Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 12 Issue 3 (2021), Pages 13-22.

ON COEFFICIENTS PROBLEMS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

AFIS SALIU, KHALIDA INAYAT NOOR

Abstract. In this present investigation, we introduce a new subclass of analytic functions initiated by both Sălăgean differential and two-parameters Komatu integral operators. Furthermore, the Fekete Szegö inequality and upper bound of the third Hankel determinant for such defined functions are obtained. For the validity of our results, relevant connections with those in earlier work are pointed out.

1. Introduction

Let A be the class of analytic functions $f(z)$ given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

Denoted by ST and CV are subclasses of A, consisting of functions that map an open unit disk E onto a star-shaped and convex domains respectively.

Sălăgean [22] introduced the operator D^n defined by

$$D^n f(z) = z\left[D^{n-1} f(z)\right]', \quad n \in \mathbb{N} \cup \{0\} \text{ and } D^0 f((z) = f(z) \tag{1.2}$$

$$= z + \sum_{k=2}^{\infty} k^n a_k z^k \tag{1.3}$$

and used it to generalized the concept of starlike and convex functions in E. The two parameters family $K_\delta^\lambda\colon A \longrightarrow A$ of integral operator defined by

$$K_\delta^\lambda f(z) = \frac{\delta^\lambda}{\Gamma(\lambda) z^{\delta-1}} \int_0^z \xi^{\delta-2} \left(\log\left(\frac{z}{\xi}\right)\right)^{\lambda-1} f(\xi) d\xi\,, \quad (z \in E,\ \delta > 0, \lambda \geq 0,\ \ f \in A),$$

$$= z + \sum_{k=2}^{\infty} \left(\frac{\delta}{\delta + k - 1}\right)^{\lambda} a_k z^k \tag{1.4}$$

was first introduced by Komatu [10]. This operator satisfies the identity

$$z\left(K^{\lambda+1} f(z)\right)' = \delta K_\delta^\lambda f(z) - (\delta - 1) K_\delta^{\lambda+1} f(z)$$

2010 *Mathematics Subject Classification.* 30C45, 30C50, 30C55.

Key words and phrases. Analytic Function, Subordination, Quasi Subordination, Hankel Determinant.

Submitted October 23, 2020. Published May 5, 2021.
Communicated by H.M.Srivastava.

and unifies several linear operators introduced by many researchers. For example:

(i) $K_1^1 f(z) = \mathcal{A}[f](z)$ is the Alexander operator [1].
(ii) $K_2^1 f(z) = \mathcal{L}[f](z)$ is the Liberal operator [12].
(iii) $K_{c+1}^1 f(z) = \mathcal{B}[f](z)$ is the Bernadi operator [4].
(iv) $K_2^\lambda f(z) = \mathcal{J}[f](z)$ is the one parameter Jung- Kim- Srivastava integral operator [8].

In order to disprove the Littlewood and Parley conjecture of 1932, that the coefficients of odd univalent functions are bounded by 1, Fekete and Szegö proved that for normalized univalent functions given by (1.1) in E,

$$\left|a_3 - \mu a_2^2\right| \leq 1 + 2e^{\frac{-2\mu}{1-\mu}}, \quad 0 \leq \mu < 1.$$

Problems of this kind are known as Fekete Szegö problems. The functional $|a_3 - \mu a_2^2|$ has been receiving attention, particularly in several subclasses of the family of univalent functions (see [16, 21, 23, 24, 25, 26, 27, 28, 29, 30]).

Noonan and Thomas [20] define for $q \geq 1, n \geq 1$, the qth Hankel determinant of $f(z) \in H$ as follows:

$$\mathcal{H}_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.5}$$

This determinant has been studied by many researchers. In particular Babalola [3] obtained the sharp bounds of $\mathcal{H}_3(1)$ for the classes ST and CV. Also, the bound of $\mathcal{H}_2(1)$ for a subclass of A defined by Komatu integral operator was obtained by Mohapatra and Panigrahi in [18].

Let $f, g \in A$. We say $f(z)$ is subordinate to $g(z)$ (written as $f(z) \prec g(z)$) if there exists an analytic function $w(z) \in E$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in E$ such that $f(z) = g(w(z))$. Further, $f(z)$ is said to be quasi-subordinate to $g(z)$ in E if there exist analytic functions $h(z) \in E$ with $|h(z)| \leq 1$ and $w(z) \in E$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = h(z)g(w(z))$, written as

$$\frac{f(z)}{h(z)} \prec g(z) \quad \text{or} \quad f(z) \prec_q g(z).$$

If we set $w(z) = z$, we say $f(z)$ is majorized by $g(z)$ written as

$$f(z) \ll g(z).$$

Motivated by the work in [2, 7, 11], we define the operator $B^n_{\lambda,\delta}: A \longrightarrow A$ as follows:

Definition 1.1. *Let $f \in A$. The operator $B^n_{\lambda,\delta}$ is defined as:*

$$\begin{aligned} B^n_{\lambda,\delta} f(z) =& D^n \left(K_\delta^\lambda f(z)\right) \\ =& z + \sum_{k=2}^{\infty} k^n \left(\frac{\delta}{\delta + k - 1}\right)^\lambda a_k z^k. \end{aligned} \tag{1.6}$$

We note that $B^0_{\lambda,\delta} f(z) = K_\delta^\lambda f(z)$, $B^n_{0,\delta} f(z) = D^n f(z)$ and $B^0_{0,\delta} f(z) = f(z)$.

Let $\phi(z)$ be analytic in E with $\phi(0) = 1$ and $\phi'(0) > 0$. Then using the operator $B^n_{\lambda,\delta}$, we define the following class of analytic functions:

Definition 1.2. *Let $f \in A$, $b \in \mathbb{C} \setminus \{0\}$, $\alpha \geq 0$, $0 \leq \beta \leq 1$. Then $f \in M^{n,b}_{\lambda,\delta}(\alpha, \beta, h, \phi)$ if it satisfies the quasi-subordination*

$$\frac{1}{b}\left\{(1-\alpha)\frac{B^{n+1}_{\lambda,\delta}F_\beta(z)}{B^{n}_{\lambda,\delta}F_\beta(z)} + \alpha\frac{B^{n+2}_{\lambda,\delta}F_\beta(z)}{B^{n+1}_{\lambda,\delta}F_\beta(z)} - 1\right\} \prec_q \phi(z) - 1, \tag{1.7}$$

where

$$F_\beta(z) = (1-\beta)f(z) + \beta z f'(z). \tag{1.8}$$

For certain values of the parameters, $h(z)$ and $\phi(z)$, we obtain the well-known subclasses of analytic functions studied in [2, 5, 7, 15, 17] and the subclass of A for which $n = 0$, $b = 1, \alpha = 0$, $h(z) = 1$, and $\phi(z) = \frac{1+z}{1-z}$ is denoted by $M^{\lambda}_{\delta}(\beta)$ [18]. We also obtain some new subclasses of A by specializing certain parameters as follows: (i) For $\alpha = 0$, we have the class $ST^{n,b}_{\lambda,\delta}(\beta, h, \phi)$ defined as:

$$ST^{n,b}_{\lambda,\delta}(\beta, h, \phi) = \left\{f \in A\colon \frac{1}{b}\left(\frac{B^{n+1}_{\lambda,\delta}F_\beta(z)}{B^{n}_{\lambda,\delta}F_\beta(z)} - 1\right) \prec_q \phi(z) - 1\right\}.$$

(ii) For $\alpha = 1$, we have the class $CV^{n,b}_{\lambda,\delta}(\beta, h, \phi)$ defined as:

$$CV^{n,b}_{\lambda,\delta}(\beta, h, \phi) = \left\{f \in A\colon \frac{1}{b}\left(\frac{B^{n+2}_{\lambda,\delta}F_\beta(z)}{B^{n+1}_{\lambda,\delta}F_\beta(z)} - 1\right) \prec_q \phi(z) - 1\right\}.$$

(iii) For $b = (1-\rho)e^{-i\theta}\cos\theta$, $0 \leq \rho < 1$, $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$, we have the class $M^{n,\rho}_{\lambda,\delta}(\beta, h, \phi)$ defined as:

$$M^{n,\rho}_{\lambda,\delta}(\beta, h, \phi) = \left\{f \in A\colon \left(\frac{(1-\alpha)\frac{B^{n+1}_{\lambda,\delta}F_\beta(z)}{B^{n}_{\lambda,\delta}F_\beta(z)} + \alpha\frac{B^{n+2}_{\lambda,\delta}F_\beta(z)}{B^{n+1}_{\lambda,\delta}F_\beta(z)}}{e^{-i\theta}(1-\rho)\cos\theta} - \frac{(\rho\cos\theta + i\sin\theta)}{(1-\rho)\cos\theta}\right) \prec_q \phi(z) - 1\right\}.$$

(iv) For $\alpha = 0$, $b = (1-\rho)e^{-i\theta}\cos\theta$, $0 \leq \rho < 1$, $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$, we have the class $ST^{n,\rho}_{\lambda,\delta}(\beta, h, \phi)$ defined as:

$$ST^{n,\rho}_{\lambda,\delta}(\beta, h, \phi) = \left\{f \in A\colon \left(\frac{e^{i\theta}}{(1-\rho)\cos\theta}\left[\frac{B^{n+1}_{\lambda,\delta}F_\beta(z)}{B^{n}_{\lambda,\delta}F_\beta(z)}\right] - \frac{(\rho\cos\theta + i\sin\theta)}{(1-\rho)\cos\theta}\right) \prec_q \phi(z) - 1\right\}.$$

(v) For $\alpha = 1$, $b = (1-\rho)e^{-i\theta}\cos\theta$, $0 \leq \rho < 1$, $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$, we have the class $CV^{n,\rho}_{\lambda,\delta}(\beta, h, \phi)$ defined as:

$$CV^{n,\rho}_{\lambda,\delta}(\beta, h, \phi) = \left\{f \in A\colon \left(\frac{e^{i\theta}}{(1-\rho)\cos\theta}\left[\frac{B^{n+2}_{\lambda,\delta}F_\beta(z)}{B^{n+1}_{\lambda,\delta}F_\beta(z)}\right] - \frac{(\rho\cos\theta + i\sin\theta)}{(1-\rho)\cos\theta}\right) \prec_q \phi(z) - 1\right\}.$$

The following lemmas are required to establish our main results.

2. A Set of Lemmas

Let $\mathcal{P}$ be the class of functions $p(z)$ of positive real part of the form

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 \cdots \in E. \tag{2.1}$$

Lemma 2.1. [9] *Let w be the analytic function in E, with $w(0) = 0$, $|w(z)| < 1$ and $w(z) = w_1 z + w_2 z^2 + \ldots$, Then*

(i)
$$|w_2 - \tau w_1^2| \leq \max\{1, |\tau|\},$$
(ii)
$$|w_n| \leq \begin{cases} 1, & n = 1 \\ 1 - |w_1|^2, & n \geq 2, \end{cases}$$
where $\tau \in \mathbb{C}$. The results are sharp for the functions $w(z) = z$ and $w(z) = z^2$.

Lemma 2.2. [19] *Let $h(z)$ be the analytic function in E, with $|h(z)| < 1$ and $h(z) = h_0 + h_1 z + h_2 z^2 + \dots,$ Then*
$$|h_n| \leq \begin{cases} 1, & n = 0 \\ 1 - |h_0|^2, & n > 0. \end{cases}$$

Lemma 2.3. [6] *Let $p \in \mathcal{P}$. Then $|c_n| \leq 2$, and the inequality is sharp.*

Lemma 2.4. [13, 14] *Let $p(z) \in \mathcal{P}$ be of the form (2.1). Then*
$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{2.2}$$
and
$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - (4 - c_1^2)c_1 x^2 + 2(4 - c_1^2)(1 - |x|^2)z, \tag{2.3}$$
for some x, z with $|x| \leq 1$ and $|z| \leq 1$.

Unless otherwise stated, we suppose throughout this work that $h(z) = h_0 + h_1 z + h_2 z^2 + \dots,$ $w(z) = w_1 z + w_2 z^2 + \dots,$ $\phi(z) = 1 + b_1 z + b_2 z^2 + \dots,$ $b_1 > 0, \delta > 0, \alpha \geq 0,$ $\beta \in [0, 1]$, $b \in \mathbb{C} \setminus \{0\}, n \in \mathbb{N} \cup \{0\}$, $\lambda \geq 0$.

3. Main Results

Theorem 3.1. *If $f \in M_{\lambda,\delta}^{n,b}(\alpha, \beta, h, \phi)$, then for $\mu \in \mathbb{C}$,*

$$|a_3 - \mu a_2^2| \leq \frac{|b|\left(b_1 + \max\left\{b_1, |b_2| + \frac{\left|2^{2n}\left(\frac{\delta}{1+\delta}\right)^{\lambda}(\beta+1)^2(3\alpha+1) - 3^n\mu(2\beta+1)(4\alpha+2)\right| b_1^2 |b|}{2^{2n}\left(\frac{\delta}{1+\delta}\right)^{\lambda}(\beta+1)^2(\alpha+1)^2}\right\}\right)}{3^n\left(\frac{\delta}{\delta+1}\right)^{\lambda}(2\beta+1)(4\alpha+2)} \tag{3.1}$$

The result is sharp.

Proof. Let $f \in M_{\lambda,\delta}^{n,b}(\alpha, \beta, h, \phi)$ Then by Definition 1.2,
$$\frac{1}{b}\left\{(1-\alpha)\frac{B_{\lambda,\delta}^{n+1}F_\beta(z)}{B_{\lambda,\delta}^{n}F_\beta(z)} + \alpha\frac{B_{\lambda,\delta}^{n+2}F_\beta(z)}{B_{\lambda,\delta}^{n+1}F_\beta(z)}\right\} = h(z)(\phi(w(z)) - 1), \tag{3.2}$$
for some analytic funtions $h(z)$ and $w(z) \in E$ and
$$(\phi(w(z)) - 1) = b_1 h_0 w_1 z + \left[b_1 h_1 w_1 + h_0 b_1 w_2 + h_0 b_2 w_1^2\right] z^2 + \dots . \tag{3.3}$$
From (3.2) and (3.3), we obtain
$$a_2 = \frac{b_1 h_0 w_1 b}{2^n\left(\frac{\delta}{\delta+1}\right)^{\lambda}(1+\beta)(1+\alpha)}, \quad a_3 = \frac{b b_1 h_0\left(\frac{h_1}{h_0}w_1 + w_2 + \left(\frac{b_2}{b_1} + \frac{b_1 h_0 b(1+3\alpha)}{(1+\alpha)^2}\right)w_1^2\right)}{3^n\left(\frac{\delta}{\delta+2}\right)^{\lambda}(1+2\beta)(2+4\alpha)}$$

and for $\mu \in \mathbb{C}$

$$|a_3 - \mu a_2^2| = \left| \frac{bb_1 h_0 \left\{ \frac{h_1}{h_0} w_1 + w_2 + \left(\left[\frac{2^{2n}\left(\frac{\delta}{\delta+2}\right)^{\lambda}(1+\beta)^2(1+3\alpha) - 3^n(2\beta+1)(4\alpha+2)\mu}{2^{2n}\left(\frac{\delta}{\delta+2}\right)^{\lambda}(1+\beta)^2(1+\alpha)^2} \right] b_1 h_1 b - \frac{b_2}{b_1} \right) w_1^2 \right\}}{3^n \left(\frac{\delta}{\delta+2}\right)^{\lambda} (2\beta+1)(4\alpha+2)} \right|$$
$$\le \frac{|bb_1| \left\{ |h_1 w_1| + |h_0| \left| w_2 + \left(\left[\frac{2^{2n}\left(\frac{\delta}{\delta+2}\right)^{\lambda}(1+\beta)^2(1+3\alpha) - 3^n(2\beta+1)(4\alpha+2)\mu}{2^{2n}\left(\frac{\delta}{\delta+2}\right)^{\lambda}(1+\beta)^2(1+\alpha)^2} \right] b_1 h_1 b - \frac{b_2}{b_1} \right) w_1^2 \right| \right\}}{3^n \left(\frac{\delta}{\delta+2}\right)^{\lambda} (2\beta+1)(4\alpha+2)}. \tag{3.4}$$

In view of Lemma 2.1 and 2.2, we obtain (3.1) and the result is sharp for the function defined by

$$\frac{1}{b}\left\{ (1-\alpha)\frac{B_{\lambda,\delta}^{n+1} F_\beta(z)}{B_{\lambda,\delta}^{n} F_\beta(z)} + \alpha \frac{B_{\lambda,\delta}^{n+2} F_\beta(z)}{B_{\lambda,\delta}^{n+1} F_\beta(z)} \right\} = h(z)(\phi(w_i(z)) - 1),\ i = 1, 2, \tag{3.5}$$

where $w_1(z) = z^2,\ w_2(z) = z.$ □

For $\alpha = 0$ and $\alpha = 1$ in Theorem 3.1, we obtain the following:

Corollary 3.2. *If $f \in ST_{\lambda,\delta}^{n,b}(\beta, h, \phi)$, then*

$$|a_3 - \mu a_2^2| \le \frac{|b| \left(b_1 + \max\left\{ b_1, |b_2| + \frac{\left| 2^{2n}\left(\frac{\delta}{1+\delta}\right)^{\lambda}(\beta+1)^2 - 3^n \mu (4\beta+2) \right| b_1^2 |b|}{2^{2n}\left(\frac{\delta}{1+\delta}\right)^{\lambda}(\beta+1)^2} \right\} \right)}{3^n \left(\frac{\delta}{\delta+1}\right)^{\lambda} (4\beta+2)}.$$

The result is sharp.

Corollary 3.3. *If $f \in CV_{\lambda,\delta}^{n,b}(\beta, h, \phi)$, then*

$$|a_3 - \mu a_2^2| \le \frac{|b| \left(b_1 + \max\left\{ b_1, |b_2| + \frac{\left| 2^{2n+2}\left(\frac{\delta}{1+\delta}\right)^{\lambda}(\beta+1)^2 - 3^{n+1} \mu (4\beta+2) \right| b_1^2 |b|}{2^{2n+2}\left(\frac{\delta}{1+\delta}\right)^{\lambda}(\beta+1)^2} \right\} \right)}{3^{n+1} \left(\frac{\delta}{\delta+1}\right)^{\lambda} (4\beta+2)}.$$

The result is sharp.

Setting $b = (1-\rho)e^{-i\theta}\cos\theta,\ o \le \rho < 1,\ |\theta| < \frac{\pi}{2}$ in Theorem 3.1, we obtain

Corollary 3.4. *If $f \in M_{\lambda,\delta}^{n,\rho}(\alpha, \beta, h, \phi)$, then*

$$|a_3 - \mu a_2^2| \le \frac{(1-\rho)\cos\theta \left(b_1 + \max\left\{ b_1, |b_2| + \frac{\left| 2^{2n}\left(\frac{\delta}{1+\delta}\right)^{\lambda}(\beta+1)^2(3\alpha+1) - 3^n \mu (2\beta+1)(4\alpha+2) \right| b_1^2 (1-\rho)\cos\theta}{2^{2n}\left(\frac{\delta}{1+\delta}\right)^{\lambda}(\beta+1)^2(\alpha+1)^2} \right\} \right)}{3^n \left(\frac{\delta}{\delta+1}\right)^{\lambda} (2\beta+1)(4\alpha+2)}.$$

The result is sharp.

Putting $\alpha = 0,\ \alpha = 1,\ b = (1-\rho)e^{-i\theta}\cos\theta$ in Theorem 3.1, we get

Corollary 3.5. *If $f \in ST^{n;\rho}_{\lambda,\delta}(\beta, h, \phi)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{(1-\rho)\cos\theta\left(b_1 + \max\left\{b_1, |b_2| + \frac{\left|2^{2n}\left(\frac{\delta}{1+\delta}\right)^{\lambda}(\beta+1)^2 - 3^n\mu(4\beta+2)\right| b_1^2 (1-\rho)\cos\theta}{2^{2n}\left(\frac{\delta}{1+\delta}\right)^{\lambda}(\beta+1)^2}\right\}\right)}{3^n\left(\frac{\delta}{\delta+1}\right)^{\lambda}(4\beta+2)}.$$

The result is sharp.

Corollary 3.6. *If $f \in CV^{n,\rho}_{\lambda,\delta}(\beta, h, \phi)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{(1-\rho)\cos\theta\left(b_1 + \max\left\{b_1, |b_2| + \frac{\left|2^{2n+2}\left(\frac{\delta}{1+\delta}\right)^{\lambda}(\beta+1)^2 - 3^{n+1}\mu(4\beta+2)\right| b_1^2 (1-\rho)\cos\theta}{2^{2n+2}\left(\frac{\delta}{1+\delta}\right)^{\lambda}(\beta+1)^2}\right\}\right)}{3^{n+1}\left(\frac{\delta}{\delta+1}\right)^{\lambda}(4\beta+2)}.$$

The result is sharp.

Theorem 3.7. *If $f(z) \in A$ satisfies the majorization condition*

$$\frac{1}{b}\left\{(1-\alpha)\frac{B^{n+1}_{\lambda,\delta}F_\beta(z)}{B^{n}_{\lambda,\delta}F_\beta(z)} + \alpha\frac{B^{n+2}_{\lambda,\delta}F_\beta(z)}{B^{n+1}_{\lambda,\delta}F_\beta(z)}\right\} \ll \phi(z) - 1,$$

then

$$|a_3 - \mu a_2^2| \leq \frac{|b|\left(b_1 + |b_2| + \frac{\left|2^{2n}\left(\frac{\delta}{1+\delta}\right)^{\lambda}(\beta+1)^2(3\alpha+1) - 3^n\mu(2\beta+1)(4\alpha+2)\right| b_1^2|b|}{2^{2n}\left(\frac{\delta}{1+\delta}\right)^{\lambda}(\beta+1)^2(\alpha+1)^2}\right)}{3^n\left(\frac{\delta}{\delta+1}\right)^{\lambda}(2\beta+1)(4\alpha+2)}.$$

The result is sharp.

Proof. Put $w(z) = z$ in the proof of Theorem 3.1. □

Theorem 3.8. *If $f \in M^{\lambda}_{\delta}(\beta)$, then*

$$|a_3| \leq \frac{3(\delta+2)^{\lambda}}{\delta^{\lambda}(1+2\beta)}, \quad |a_4| \leq \frac{4(\delta+3)^{\lambda}}{\delta^{\lambda}(1+3\beta)}, \quad |a_5| \leq \frac{5(\delta+4)^{\lambda}}{\delta^{\lambda}(1+4\beta)} \tag{3.6}$$

and

$$|\mathcal{H}_3(1)| \leq \begin{cases} \mathcal{R}_1 + \mathcal{R}_2 + \frac{16(\delta+3)^{\lambda}(6q-1)Q(\delta,\lambda,\beta)}{3\delta^{\lambda}(1+3\beta)}\sqrt{\frac{6q-1}{3(3q-1)}}, & if \quad q \neq \frac{1}{3}, \\ \mathcal{R}_1 + \mathcal{R}_2 + \frac{16(\delta+3)^{\lambda}Q(\delta,\lambda,\beta)}{\delta^{\lambda}(1+3\beta)}, & if \quad q = \frac{1}{3}, \end{cases} \tag{3.7}$$

where

$$\mathcal{R}_1 = \left[\frac{(\delta+1)(\delta+2)(\delta+3)}{\delta^3}\right]^{\lambda}\frac{3}{(1+\beta)(1+2\beta)(1+3\beta)}, \quad \mathcal{R}_2 = \frac{10(\delta+4)^{\lambda}Q_1(\delta,\lambda,\beta)}{\delta^{\lambda}(1+4\beta)}, \tag{3.8}$$

and

$$Q(\delta,\lambda,\beta) = \frac{[(1+\delta)(2+\delta)]^{\lambda}}{2\delta^{2\lambda}(1+\beta)(1+2\beta)}, \quad Q_1(\delta,\lambda,\beta) = \frac{(\delta+2)^{\lambda}}{2\delta^{\lambda}(1+2\beta)}, \quad q = \frac{\delta^{\lambda}(\delta+3)^{\lambda}(1+\beta)(1+2\beta)}{3\left[(\delta+1)(\delta+2)\right]^{\lambda}(1+3\beta)}. \tag{3.9}$$

Proof. Let $f \in M_\delta^\lambda(\beta)$. Then there exists a function $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 \cdots \in \mathcal{P}$ such that

$$\frac{z\Big(\beta z(K_\delta^\lambda f(z))' + (1-\beta)K_\delta^\lambda f(z)\Big)'}{\beta z(K_\delta^\lambda f(z))' + (1-\beta)K_\delta^\lambda f(z)} = p(z). \tag{3.10}$$

By coefficients comparison of (3.10), it follows that

$$\begin{aligned} a_2 =& \frac{c_1(\delta+1)^\lambda}{\delta^\lambda(1+\beta)}, \quad a_3 = \frac{(\delta+2)^\lambda(c_1^2+c_2)}{2\delta^\lambda(1+2\beta)}, \quad a_4 = \frac{(\delta+3)^\lambda(2c_3+3c_1c_2+c_1^3)}{6\delta^\lambda(1+3\beta)}, \\ a_5 =& \frac{(\delta+4)^\lambda(8c_1c_3+6c_1^2c_2+c_1^4+3c_2^2+6c_4)}{24\delta^\lambda(1+4\beta)}. \end{aligned}$$

On using Lemma 2.3, we obtain (3.6). Next,

$$|a_2a_3 - a_4| = \left| \frac{c_1^3 + c_1c_2}{2\left(\frac{\delta}{\delta+1}\right)^\lambda \left(\frac{\delta}{\delta+2}\right)^\lambda (1+\beta)(1+2\beta)} - \frac{c_1^3 + 3c_1c_2 + 2c_3}{6\left(\frac{\delta}{\delta+3}\right)^\lambda (1+3\beta)} \right|. \tag{3.11}$$

Substituting for c_2, c_3 from Lemma 2.4 in (3.11) and by careful simplification, and applying triangular inequality with $c_1 = c$ $(0 \le c \le 2)$, $|x| = \rho$ $(0 \le \rho \le 1)$, we get

$$\begin{aligned} |a_2a_3 - a_4| \le& \frac{Q(\delta,\lambda,\beta)}{2}\Big\{(3-6q)c^3 + (5q-1)(4-c^2)c\rho + qc(4-c^2)\rho^2 + 2q(4-c^2)(1-\rho^2)\Big\} \\ :=& F(c,\rho), \end{aligned} \tag{3.12}$$

where $Q(\delta,\lambda,\beta)$, q are given by (3.9) and $q \in [\frac{1}{3}, \frac{1}{2}]$ for $0 \le \beta \le 1$ and $\lambda = 0$.

$$\therefore \frac{\partial F(c,\rho)}{\partial \rho} = \frac{Q(\delta,\lambda,\beta)}{2}\left\{(5q-1)(4-c^2)c + 2q(c-2)(4-c^2)\rho\right\} > 0 \quad \text{for} \quad c \in [1,2].$$

This means that $F(c,\rho)$ is an increasing function of ρ on the interval $[0,1]$. Thus

$$F(c,\rho) \le F(c,1).$$

Therefore,

$$\begin{aligned} |a_2a_3 - a_4| \le& \frac{Q(\delta,\lambda,\beta)}{2}\left\{(4-12q)c^3 + 4(6q-1)c\right\} \\ :=& G(c) \end{aligned}$$

where

$$G'(c) = Q(\delta,\lambda,\beta)\left\{3(2-6q)c^2 + 2(6q-1)\right\} \quad \text{and} \quad G''(c) = 3Q(\delta,\lambda,\beta)(4-12q)c \le 0, \quad \text{since} \quad q \in \left[\frac{1}{3}, \frac{1}{2}\right].$$

For $c \in [1,2]$, it follows that $G(c)$ attains maximum at $c = \sqrt{\frac{6q-1}{3(3q-1)}}$. Thus

$$|a_2a_3 - a_4| \le \begin{cases} 4Q(\delta,\lambda,\beta) & \text{if } q = \frac{1}{3}, \\ \frac{4}{3}(6q-1)Q(\delta,\lambda,\beta)\sqrt{\frac{6q-1}{3(3q-1)}} & \text{if } q \ne \frac{1}{3}. \end{cases} \tag{3.13}$$

Using Lemma 2.4 and performing some simplifications, we have that

$$|a_3 - a_2^2| = \frac{Q_1(\delta,\lambda,\beta)}{2}\left|-(2q_1-3)c_1^2 + x(4-c_1^2)\right|,$$

where $Q_1(\delta,\lambda,\beta)$, is given by (3.9) and

$$q_1 = \frac{2(\delta+1)^{2\lambda}(1+2\beta)}{\delta^\lambda(\delta+2)^\lambda(1+\beta)^2} \in \left[\frac{3}{2}, 2\right]$$

for $0 \le \beta \le 1$ and $\lambda = 0$. Therefore,

$$|a_3 - a_2^2| \le \frac{Q_1(\delta,\lambda,\beta)}{2}\left\{(2q_1-3)c^2 + \rho(4-c^2)\right\} := D(c,\rho),$$

where $c = c_1$ $(0 \le c \le 1)$ and $\rho = |x|$ $(0 \le \rho \le 1)$. But

$$\frac{\partial D(c,\rho)}{\partial \rho} = \frac{Q_1(\delta,\lambda,\beta)}{2}(4-c^2) > 0$$

shows that $D(c,\,\rho)$ is an increasing function of ρ on $[0,1]$. Hence

$$\begin{aligned} D(c,\,\rho) \le& D(c,\,1)\\ =& Q_1(\delta,\lambda,\beta)\left\{(q_1-2)c^2+2\right\}\\ :=& V(c). \end{aligned}$$

Since $q_1 \in [\frac{3}{2}, 2]$, then $V(c)$ is a decreasing function of c on $[0,2]$. Hence, $V(c) \le V(0)$, which in turn implies that

$$|a_3 - a_2^2| \le 2Q_1(\delta,\lambda,\beta). \tag{3.14}$$

From the definition of Hankel determinant given by (1.5) and triangular inequality, we have

$$|\mathcal{H}_3(1)| \le |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|. \tag{3.15}$$

It is known in [18] that if $f \in M_\delta^\lambda(\beta)$, then

$$|a_2a_4 - a_3^2| \le \left[\frac{(\delta+1)(\delta+3)}{\delta^2}\right]^\lambda \frac{1}{(1+\beta)(1+3\beta)}. \tag{3.16}$$

Using this result together with (3.6), (3.13) and (3.14) in (3.15), we obtain (3.7) $\square$

For $\lambda = 0$, $\beta = 0$ and $\lambda = 0$, $\beta = 1$, we obtain the following corollaries:

Corollary 3.9. [3] *If $f \in ST$, then*

$$|\mathcal{H}_3(1)| \le 16.$$

Corollary 3.10. [3] *If $f \in CV$, then*

$$|\mathcal{H}_3(1)| \le \frac{32 + 33\sqrt{3}}{72\sqrt{3}}.$$

4. Conclusion

By making use of both Sălăgean differential and two-parameters Komatu integral operators, a class of analytic functions in the open unit disk E was introduced. We have successfully obtained the bound of the Fekete Szegö functional for this class and for a particular case, the upper bound of the third Hankel determinant was obtained. These general results are motivated essentially by their several special cases and consequences, some of which were pointed out in this presentation.

Acknowledgments. The authors would like to thank the Rector of COMSATS University Islamabad, Islamabad, Pakistan, for providing excellent research and academic environment.

References

[1] J. W. Alexander, *Functions which map the interior of the unit circle upon simple regions*. Ann. Math. Second series **17** **1**(1915), 12-22. (1915).

[2] M. K. Aouf, A. O. Mostafa, S. M. Madian, *Fekete Szegö properties for quasi-subordinations of complex order defined by Sălăgean operator*. Afr. Mat. (2019), 1-10.

[3] K. O. Babalola, *On $H_3(1)$ Hankel determinant for some classes of univalent functions*. Inequality Theory and Applications, **6** (2010), 1-7.

[4] S. D. Bernardi, *Convex and starlike univalent functions*. Trans. Amer. Math. Soc. **135** (1969), 429-446 .

[5] H. Darwish, A. M. Lashin, S. Sowileh, *Fekete Szegö type coefficient inequalities for ceertain subclasses ofn analytic functions involving Sălăgean operator*. Punjab Univ. J. Math. **48** (2016), 65-80.

[6] P. L. Duren, *Univalent functions*. Grundlehren der Mathematischen Wissenschaften, Springer, New york, 1983.

[7] R. El-Ashwah, S. Kanas, *Fekete Szegö inequalities for quasi-subordination functions classes of complex order*. Kyungpook Math. J. **55** (2015), 679-688.

[8] I. B. Jung, Y. C. Kim, H. M. Srivastava, *The Hardy space of analytic functions associated with certain one-parameter families of integral operators*. J. Math. Anal. Appl. **176** (1993), 138-147.

[9] F. R. Keogh, E. P. Merkes, *A coefficient inequality for certain classes of univalent functions*. Proc. Amer. Math. Soc. **20** (1969), 812 .

[10] Y. Komatu, *On Analytic prolongation of a family of integral operator*. Mathematica (Cluj), **32** **55** (1990), 141-145.

[11] B. Kowalczyk, A. Lecko, H. M. Srivastava, *A note on the Fekete-Szeg problem for close-to-convex functions with respect to convex functions*. Publications de l'Institut Mathematique, **101** **115** (2017), 143-149.

[12] R. J. Libera, *Some classes of regular univalent functions*. Proc. Amer. Math. Soc. **16** (1965), 755-758.

[13] R. J. Libera, E. J. Zlotkiewicz, *Early coefficient of the inverse of a regular convex function*. Proc. Amer. Math. Soc. **8** **2** (1982), 225-230.

[14] R. J. Libera, E. J. Zlotkiewicz, *Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$*. Proc. Amer. Math. Soc. **87** **2** (1983), 251-257.

[15] N. Magesh, V. K. Balaji, C. Abirami, *Fekete Szegö inequalities for certain subclasses of starlike and convex functions of complex order associated with quasi- subordinations*. Khayyam J. Math. **2** **2** (2016), 112-119.

[16] N. Magesh and J. Yamini, *Fekete-Szegö inequalities associated with k-th root transformation based on quasi-subordination*, Ann. Univ. Paedagog. Crac. Stud. Math. **16** (2017), 5–13.

[17] P. T. Mocanu., *Une propriete de convexite generlise dans la theorie de la representation conforme*, Mathematica (Cluj). **11** (1969), 127-133.

[18] R. M. Mohapatra, T. Panigrahi, *Second Hankel determinant for a class of analytic functions defined by komatu integral operator*. Rend. Math. Appl. **41** **7** (2020), 51-58.

[19] Z. Nehari, *Conformal mapping*. McGraw-Hill, New York, (1952).

[20] J. W. Noonan, D. K. Thomas, *On the Hankel determinants of areally mean p-valent functions*. Proc. Lond. Math. Soc., **3** **3** (1972), 503-524.

[21] H. Orhan, N. Magesh, V. K. Balaji. *Second Hankel determinant for certain class of bi-univalent functions defined by Chebyshev polynomials*. Asian-European Journal of Mathematics, (2018), 1950017.

[22] G. S. Salagean, *Subclasses of univalent functions*. Lecture Notes in Math. vol. 1013, 362-372, Springer- Verlag, Berlin, Heidelberg, New York, 1983.

[23] A. Saliu, K. I. Noor, S. Hussain, M. Darus, M. *On Quantum Differential Subordination Related with Certain Family of Analytic Functions* .J. Math., **2020** (2020).

[24] A. Saliu, K. I. Noor, S. Hussain, M. Darus, *Some results for the family of univalent functions related with Limaçon domain*. AIMS Math., **6** **4** (2021), 3410-3431.

[25] H. M. Srivastava, A. K. Mishra, M. K. Das, *The Fekete Szegö problem for a subclasses of close-to-convex functions*. Complex Var. Elliptic Equ. **44** (2001), 145-163.

[26] H. M. Srivastava, S. Hussain, A. Raziq, M. Raza, *The Fekete Szegö functional for a subclass of analytic functions associated with quasi-subordination*. Carpathian Journal of Mathematics, **34 1**, (2018), 103-113.

[27] H. M. Srivastava, N. Raza, E. S. AbuJarad, G. Srivastava, M. H. Abujarad, *Fekete Szegö inequality for classes of (p,q)-Starlike and (p,q)-convex functions*. RACSAM **113** (2019), 35633584. https://doi.org/10.1007/s13398-019-00713-5 .

[28] H. M. Srivastava, N. Khan, D. Darus, S. Khan, Q. Z. Ahmad, S. Hussain, *Fekete-Szeg type problems and their applications for a subclass of q-starlike functions with respect to symmetrical points*. Mathematics, **8 5**, (2020), 842.

[29] H. S. Srivastava, *Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis*. Iran. J. Sci. Technol., Trans. A, Sci., **44 1** (2020), 327-344.

[30] H. Tang, H. M. Srivastava, S. Sivasubramanian, P. Gurusamy, *The Fekete Szegö functional problems for some subclasses of m-fold symmetric bi-univalent functions*. J. Math. Inequal, **10 4** (2016), 1063-1092.

Afis Saliu
Department of Mathematics, COMSATS University Islamabad, Park Road, Tarlai Kalan, Islamabad 45550, Pakistan
E-mail address: `saliugsu@gmail.com`

Khalida Inayat Noor
Department of Mathematics, COMSATS University Islamabad, Park Road, Tarlai Kalan, Islamabad 45550, Pakistan
E-mail address: `khalidainayat@comsats.edu.pk`