(x^*, x_{k_m})) \ge 0$. Suppose that $\{\phi(x^*, x_{k_m})\}$ is a subsequence of $\{\phi(x^*, x_k)\}$ such that

$\liminf_{m\to\infty}(\phi(x^*, x_{k_m+1}) - \phi(x^*, x_{k_m})) \geq 0$. Then

$$\begin{aligned}
0 &\leq \liminf_{m\to\infty}(\phi(x^*, x_{k_m+1}) - \phi(x^*, x_{k_m})) \\
&= \liminf_{m\to\infty}(V(x^*, \alpha_{k_m} Ju + (1-\alpha_{k_m})JT_{k_m}x_{k_m}) - \phi(x^*, x_{k_m})) \\
&\leq \liminf_{m\to\infty}(\alpha_{k_m} V(x^*, Ju) + (1-\alpha_{k_m})V(x^*, JT_{k_m}x_{k_m}) - \phi(x^*, x_{k_m})) \\
&= \liminf_{m\to\infty}(\alpha_{k_m}\phi(x^*, u) + (1-\alpha_{k_m})\phi(x^*, T_{k_m}x_{k_m}) - \phi(x^*, x_{k_m})) \\
&= \liminf_{m\to\infty}(\alpha_{k_m}(\phi(x^*, u) - \phi(x^*, T_{k_m}x_{k_m})) + \phi(x^*, T_{k_m}x_{k_m}) - \phi(x^*, x_{k_m})) \\
&\leq \limsup_{m\to\infty}\alpha_{k_m}(\phi(x^*, u) - \phi(x^*, T_{k_m}x_{k_m})) + \liminf_{m\to\infty}(\phi(x^*, T_{k_m}x_{k_m}) - \phi(x^*, x_{k_m})) \\
&= \liminf_{m\to\infty}(\phi(x^*, T_{k_m}x_{k_m}) - \phi(x^*, x_{k_m})) \\
&\leq \limsup_{m\to\infty}(\phi(x^*, T_{k_m}x_{k_m}) - \phi(x^*, x_{k_m})) \leq 0.
\end{aligned}$$

So, we have

$$\lim_{m\to\infty}(\phi(x^*, T_{k_m}x_{k_m}) - \phi(x^*, x_{k_m})) = 0. \tag{3.4}$$

Hence, by the definition of strongly quasi ϕ-nonexpansive sequence,

$$\lim_{m\to\infty}\phi(T_{k_m}x_{k_m}, x_{k_m}) = 0. \tag{3.5}$$

In the sequel, by Proposition 1.1, we have

$$\lim_{m\to\infty}\|T_{k_m}x_{k_m} - x_{k_m}\| = 0. \tag{3.6}$$

Note that

$$\begin{aligned}
\phi(T_{k_m}x_{k_m}, x_{k_m+1}) &= V(T_{k_m}x_{k_m}, \alpha_{k_m}Ju + (1-\alpha_{k_m})JT_{k_m}x_{k_m}) \\
&\leq \alpha_{k_m}V(T_{k_m}x_{k_m}, Ju) + (1-\alpha_{k_m})V(T_{k_m}x_{k_m}, JT_{k_m}x_{k_m}) \\
&= \alpha_{k_m}\phi(T_{k_m}x_{k_m}, u).
\end{aligned}$$

Taking the limit we get,

$$\lim_{m\to\infty}\phi(T_{k_m}x_{k_m}, x_{k_m+1}) = 0.$$

Then Proposition 1.1 implies that

$$\lim_{m\to\infty}\|T_{k_m}x_{k_m} - x_{k_m+1}\| = 0. \tag{3.7}$$

Now, (3.6) and (3.7) implies that

$$\lim_{m\to\infty}\|x_{k_m} - x_{k_m+1}\| = 0. \tag{3.8}$$

On the other hand, there exists a subsequence $\{x_{k_{m_t}}\}$ of $\{x_{k_m}\}$ and $p \in E$ such that $x_{k_{m_t}} \rightharpoonup p$ and

$$\limsup_{m\to\infty}\langle x_{k_m} - x^*, Ju - Jx^*\rangle = \lim_{t\to\infty}\langle x_{k_{m_t}} - x^*, Ju - Jx^*\rangle = \langle p - x^*, Ju - Jx^*\rangle. \tag{3.9}$$

(2.1) implies that $p \in \bigcap_k F(T_k)$ because $x_{k_{m_t}} \rightharpoonup p$ and $\lim_{t\to\infty}\|T_{k_{m_t}}x_{k_{m_t}} - x_{k_{m_t}}\| = 0$ by (3.6). Now, since $\bigcap_k F(T_k)$ is closed and convex and $x^* = P_{\bigcap_k F(T_k)}u$, by

Proposition 1.3, we have $\langle p - x^*, Ju - Jx^* \rangle \leq 0$. From (3.8) and (3.9), we have

$$\begin{aligned}
&\limsup_{m\to\infty}\langle x_{k_m+1} - x^*, Ju - Jx^*\rangle = \limsup_{m\to\infty}\langle x_{k_m+1} - x_{k_m} + x_{k_m} - x^*, Ju - Jx^*\rangle \\
&\leq \limsup_{m\to\infty}\langle x_{k_m+1} - x_{k_m}, Ju - Jx^*\rangle + \limsup_{m\to\infty}\langle x_{k_m} - x^*, Ju - Jx^*\rangle \\
&= 0 + \langle p - x^*, Ju - Jx^*\rangle \leq 0.
\end{aligned}$$

Now, Lemma 3.2 implies that $\phi(x^*, x_k) \to 0$ and so, by Proposition 1.1, we get $x_k \to x^* = P_{\bigcap_k F(T_k)}u$. □

4. Applications

In this section, we apply our main results to approximate a minimizer of a convex function and a solution of an equilibrium problem. We also present some numerical examples.

4.1. **Application in convex minimization problems.** Assume that E is a uniformly convex and uniformly smooth Banach space. Let $f : E \to (-\infty, \infty]$ be a function, the domain of f is denoted by $D(f) := \{x \in E : f(x) < \infty\}$. f is said to be (weakly) lower semicontinuous at $x \in D(f)$ whenever

$$f(x) \leq \liminf_{k\to\infty} f(x_k)$$

for each sequence $x_k \to x$ ($x_k \rightharpoonup x$). In addition, f is called convex whenever

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \qquad \forall\, x, y \in E\ , 0 \leq t \leq 1.$$

It is known that lower semicontinuity together with convexity imply weak lower semicontinuity. We denote the set of all minimizers of f by Argmin$f$. It is well known that if f is convex and lsc, then Argmin$f$ is closed and convex. We denote the resolvent of f of order $\lambda > 0$ at $x \in E$ by

$$J_\lambda^f x := \mathrm{Argmin}_{y\in E}\{f(y) + \frac{1}{2\lambda}\|y\|^2 - \frac{1}{\lambda}\langle y, Jx\rangle\}. \tag{4.1}$$

By Rockafellar's theorem [16, 17], the subdifferential operator is maximal monotone. Hence for each $x \in E$, $J_\lambda^f x$ exists (see [11]). The proximal point method to approximate a minimum point of f is obtained by the iterations of J_λ^f on a given point $x \in E$. In other words, for a given $x_0 \in E$ and a sequence $\{\lambda_k\}$, we have

$$x_{k+1} = J_{\lambda_k}^f x_k. \tag{4.2}$$

This algorithm was first introduced by Martinet [13]. He studied the weak convergence of generated sequence to a minimizer of f(see also [4]). Xu [21] proved the strong convergence of the proximal point algorithm by the Halpern regularization method [5] in Hilbert spaces. Kohsaka and Takahashi [11] showed strong convergence of the proximal point method

$$y_k = \mathrm{Argmin}_{y\in E}\{f(y) + \frac{1}{2\lambda_k}\|y\|^2 - \frac{1}{\lambda_k}\langle y, Jx_k\rangle\} \tag{4.3}$$

$$x_{k+1} = J^{-1}(\alpha_k Ju + (1-\alpha_k)Jy_k),$$

to a minimizer of f where $u, x_1 \in E$ and $\{\alpha_k\} \subset (0,1)$ such that $\lim_{k\to\infty} \alpha_k = 0$ and $\sum_{k=1}^\infty \alpha_k = \infty$. (see Theorem 4.1 in [11]). In the sequel, we show that the above result can be obtained as a consequence of Theorem 3.3.

Lemma 4.1. *Let E be a uniformly convex and uniformly smooth Banach space and $f : E \to (-\infty, \infty]$ is a convex, proper and lsc function. If* $\text{Argmin} f \neq \varnothing$*, then* J^f_λ *is a strongly quasi ϕ-nonexpansive mapping.*

Proof. Since $J^f_\lambda x$ solves the minimization problem in (4.1), we have:

$$J^f_\lambda x \in \text{Argmin}_{y\in E}\{f(y) + \frac{1}{2\lambda}\|y\|^2 - \frac{1}{\lambda}\langle y, Jx\rangle\}.$$

Therefore

$$0 \in \partial\{f(\cdot) + \frac{1}{2\lambda}\|\cdot\|^2 - \frac{1}{\lambda}\langle \cdot, Jx\rangle\}(J^f_\lambda x).$$

Thus, there exist $w \in \partial f(J^f_\lambda x)$ such that

$$0 = w + \frac{1}{\lambda}J(J^f_\lambda x) - \frac{1}{\lambda}Jx. \tag{4.4}$$

On the other hand, we have

$$\langle y - J^f_\lambda x, w\rangle \leq f(y) - f(J^f_\lambda x). \tag{4.5}$$

By (4.4) and (4.5), we get

$$\frac{1}{\lambda}\langle y - J^f_\lambda x, Jx - J(J^f_\lambda x)\rangle \leq f(y) - f(J^f_\lambda x). \tag{4.6}$$

Now, take $p \in \text{Argmin} f$ and set $y = p$ in (4.6). Thus we have

$$0 \leq f(J^f_\lambda x) - f(p) \leq \frac{1}{\lambda}\langle p - J^f_\lambda x, J(J^f_\lambda x) - Jx\rangle. \tag{4.7}$$

In other words,

$$0 \leq 2\langle p - J^f_\lambda x, J(J^f_\lambda x) - Jx\rangle = \phi(p, x) - \phi(p, J^f_\lambda x) - \phi(J^f_\lambda x, x), \tag{4.8}$$

which implies that J^f_λ is a strongly quasi ϕ-nonexpansive mapping. □

Remark 1. If $x \in F(J^f_\lambda)$, i.e. x is a fixed point of the strongly quasi ϕ-nonexpansive mapping J^f_λ, then $x \in \text{Argmin} f$ by (4.6). Also, if $x \in \text{Argmin} f$, then by taking $y = x$ in (4.6), we have

$$\langle x - J^f_\lambda x, Jx - J(J^f_\lambda x)\rangle \leq 0.$$

This implies that $x = J^f_\lambda x$, i.e. $x \in F(J^f_\lambda)$. Therefore $\text{Argmin} f = F(J^f_\lambda)$.

Lemma 4.2. *Let E be a uniformly convex and uniformly smooth Banach space and $f : E \to (-\infty, \infty]$ is a convex, proper and lsc function. If* $\liminf_{k\to\infty} \lambda_k > 0$*, then* $J^f_{\lambda_k}$ *is demiclosed.*

Proof. Suppose that the sequence $\{x_k\}$ is arbitrary such that $x_k \rightharpoonup p$ and $\|J^f_{\lambda_k} x_k - x_k\| \to 0$. We will show that $p \in \bigcap_k F(J^f_{\lambda_k})$. Note that

$$J^f_{\lambda_k} x_k \in \text{Argmin}_{y\in E}\{f(y) + \frac{1}{2\lambda_k}\|y\|^2 - \frac{1}{\lambda_k}\langle y, Jx_k\rangle\}.$$

Therefore

$$0 \in \partial\{f(\cdot) + \frac{1}{2\lambda_k}\|\cdot\|^2 - \frac{1}{\lambda_k}\langle \cdot, Jx_k\rangle\}(J^f_{\lambda_k} x_k).$$

Thus, there exist $w_k \in \partial f(J^f_{\lambda_k} x_k)$ such that

$$0 = w_k + \frac{1}{\lambda_k}J(J^f_{\lambda_k} x_k) - \frac{1}{\lambda_k}Jx_k. \tag{4.9}$$

On the other hand,

$$\langle y - J^f_{\lambda_k} x, w_k\rangle \le f(y) - f(J^f_{\lambda_k} x_k). \tag{4.10}$$

By (4.9) and (4.10), we have

$$\frac{1}{\lambda_k}\langle y - J^f_{\lambda_k} x_k, Jx_k - J(J^f_{\lambda_k} x_k)\rangle \le f(y) - f(J^f_{\lambda_k} x_k). \tag{4.11}$$

Thus,

$$f(J^f_{\lambda_k} x_k) - \frac{1}{\lambda_k}\|y - J^f_{\lambda_k} x_k\|\|Jx_k - J(J^f_{\lambda_k} x_k)\| \le f(y). \tag{4.12}$$

Note that $x_k \rightharpoonup p$ and $\|J^f_{\lambda_k} x_k - x_k\| \to 0$ imply that $J^f_{\lambda_k} x_k \rightharpoonup p$. On the other hand, since E is uniformly smooth, the duality mapping J from E into E^* is uniformly norm-to-norm continuous on bounded sets. Therefore, $\|J^f_{\lambda_k} x_k - x_k\| \to 0$ implies that $\|Jx_k - J(J^f_{\lambda_k} x_k)\| \to 0$. Note that $\liminf_{k\to\infty} \lambda_k > 0$ and f is weakly lower semicontinuous. Taking liminf in (4.12) implies that $f(p) \le f(y)$ for all $y \in E$ and so, $p \in \mathrm{Argmin} f$. Now, Remark 1 shows that $p \in \bigcap_k F(J^f_{\lambda_k})$. □

Theorem 4.3. *Suppose that E is a uniformly convex and uniformly smooth Banach space and $f : E \to (-\infty, \infty]$ is a convex, proper and lsc function.*
If $\liminf_{k\to\infty} \lambda_k > 0$ and $\mathrm{Argmin} f \neq \varnothing$, then the sequence $\{x_k\}$ generated by

$$x_{k+1} = J^{-1}(\alpha_k Ju + (1-\alpha_k)J(J^f_{\lambda_k} x_k)), \tag{4.13}$$

is strongly convergent to $P_{\mathrm{Argmin} f}u$, where $u, x_0 \in E$ and $\{\alpha_k\} \subset (0,1)$ such that $\lim_{k\to\infty} \alpha_k = 0$ and $\sum_{k=0}^{\infty} \alpha_k = \infty$.

Proof. Lemma 4.2 implies that $J^f_{\lambda_k}$ satisfies (2.1). Also by Lemma 4.1, $J^f_{\lambda_k}$ is a strongly quasi ϕ-nonexpansive sequence. Now, Theorem 3.3 and Remark 1 imply that $\{x_k\}$ converges strongly to $P_{\mathrm{Argmin} f}u$. □

Now, we give a numerical example to illustrate an application of Theorem 4.3.

Example 2. We define the function $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) = xAx^t + xB$ where the matrix A is square matrix of order n such that A is positive semidefinite, the matrix B is a matrix of order $n \times 1$ and x^t denotes the transpose of the matrix x. Note that f is a convex function, because A is positive semidefinite. Also, f is proper and continuous and $\mathrm{Argmin} f \neq \varnothing$.

Now, in order to illustrate the application of Theorem 4.3 for this example, we take $n = 3$, $\lambda_k = \frac{k+1}{3k+1}$, $\alpha_k = \frac{1}{k+1}$, $u = (1, -2, 3) \in \mathbb{R}^3$ and $x_0 = (0, 0, 0) \in \mathbb{R}^3$. We also consider

$$A = \begin{bmatrix} 4 & 3 & 0 \\ -3 & 2 & 1 \\ 0 & -1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$$

If $\{x_k\}$ is the sequence generated by (4.13), then Theorem 4.3 ensures that $\{x_k\}$ converges to $P_{\mathrm{Argmin} f}u$. We performed our numerical experiment for this example. Our stopping criterion is $\|x_{k-1} - x_k\| < 10^{-6}$. The numerical results are displayed in the following table:

The sequence $x_k = (x_k^{(1)}, x_k^{(2)}, x_k^{(3)})$ generated by (4.13)			
k	$x_k^{(1)}$	$x_k^{(2)}$	$x_k^{(2)}$
1	0.666252888824739	-1.697935942128588	1.496124330268674
2	0.555500379244165	-1.599541316472595	0.999354063570689
3	0.480221068854458	-1.519447232569119	0.637470086251568
10	0.368013214249387	-1.412394122092918	0.122318208755844
100	0.287864300369334	-1.323884045982726	-0.268430483081751
200	0.254982343281192	-1.256991666280909	-0.474423108989319
1911	0.250628911992376	-1.251180446088091	-0.496413241298491

As we can see the sequence $\{x_k\}$ converges to the point $(0.25, -1.25, -0.5)$ which is the unique minimizer of the function f. Also, all tests for this problem corresponding to each starting point were successful, meaning that the sequence $\{x_k\}$ converges to $(0.25, -1.25, -0.5)$, which is the unique solution of the problem.
This problem was solved by the Optimization Toolbox in Matlab R2020a and performed on a Laptop with Intel(R) Core(TM) i3-4005U CPU @ 1.70 GHz, 1700 Mhz, 2 Core(s), 4 Logical Processor(s), Ram 4.00 GB.

4.2. **Application in equilibrium problems.** Let K be a nonempty, closed and convex subset of the Banach space E. Suppose that $f : K \times K \to \mathbb{R}$ is a bifunction. An equilibrium problem for f and K (shortly $EP(f;K)$) is to find $x^* \in K$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in K. \tag{4.14}$$

x^* is called an equilibrium point. We denote the set of all equilibrium points for (4.14) by $S(f;K)$. We now recall the definition of the monotone bifunction, $f : K \times K \to \mathbb{R}$ is called monotone, whenever $f(x,y) + f(y,x) \leq 0, \quad \forall x, y \in K$.

In [7], has been shown that if f satisfies in the following conditions;
(P1): $f(x,x) = 0$ for all $x \in K$,
(P2): $f(x,\cdot)$ is lower semi-continuous (lsc) and convex for all $x \in K$,
(P3): $f(\cdot,y)$ is upper semi-continuous for all $y \in K$,
(P4): f is monotone,
then for a given $x \in E$ and $\lambda > 0$, there is a unique point $J_\lambda^f x$ in K such that

$$f(J_\lambda^f x, y) + \lambda\langle y - J_\lambda^f x, J(J_\lambda^f x) - Jx\rangle \geq 0, \quad \forall y \in K. \tag{4.15}$$

$J_\lambda^f x$ is said to be the resolvent of f of order λ at $x \in E$ (see also [10]). In (4.15), it is easy to see that $F(J_\lambda^f) \subseteq S(f,K)$ and since f is a monotone bifunction, we have $S(f,K) \subseteq F(J_\lambda^f)$.

Consider $\{\lambda_k\} \subset (0, \alpha]$, for some $\alpha > 0$ and $x_0 \in E$. The proximal point method to approximate a solution of the problem is defined by $x_{k+1} = J_{\lambda_k}^f x_k$. We will prove weak convergence of the sequence generated by the algorithm to a solution of the problem.

Lemma 4.4. *Suppose that $f : K \times K \to \mathbb{R}$ satisfies P1-P4. If $S(f,K) \neq \varnothing$, then $J_{\lambda_k}^f$ is a strongly quasi ϕ-nonexpansive sequence.*

Proof. Take $p \in S(f,K)$ and set $y = p$ in (4.15). Then

$$f(J_{\lambda_k}^f x_k, p) + \lambda_k\langle p - J_{\lambda_k}^f x_k, J(J_{\lambda_k}^f x_k) - Jx_k\rangle \geq 0.$$

Since $p \in S(f,K)$ and f is monotone, we have $f(J^f_{\lambda_k}x_k, p) \leq 0$. Hence we get

$$\langle p - J^f_{\lambda_k}x_k, J(J^f_{\lambda_k}x_k) - Jx_k\rangle \geq 0,$$

which implies that

$$\phi(J^f_{\lambda_k}x_k, x_k) \leq \phi(p, x_k) - \phi(p, J^f_{\lambda_k}x_k).$$

Therefore, $J^f_{\lambda_k}$ is strongly quasi ϕ-nonexpansive. □

Lemma 4.5. *Suppose that $f : K \times K \to \mathbb{R}$ satisfies P1-P4, and $f(\cdot, y)$ is weakly upper semicontinuous for all $y \in K$. If $S(f,K) \neq \varnothing$, then $J^f_{\lambda_k}$ is demiclosed.*

Proof. Take $y \in K$. Assume that the sequence $\{x_k\}$ is arbitrary such that $x_k \rightharpoonup p$ and $\|J^f_{\lambda_k}x_k - x_k\| \to 0$. We will prove $p \in \bigcap_k F(J^f_{\lambda_k})$. We know that

$$\begin{aligned} 0 &\leq f(J^f_{\lambda_k}x_k, y) + \lambda_k\langle y - J^f_{\lambda_k}x_k, J(J^f_{\lambda_k}x_k) - Jx_k\rangle \\ &\leq f(J^f_{\lambda_k}x_k, y) + \lambda_k\|y - J^f_{\lambda_k}x_k\|\|J(J^f_{\lambda_k}x_k) - Jx_k\|. \end{aligned}$$

Note that $\lim_{k\to\infty}\|J(J^f_{\lambda_k}x_k) - Jx_k\| = 0$ and the sequences $\{x_k\}$ and $\{\lambda_k\}$ are bounded. Taking liminf, we have

$$0 \leq \liminf_{k\to\infty} f(J^f_{\lambda_k}x_k, y), \quad \forall y \in K. \tag{4.16}$$

On the other hand, since $\lim_{k\to\infty}\|J^f_{\lambda_k}x_k - x_k\| = 0$, we get $J^f_{\lambda_k}x_k \rightharpoonup p$. Now, since $f(\cdot, y)$ is weakly upper semicontinuous for every $y \in K$, we have

$$0 \leq \liminf_{k\to\infty} f(J^f_{\lambda_k}x_k, y) \leq \limsup_{k\to\infty} f(J^f_{\lambda_k}x_k, y) \leq f(p, y),$$

for every $y \in K$. Therefore, $p \in S(f,K)$ and since f is monotone, $p \in \bigcap_k F(J^f_{\lambda_k})$. □

Theorem 4.6. *Suppose that $f : K \times K \to \mathbb{R}$ satisfies P1-P4, and $f(\cdot, y)$ is weakly upper semicontinuous for all $y \in K$. If $S(f,K) \neq \varnothing$, then the sequence $\{x_k\}$ generated by (4.15) is weakly convergent to an element of $S(f,K)$.*

Proof. It follows from Lemmas 4.4, 4.5 and Theorem 2.2. □

Suppose that $\{\lambda_k\} \subset (0, \alpha]$ for some $\alpha > 0$ and $x_0 \in E$. The Halpern regularization method for the equilibrium problem $EP(f,K)$ is defined by

$$\begin{aligned} &f(J^f_{\lambda_k}x_k, y) + \lambda_k\langle y - J^f_{\lambda_k}x_k, J(J^f_{\lambda_k}x_k) - Jx_k\rangle \geq 0, \quad \forall y \in K, \\ &x_{k+1} = J^{-1}(\alpha_k Ju + (1-\alpha_k)J(J^f_{\lambda_k}x_k)), \end{aligned} \tag{4.17}$$

where $u \in E$ and $\{\alpha_k\} \subset (0,1)$ such that $\lim_{k\to\infty}\alpha_k = 0$ and $\sum_{k=1}^{\infty}\alpha_k = \infty$ (see [10]). In the following theorem, we prove the strong convergence of the sequence $\{x_k\}$ generated by (4.17) to a solution of $EP(f,K)$. In fact, we prove $x_k \to x^* = P_{S(f,K)}u$.

Theorem 4.7. *Suppose that $f : K \times K \to \mathbb{R}$ satisfies P1-P4, and $f(\cdot, y)$ is weakly upper semicontinuous for all $y \in K$. If $S(f,K) \neq \varnothing$, then the sequence $\{x_k\}$ generated by (4.17) is strongly convergent to $P_{S(f,K)}u$.*

Proof. It follows from Lemmas 4.4, 4.5 and Theorem 3.3. □

Remark 2. Suppose that $\psi : E \to (-\infty, \infty]$ is a convex, proper and lsc function. If we define $f(x,y) = \psi(y) - \psi(x)$ and $K = E$, then it is obvious that each equilibrium point of f coincides with a minimizer of ψ and vice versa. Also, Theorem 4.3 is a consequence of Theorem 4.7.

In the following example, we illustrate an application of Theorem 4.7.

Example 3. Consider $K = [-10,10]\times[-10,10]\times[-10,10]$ and define the bifunction $f : K \times K \to \mathbb{R}$ by

$$f(x,y) = y_1^2 - 4y_1 + 4y_2^2 + 12y_2 + y_3^2 + 2y_3 - (x_1^2 - 4x_1 + 4x_2^2 + 12x_2 + x_3^2 + 2x_3).$$

It is easy to see that the conditions B1-B4 are satisfied and $S(f,K) \neq \varnothing$. Now, in order to illustrate the application of Theorem 4.7 for this example, we take $\lambda_k = \frac{4k}{k+1}$, $\alpha_k = \frac{1}{k+1}$, $u = (2,5,-3)$ and $x_0 = (0,0,0)$. If $\{x_k\}$ is the sequence generated by (4.17), then Theorem 4.7 ensures that $\{x_k\}$ converges to an element of $S(f,K)$. We performed our numerical experiment for this example. Our stopping criterion is $\|x_{k-1} - x_k\| < 10^{-6}$. The numerical results are displayed in the following table:

The sequence $x_k = (x_k^{(1)}, x_k^{(2)}, x_k^{(3)})$ generated by (4.17)			
k	$x_k^{(1)}$	$x_k^{(2)}$	$x_k^{(2)}$
1	1.983867940406904	1.842430813155550	-2.050808110550836
2	1.994622561587510	1.112324059072868	-2.016936040457454
3	1.997695352885247	0.614810768998420	-1.935829709932863
10	1.998742889206285	-0.384757654195337	-1.692270714833297
100	1.999197008151127	-1.090580904077409	-1.462006175733657
1000	1.999843209005453	-1.486155358408404	-1.096048403571341
2604	1.999873160521294	-1.493532995035196	-1.037381383434542

Note that the sequence $\{x_k\}$ converges to the point $(2,-1.5,-1)$ which is the unique equilibrium point of the bifunction f. Also, all tests for this problem corresponding to each starting point were successful, meaning that the sequence $\{x_k\}$ converges to $(2,-1.5,-1)$, which is the unique solution of the problem.
This problem was solved by the Optimization Toolbox in Matlab R2020a and performed on a Laptop with Intel(R) Core(TM) i3-4005U CPU @ 1.70 GHz, 1700 Mhz, 2 Core(s), 4 Logical Processor(s), Ram 4.00 GB.

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