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EXISTENCE OF SOLUTIONS OF INFINITE SYSTEMS OF NONLINEAR CONVOLUTION TYPE INTEGRAL EQUATIONS OF N-VARIABLES IN $C(I_1 \times I_2 \times \ldots \times I_N, l_p)$ AND NUMERICAL RESULTS

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Abstract. The aim of the present paper is to investigate the solvability of infinite systems of nonlinear convolution type integral equations of N-variables in $C(I_1 \times I_2 \times \ldots \times I_N, l_p)$ by using Hausdorff measure of noncompactness with the help of Meir-Keeler condensing operators. Finally, to credibility we propose a numerical method to find solutions of the problem with the high accuracy.

1. Introduction and Auxiliary facts

Infinite system of ordinary differential equations or integral equations are closely related to several important problems appearing naturally in applications (cf. [8, 12, 13]). Apart from this, infinite systems of ordinary differential equations or integral equations are connected with many real life problems considered in engineering, mechanics, in the theory of branching processes, the theory of dissociation of polymers, the theory of neural nets and so on (see [7, 8, 12, 20]). Several researchers have studied the various infinite systems of differential (or integral) equations by applying the measure of noncompactness (see [1, 2, 3, 4, 5, 14, 20, 21, 22], for example). Besides, many mathematicians studied the infinite systems of integral equations in two variables by using measure of noncompactness and some fixed point theorems (see [4, 9, 10, 11, 16, 23]).

The aim of this paper is to investigate the solvability of the following infinite system of nonlinear convolution type integral equations of N-variables

$$x_i(t_1,\ldots,t_N) = f_i(t_1,\ldots,t_N,x(t_1,\ldots,t_N))+\lambda_i \int_{I_1\times I_2\times\ldots\times I_N} k(t_1-s_1,\ldots,t_N-s_N)Qx_i(s_1,\ldots,s_N)ds_1\ldots ds_N, \tag{1.1}$$

where $I_i = [a_i, b_i]$, $f_i : I_1 \times I_2 \times \ldots \times I_N \times \mathbb{R}^\infty \to \mathbb{R}$, $\lambda_i \in \mathbb{R}$ $(i = 1, 2, \ldots)$, $k : [a_1-b_1, b_1-a_1] \times \ldots \times [a_N-b_N, b_N-a_N] \to \mathbb{R}$ and $Q : C(I_1 \times I_2 \times \ldots \times I_N) \to \mathbb{R}$ are arbitrary functions. We also provide an illustrative example in support of our

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existence theorem. Finally, to credibility we propose a numerical method to find solutions of the above problem with the high accuracy.

In the following, we give a few auxiliary facts which will be used in our further considerations.

Let $(E, \|.\|)$ be a real Banach space with zero element 0. Let $\mathbb{R}_+ = [0, +\infty)$, $\overline{X}$ and Conv$X$ denote the closure and closed convex hull of a set X, respectively. Moreover, let $\mathfrak{M}_E$ indicate the family of nonempty and bounded subsets of E and $\mathfrak{N}_E$ indicate the family of all nonempty and relatively compact subsets of E.
The following axiomatic definition of a measure of noncompactness was found in [6].

Definition 1.1. A function $\mu : \mathfrak{M}_E \to \mathbb{R}_+$, is said to be a measure of noncompactness if it fulfils the following conditions:

1° The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subseteq \mathfrak{N}_E$.
2° $X \subset Y$ implies that $\mu(X) \leq \mu(Y)$.
3° $\mu(\overline{X}) = \mu(X)$.
4° $\mu(\text{Conv}X) = \mu(X)$.
5° $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
6° If $\{X_n\}$ is a sequence of closed chains of $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

Definition 1.2. [6] Let (X, d) be a metric space and $Q \in \mathfrak{M}_X$. Then the Kuratowski measure of noncompactness of Q, denoted by $\alpha(Q)$, is the infimum of the set of all numbers $\varepsilon > 0$ such that Q can be covered by a finite number of sets with diameters ε, that is

$$\alpha(Q) = \inf\Big\{\varepsilon > 0 : Q \subset \bigcup_{i=1}^{n} S_i, S_i \subset X, \text{diam}(S_i) < \varepsilon \ (i = 1, 2, \ldots, n);\ n \in \mathbb{N}\Big\},$$

where $\text{diam}(S_i) = \sup\{d(x, y) : x, y \in S_i\}$.

The Hausdorff measure of noncompactness for a bounded set Q is defined by

$$\chi(Q) = \inf\Big\{\varepsilon > 0 : Q \subset \bigcup_{i=1}^{n} B(x_i, r_i), x_i \in X, r_i < \varepsilon \ (i = 1, 2, \ldots, n);\ n \in \mathbb{N}\Big\}.$$

The Hausdorff measure of noncompactness is often called ball measure of noncompactness .

In 1969, Meir and Keeler [19] introduced the concept of Meir-Keeler contractive mapping and proved some fixed point theorems for this kind of mappings. Thereafter, Aghajani et al. [1] generalized some fixed point and coupled fixed point theorems for Meir-Keeler condensing operators via measures of noncompactness.

Definition 1.3. [1] Let C be a nonempty subset of a Banach space E and μ be an arbitrary measure of noncompactness on E. An operator $T : C \to C$ is called a Meir-Keeler condensing operator if for any $\varepsilon > 0$, $\delta > 0$ exists such that

$$\varepsilon \leq \mu(X) < \varepsilon + \delta \quad \text{implies} \quad \mu(T(X)) < \varepsilon$$

for any bounded subset X of C.

Theorem 1.4. [1] *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and μ be an arbitrary measure of noncompactness on E. If $T : C \to C$ is a continuous and Meir-Keeler condensing operator, then T has at least one fixed point and the set of all fixed points of T in C is compact.*

2. Hausdorff measure of noncompactness in sequence spaces

In the Banach space $(l_p, \|.\|_p)$ $(1 \leq p < \infty)$, the Hausdorff measure of noncompactness χ is defined as follows (see [6]):

$$\chi_{l_p}(D) = \lim_{n\to\infty} \big[\sup_{u\in D} \big(\sum_{k=n}^{\infty} |u_k|^p \big)^{\frac{1}{p}} \big], \tag{2.1}$$

where $u = (u_i) \in l_p$ and $D \in \mathfrak{M}_{l_p}$.
Let $I_i = [a_i, b_i]$, $1 \leq i \leq N$ be compact intervals in $\mathbb{R}$. Let us define $C(I_1 \times I_2 \times \ldots \times I_N, l_p)$ denotes the space of all continuous functions defined on $I_1 \times I_2 \times \ldots \times I_N$ with values in l_p $(1 \leq p < \infty)$. Then $C(I_1 \times I_2 \times \ldots \times I_N, l_p)$ is also a Banach space with the norm

$$\|x\|_{C(I_1\times I_2\times\ldots\times I_N, l_p)} = \sup\{\|x(t_1,\ldots,t_N)\|_{l_p} : \ (t_1, t_2, \ldots, t_N) \in I_1 \times I_2 \times \ldots \times I_N\},$$

where $x \in C(I_1 \times I_2 \times \ldots \times I_N, l_p)$. For any non-empty bounded subset $\hat{E}$ of $C(I_1 \times \ldots \times I_N, l_p)$ and $(t_1, \ldots, t_N) \in I_1 \times \ldots \times I_N$, let $\hat{E}(t_1, \ldots, t_N) = \{x(t_1, \ldots, t_N) : x \in \hat{E}\}$.
Now, using (2.1), we conclude that the Hausdorff measure of noncompactness for $\hat{E} \subset C(I_1 \times \ldots \times I_N, l_p)$ can be defined by

$$\chi_{C(I_1\times\ldots\times I_N, l_p)}(\hat{E}) = \sup\{\chi_{l_p}(\hat{E}(t_1,\ldots,t_N)) : \ (t_1,\ldots,t_N) \in I_1 \times \ldots \times I_N\}.$$

3. Solvability of infinite systems of nonlinear convolution type integral equations of N-variables in $C(I_1 \times I_2 \times \ldots \times I_N, l_p)$

In this section, we investigate the solvability of the infinite system of nonlinear convolution type integral equations (1.1). We provide an illustrative example to show the effectiveness and applicability of our results.

Consider the following conditions:
(i) The functions $f_i : I_1 \times I_2 \times \ldots \times I_N \times \mathbb{R}^\infty \to \mathbb{R}$, $i \in \mathbb{N}$ are continuous and for each $i \in \mathbb{N}$, there exists a positive real number M such that for each $x, y \in C(I_1 \times I_2 \times \ldots \times I_N, l_p)$ and $(t_1, \ldots, t_N), (s_1, \ldots, s_N) \in I_1 \times \ldots \times I_N$ we have

$$|f_i(t_1,\ldots,t_N, x(t_1,\ldots,t_N)) - f_i(s_1,\ldots,s_N, y(s_1,\ldots,s_N))|^p \leq M|x_i(t_1,\ldots,t_N)) - y_i(s_1,\ldots,s_N)|^p.$$

(ii) $\displaystyle\sum_{i=1}^{\infty}|f_i(t_1,\ldots,t_N, x^0(t_1,\ldots,t_N))|^p$ uniformly converges to zero, the sequence (λ_i) is bounded with $|\lambda_i| \leq \lambda$ and for each $i \in \mathbb{N}$, $Qx_i^0 = 0$, where $x^0(t_1,\ldots,t_N) = (x_i^0(t_1,\ldots,t_N))$, $x_i^0(t_1,\ldots,t_N) = 0$ for all $(t_1,\ldots,t_N) \in I_1 \times \ldots \times I_N$.
(iii) There exists a positive real number L such that

$$\sum_{i=1}^{\infty}|Qx_i(t_1,\ldots,t_N) - Qy_i(t_1,\ldots,t_N)|^p \leq L\sum_{i=1}^{\infty}|x_i(t_1,\ldots,t_N) - y_i(t_1,\ldots,t_N)|^p.$$

(iv) The mapping $k : [a_1 - b_1, b_1 - a_1] \times \ldots \times [a_N - b_N, b_N - a_N] \to \mathbb{R}$ is continuous.

(v) $4^p M + \lambda 2^p K^p L(\prod_{i=1}^{N}(b_i - a_i))^p < 1$, where $K = \sup\{k(t_1, \ldots, t_N) : (t_1, \ldots, t_N) \in [a_1 - b_1, b_1 - a_1] \ldots \times [a_N - b_N, b_N - a_N]\}$.

Theorem 3.1. *Suppose that the assumptions* (i)*-*(v) *are satisfied. Then the infinite system (1.1) has at least one solution in* $C(I_1 \times \ldots \times I_N, l_p)$*.*

Proof. We define the mapping $F : (C(I_1 \times \ldots \times I_N, l_p))^\infty \to (C(I_1 \times \ldots \times I_N, l_p))^\infty$ by

$$Fx(t_1, \ldots, t_N) = (F_i(x)(t_1, \ldots, t_N))$$

$$= \big(f_i(t_1, \ldots, t_N, x(t_1, \ldots, t_N)) + \lambda_i \int_{I_1 \times \ldots \times I_N} k(t_1 - s_1, \ldots, t_N - s_N) Q x_i(s_1, \ldots, s_N) ds_1 \ldots ds_N\big),$$

where $x = (x_i) \in C(I_1 \times I_2 \times \ldots \times I_N, l_p)$. By using our assumptions and Lebesgue dominated convergence theorem, for arbitrarily fixed $(t_1, \ldots, t_N) \in I_1 \times \ldots \times I_N$, we deduce

$$\begin{aligned}
&\|Fx(t_1, \ldots, t_N)\|_{l_p}^p \\
&= \sum_{i=1}^{\infty} |f_i(t_1, \ldots, t_N, x(t_1, \ldots, t_N)) + \lambda_i \int_{I_1 \times \ldots \times I_N} k(t_1 - s_1, \ldots, t_N - s_N) Q x_i(s_1, \ldots, s_N) ds_1 \ldots ds_N|^p \\
&\leq \sum_{i=1}^{\infty} 4^p |f_i(t_1, \ldots, t_N, x(t_1, \ldots, t_N)) - f_i(s_1, \ldots, s_N, x^0(s_1, \ldots, s_N))|^p \\
&\quad + \sum_{i=1}^{\infty} 4^p |f_i(s_1, \ldots, s_N, x^0(s_1, \ldots, s_N))|^p \\
&\quad + \sum_{i=1}^{\infty} \lambda_i 2^p |\int_{I_1 \times \ldots \times I_N} k(t_1 - s_1, \ldots, t_N - s_N) Q x_i(s_1, \ldots, s_N) ds_1 \ldots ds_N|^p \\
&\leq \sum_{i=1}^{\infty} 4^p M |x_i(t_1, \ldots, t_N)|^p + \sum_{i=1}^{\infty} 4^p |f_i(s_1, \ldots, s_N, x^0(s_1, \ldots, s_N))|^p \\
&\quad + 2^p \lambda \sum_{i=1}^{\infty} (\int_{I_1 \times \ldots \times I_N} |k(t_1 - s_1, \ldots, t_N - s_N)|^p |Q x_i(s_1, \ldots, s_N)|^p ds_1 \ldots ds_N)(\int_{I_1 \times \ldots \times I_N} ds_1 \ldots ds_N)^{p-1} \\
&\leq 4^p M \|x(t_1, \ldots, t_N)\|_{l_p}^p + \lambda 2^p (\prod_{i=1}^{N}(b_i - a_i))^{p-1} K^p \int_{I_1 \times \ldots \times I_N} \sum_{i=1}^{\infty} |Q x_i(s_1, \ldots, s_N)|^p ds_1 \ldots ds_N \\
&\leq \|x\|_{C(I_1 \times \ldots \times I_N, l_p)}^p (4^p M + \lambda 2^p K^p L(\prod_{i=1}^{N}(b_i - a_i))^p),
\end{aligned}$$

Taking the supremum over $(t_1, \ldots, t_N) \in I_1 \times \ldots \times I_N$, we obtain

$$\|Fx\|_{C(I_1 \times \ldots \times I_N, l_p)}^p \leq \|x\|_{C(I_1 \times \ldots \times I_N, l_p)}^p (4^p M + \lambda 2^p K^p L(\prod_{i=1}^{N}(b_i - a_i))^p).$$

Above inequality can be written as

$$r^p \leq r^p (4^p M + \lambda 2^p K^p L(\prod_{i=1}^{N}(b_i - a_i))^p).$$

Let r_0 denotes the optimal solution of the above inequality. Now, consider the set $B = \{x = (x_i) \in (C(I_1 \times \ldots \times I_N, l_p))^\infty : \|x\|_{C(I_1 \times \ldots \times I_N, l_p)} \leq r_0\}$ which is closed, bounded and convex.

We show that for each $x \in C(I_1 \times \ldots \times I_N, l_p)$, Fx is continuous. Let $\varepsilon > 0$ and $(t_1, \ldots, t_N), (s_1, \ldots, s_N) \in I_1 \times \ldots \times I_N$ be arbitrary and fixed. Since k is continuous, so there exists $\delta_1 > 0$ such that if $\|(t_1,\dots,t_N)-(s_1,\dots,s_N)\| < \delta_1$, then

$$|k(t_1,\dots,t_N)-k(s_1,\dots,s_N)| < \frac{\varepsilon^{\frac{1}{p}}}{L^{\frac{1}{p}}\lambda 2^{\frac{p+1}{p}}(\prod_{i=1}^{N}(b_i-a_i))\|x\|_{C(I_1\times\ldots\times I_N,l_p)}}.$$

On the other hand, since $x\in C(I_1\times\ldots\times I_N,l_p)$, then there exists $\delta_2>0$, such that if $\|(t_1,\dots,t_N)-(s_1,\dots,s_N)\| < \delta_2$, then $\|x(t_1,\dots,t_N)-x(s_1,\dots,s_N)\|_{l_p} < (\frac{\varepsilon}{2^{p+1}M})^{\frac{1}{p}}$. Take $\delta=\min\{\delta_1,\delta_2\}$. Then, using again Lebesgue dominated convergent theorem we have

$$\begin{aligned}
&\|Fx(t_1,\dots,t_N)-Fx(s_1,\dots,s_N)\|_{l_p}^p\\
&= \sum_{i=1}^{\infty}|f_i(t_1,\dots,t_N,x(t_1,\dots,t_N))-f_i(s_1,\dots,s_N,x(s_1,\dots,s_N))\\
&\quad +\lambda_i\int_{I_1\times I_2\times\ldots\times I_N}(k(t_1-t_1',\dots,t_N-t_N')-k(s_1-t_1',\dots,s_N-t_N'))Qx_i(t_1',\dots,t_N')dt_1'\dots dt_N'|^p\\
&\le \sum_{i=1}^{\infty}2^p|f_i(t_1,\dots,t_N,x(t_1,\dots,t_N))-f_i(s_1,\dots,s_N,x(s_1,\dots,s_N))|^p\\
&\quad +\sum_{i=1}^{\infty}\lambda_i 2^p|\int_{I_1\times I_2\times\ldots\times I_N}(k(t_1-t_1',\dots,t_N-t_N')-k(s_1-t_1',\dots,s_N-t_N'))Qx_i(t_1',\dots,t_N')dt_1'\dots dt_N'|^p\\
&\le \sum_{i=1}^{\infty}2^pM|x_i(t_1,\dots,t_N)-x_i(s_1,\dots,s_N)|^p\\
&\quad +\lambda\sum_{i=1}^{\infty}2^p\int_{I_1\times I_2\times\ldots\times I_N}|k(t_1-t_1',\dots,t_N-t_N')-k(s_1-t_1',\dots,s_N-t_N')|^p|Qx_i(t_1',\dots,t_N')|^pdt_1'\dots dt_N'\\
&\quad (\int_{I_1\times I_2\times\ldots\times I_N}ds_1\dots ds_N)^{p-1}\\
&\le 2^pM\sum_{i=1}^{\infty}|x_i(t_1,\dots,t_N)-x_i(s_1,\dots,s_N)|^p\\
&\quad +2^p\lambda\frac{\varepsilon(\prod_{i=1}^{N}(b_i-a_i))^{p-1}}{2^{p+1}L\lambda(\prod_{i=1}^{N}(b_i-a_i))^p\|x\|_{C(I_1\times\ldots\times I_N,l_p)}^p}\int_{I_1\times I_2\times\ldots\times I_N}\sum_{i=1}^{\infty}|Qx_i(t_1',\dots,t_N')|^pdt_1'\dots dt_N'\\
&\le \frac{\varepsilon}{2}+\frac{\varepsilon}{2L\|x\|_{C(I_1\times\ldots\times I_N,l_p)}^p(\prod_{i=1}^{N}(b_i-a_i))}\int_{I_1\times I_2\times\ldots\times I_N}\sum_{i=1}^{\infty}|Qx_i(t_1',\dots,t_N')-Qx_i^0(t_1',\dots,t_N')|^pdt_1'\dots dt_N'\\
&\le \frac{\varepsilon}{2}+\frac{\varepsilon}{2L\|x\|_{C(I_1\times\ldots\times I_N,l_p)}^p(\prod_{i=1}^{N}(b_i-a_i))}\int_{I_1\times I_2\times\ldots\times I_N}L\sum_{i=1}^{\infty}|x_i(t_1',\dots,t_N')|^pdt_1'\dots dt_N'\\
&\le \frac{\varepsilon}{2}+\frac{\varepsilon L\|x\|_{C(I_1\times\ldots\times I_N,l_p)}^p(\prod_{i=1}^{N}(b_i-a_i))}{2L\|x\|_{C(I_1\times\ldots\times I_N,l_p)}^p(\prod_{i=1}^{N}(b_i-a_i))}\\
&= \varepsilon.
\end{aligned}$$

Next, we show that the operator F is continuous on B. For this purpose, let us fix arbitrarily a number $\varepsilon > 0$ and a function $x\in B$. Then, for each $y\in B$

such that $\|x-y\|^p_{C(I_1\times\ldots\times I_N,l_p)} < \Big(\frac{\varepsilon}{2^p(M+\lambda K^p(\prod_{i=1}^{N}(b_i-a_i))^p L}\Big)$ and for a fixed N-tuple $(t_1,\ldots,t_N)\in I_1\times\ldots\times I_N$, we get

$$
\begin{aligned}
&\|Fx(t_1,\ldots,t_N)-Fy(t_1,\ldots,t_N)\|^p_{C(I_1\times\ldots\times I_N,l_p)}\\
\leq\ & \sum_{i=1}^{\infty}2^p|f_i(t_1,\ldots,t_N,x(t_1,\ldots,t_N))-f_i(t_1,\ldots,t_N,y(t_1,\ldots,t_N))|^p\\
&+\sum_{i=1}^{\infty}2^p\lambda_i|\int_{I_1\times I_2\times\ldots\times I_N}k(t_1-t_1',\ldots,t_N-t_N')(Qx_i(t_1',\ldots,t_N')-Qy_i(t_1',\ldots,t_N'))dt_1'\ldots dt_N'|^p\\
\leq\ & \sum_{i=1}^{\infty}2^pM|x_i(t_1,\ldots,t_N)-y_i(t_1,\ldots,t_N)|^p\\
&+\lambda 2^p(\prod_{i=1}^{N}(b_i-a_i))^{p-1}\int_{I_1\times\ldots\times I_N}\sum_{i=1}^{\infty}|k(t_1-t_1',\ldots,t_N-t_N')|^p|Qx_i(t_1',\ldots,t_N')-Qy_i(t_1',\ldots,t_N')|^p dt_1'\ldots dt_N'\\
\leq\ & 2^pM\|x(t_1,\ldots,t_N)-y(t_1,\ldots,t_N)\|^p_{l_p}\\
&+\lambda 2^pK^p(\prod_{i=1}^{N}(b_i-a_i))^{p-1}\int_{I_1\times\ldots\times I_N}L(\sum_{i=1}^{\infty}|x_i(t_1',\ldots,t_N')-y_i(t_1',\ldots,t_N')|^p)dt_1'\ldots dt_N'\\
\leq\ & 2^p\|x-y\|^p_{C(I_1\times\ldots\times I_N,l_p)}(M+\lambda K^pL(\prod_{i=1}^{N}(b_i-a_i))^p)\\
\leq\ & \varepsilon.
\end{aligned}
$$

Thus F is continuous on B.

Finally, we show that F is a Meir-Keeler condensing operator. For arbitrary fixed $(t_1,\ldots,t_N),(s_1,\ldots,s_N)\in I_1\times\ldots\times I_N$, we obtain

$$
\begin{aligned}
&(\chi_{C(I_1\times\ldots\times I_N,l_p)}(F(B)))^p\\
=\ & \sup\{(\chi_{l_p}(F(B)(t_1,\ldots,t_N)))^p:\ (t_1,\ldots,t_N)\in I_1\times\ldots\times I_N\}\\
=\ & \sup_{(t_1,\ldots,t_N)\in I_1\times\ldots\times I_N}\lim_{n\to\infty}\sup_{u\in B}\Big(\sum_{i=n}^{\infty}|f_i(t_1,\ldots,t_N,x(t_1,\ldots,t_N))\\
&+\lambda_i\int_{I_1\times\ldots\times I_N}k(t_1-s_1,\ldots,t_N-s_N)Qx_i(s_1,\ldots,s_N)ds_1\ldots ds_N|^p\Big)\\
\leq\ & \sup_{(t_1,\ldots,t_N)\in I_1\times\ldots\times I_N}\lim_{n\to\infty}\sup_{u\in B}\Big(\sum_{i=n}^{\infty}2^p|f_i(t_1,\ldots,t_N,x(t_1,\ldots,t_N))|^p\\
&+\sum_{i=n}^{\infty}\lambda_i2^p|\int_{I_1\times\ldots\times I_N}k(t_1-s_1,\ldots,t_N-s_N)Qx_i(s_1,\ldots,s_N)ds_1\ldots ds_N|^p\Big)\\
\leq\ & \sup_{(t_1,\ldots,t_N)\in I_1\times\ldots\times I_N}\lim_{n\to\infty}\sup_{u\in B}\Big[\sum_{i=n}^{\infty}4^p|f_i(t_1,\ldots,t_N,x(t_1,\ldots,t_N))-f_i(s_1,\ldots,s_N,x^0(s_1,\ldots,s_N))|^p\\
&+\sum_{i=n}^{\infty}4^p|f_i(s_1,\ldots,s_N,x^0(s_1,\ldots,s_N))|^p+\lambda 2^pK^p(\prod_{i=1}^{N}(b_i-a_i))^{p-1}\int_{I_1\times\ldots\times I_N}\sum_{i=n}^{\infty}|Qx_i(s_1,\ldots,s_N)|^p ds_1\ldots ds_N\Big]
\end{aligned}
$$

$$\begin{aligned}
&\leq \sup_{(t_1,\dots,t_N)\in I_1\times\ldots\times I_N} \lim_{n\to\infty} \sup_{u\in B} \Big[\sum_{i=n}^{\infty} 4^p M|x_i(t_1,\dots,t_N)|^p + 4^p \sum_{i=n}^{\infty} |f_i(s_1,\dots,s_N,x^0(s_1,\dots,s_N))|^p \\
&\quad +\lambda 2^p K^p (\prod_{i=1}^{N}(b_i-a_i))^{p-1} \int_{I_1\times\ldots\times I_N} L\sum_{i=n}^{\infty} |x_i(s_1,\dots,s_N)|^p ds_1\ldots ds_N\Big] \\
&\leq 4^p M(\chi_{C(I_1\times\ldots\times I_N,l_p)}(B))^p + \lambda 2^p K^p L \chi_{C(I_1\times\ldots\times I_N,l_p)}(B)(\prod_{i=1}^{N}(b_i-a_i))^p \\
&= (\chi_{C(I_1\times\ldots\times I_N,l_p)}(B))^p\big(4^p M + \lambda 2^p K^p L(\prod_{i=1}^{N}(b_i-a_i))^p\big).
\end{aligned}$$

We observe that $\chi_{C(I_1\times\ldots\times I_N,l_p)}(F(B)) \leq \chi_{C(I_1\times\ldots\times I_N,l_p)}(B)[4^pM + \lambda 2^p K^p L(\prod_{i=1}^{N}(b_i - a_i))^p]^{\frac{1}{p}}$. Now, let $\varepsilon > 0$ be given. Taking $\delta = \varepsilon\Big((4^pM + \lambda 2^pK^pL(\prod_{i=1}^{N}(b_i-a_i))^p)^{-\frac{1}{p}} - 1\Big)$. Hence, we get

$$\varepsilon < \chi_{C(I_1\times\ldots\times I_N,l_p)}(B) < \varepsilon + \delta \Rightarrow \chi_{C(I_1\times\ldots\times I_N,l_p)}(F(B)) < \varepsilon,$$

which implies that F is a Meir-Keeler condensing operator defined on the set B. So F satisfies all the conditions of Theorem 1.4 which shows that the mapping F has a fixed point in B. Hence the infinite system (1.1) has a solution in $C(I_1 \times \ldots \times I_N, l_p)$. □

Example 3.2. Let us consider the infinite system of integral equations

$$x_i(t_1,t_2) = \sum_{j=i}^{i+1} \frac{x_j(t_1,t_2)}{101j-1} + \frac{i}{2i+1} \int_{[\frac{1}{2},\frac{4}{3}]\times[\frac{2}{3},\frac{5}{4}]} \cos((t_1+t_2)-(s_1+s_2)) \ln(1+|\arctan x_i(s_1,s_2)|) ds_2 ds_1. \tag{3.1}$$

Here $I_j = [\frac{j}{j+1}, \frac{j+3}{j+2}]$ $(j = 1,2)$, $k(t_1,t_2) = \cos((t_1+t_2)$, $f_i(t_1,t_2,x(t_1,t_2)) = \sum_{j=i}^{i+1} \frac{x_j(t_1,t_2)}{101j-1}$, and $Qx_i(t_1,t_2) = \ln(1 + |\arctan x_i(t_1,t_2)|)$ and $\lambda_i = \frac{i}{2i+1}$, $i = 1,\ldots$. So (3.1) is a special case of the infinite system (1.1). Clearly, f_i s' are continuous on $I_1 \times I_2$ and we have

$$\begin{aligned}
|f_i(t_1,t_2,x(t_1,t_2)) - f_i(s_1,s_2,y(s_1,s_2))|^2 &= \Big|\sum_{j=i}^{i+1} \frac{x_j(t_1,t_2)-y_j(s_1,s_2)}{101j-1}\Big|^2 \\
&\leq \Big|\sum_{j=i}^{i+1} \frac{|x_j(t_1,t_2)-y_j(s_1,s_2)|}{101j-1}\Big|^2 \\
&\leq 4\sum_{j=i}^{i+1} \frac{|x_j(t_1,t_2)-y_j(s_1,s_2)|^2}{(101j-1)^2}.
\end{aligned}$$

We may assume that the right hand side of the above inequality is less than or equal $\frac{8}{10^4}|x_i(t_1,t_2) - y_i(s_1,s_2)|^2$, where $M = \frac{8}{10^4}$. Also, $\sum_{i=1}^{\infty}|f_i(t_1,t_2,x^0(t_1,t_2))|^2$ uniformly converges to 0 and $Qx_i^0 = 0$. Further, we can write

$$\begin{aligned}
&\sum_{i=1}^{\infty}|Qx_i(t_1,t_2)-Qy_i(t_1,t_2)|^2 \\
&= \sum_{i=1}^{\infty}\Big|\ln(1+|\arctan x_i(t_1,t_2)|)-\ln(1+|\arctan y_i(t_1,t_2)|)\Big|^2 \\
&= \sum_{i=1}^{\infty}\Big|\ln\frac{1+|\arctan x_i(t_1,t_2)|}{1+|\arctan y_i(t_1,t_2)|}\Big|^2 \\
&= \sum_{i=1}^{\infty}\Big|\ln(1+\frac{|\arctan x_i(t_1,t_2)|-|\arctan y_i(t_1,t_2)|}{1+|\arctan y_i(t_1,t_2)|})\Big|^2 \\
&\le \sum_{i=1}^{\infty}\Big|\ln(1+\frac{|\arctan x_i(t_1,t_2)-\arctan y_i(t_1,t_2)|}{1+|\arctan y_i(t_1,t_2)|})\Big|^2 \\
&\le \sum_{i=1}^{\infty}\Big|\ln(1+|\arctan x_i(t_1,t_2)-\arctan y_i(t_1,t_2)|)\Big|^2 \\
&\le \sum_{i=1}^{\infty}|x_i(t_1,t_2)-y_i(t_1,t_2)|^2.
\end{aligned}$$

Moreover, $4^2M+\frac{1}{2}2^2K^2L^2(\prod_{i=1}^{N}(b_i-a_i))^2=0.4854080247<1$. Thus the infinite system (3.1) satisfies the hypotheses of the Theorem 3.1. Therefore (3.1) has at least one solution in $C(I_1\times I_2,l_2)$.

4. Numerical solution of the infinite system (3.1)

Finding the solution of nonlinear integral equations is an important subject in various scientific problems. Numerical methods can help us to achieve this goal (see [15, 18, 24]). In this section, we propose a numerical method to solve the infinite system (3.1) approximately.

Example 4.1. Consider the infinite system of integral equations

$$x_i(t_1,t_2)=\sum_{j=i}^{i+1}\frac{x_j(t_1,t_2)}{101j-1}+\frac{i\lambda_i}{2i+1}\int_{[\frac{1}{2},\frac{4}{3}]\times[\frac{2}{3},\frac{5}{4}]}\cos((t_1+t_2)-(s_1+s_2))\ln(1+|\arctan x_i(s_1,s_2)|)ds_2ds_1,\ i=1,2,\dots. \tag{4.1}$$

This system can be written as follows:

$$x_i(t_1,t_2)=\delta_i x_i(t_1,t_2)+\delta_{i+1}x_{i+1}(t_1,t_2)+\frac{i\lambda_i}{2i+1}\int_{\frac{1}{2}}^{\frac{4}{3}}\int_{\frac{2}{3}}^{\frac{5}{4}}F_i(t_1,t_2,s_1,s_2)ds_2ds_1, \tag{4.2}$$

where

$$\delta_i=\frac{1}{101i-1}, \tag{4.3}$$

and

$$F_i(t_1,t_2,s_1,s_2)=\cos((t_1+t_2)-(s_1+s_2))\ln(1+|\arctan x_i(s_1,s_2)|). \tag{4.4}$$

Clearly, when i tends to infinity, δ_i vanishes. Hence for a big enough n, we let

$$x_n(t_1,t_2)\simeq\frac{n\lambda_n}{2n+1}\int_{\frac{1}{2}}^{\frac{4}{3}}\int_{\frac{2}{3}}^{\frac{5}{4}}F_n(t_1,t_2,s_1,s_2)ds_2ds_1. \tag{4.5}$$

Equation (4.5) is an eigenvalue problem. For more details about an eigenvalue problem one can see [17]. To solve this problem, first we expand the integral term in (4.5) as follows:

$\int_{\frac{1}{2}}^{\frac{4}{3}} \int_{\frac{2}{3}}^{\frac{5}{4}} F_i(t_1, t_2, s_1, s_2) ds_2 ds_1$

$$= \int_{\frac{1}{2}}^{\frac{4}{3}} \int_{\frac{2}{3}}^{\frac{5}{4}} \cos((t_1 + t_2) - (s_1 + s_2) G(x_n(s_1, s_2)) ds_2 ds_1,$$

where $G(x_n(s_1, s_2)) = \ln(1 + |\arctan x_n(s_1, s_2)|)$. Hence

$\int_{\frac{1}{2}}^{\frac{4}{3}} \int_{\frac{2}{3}}^{\frac{5}{4}} F_i(t_1, t_2, s_1, s_2) ds_2 ds_1$

$$= \cos(t_1+t_2) \int_{\frac{1}{2}}^{\frac{4}{3}} \int_{\frac{2}{3}}^{\frac{5}{4}} \cos(s_1+s_2) G(x_n(s_1, s_2)) ds_2 ds_1 + \sin(t_1+t_2) \int_{\frac{1}{2}}^{\frac{4}{3}} \int_{\frac{2}{3}}^{\frac{5}{4}} \sin(s_1+s_2) G(x_n(s_1, s_2)) ds_2 ds_1. \tag{4.6}$$

From (4.5) and (4.6) we deduce

$$x_n(s_1, s_2) \simeq \frac{n}{2n+1} \lambda_n \big(C_{1,n} \cos(t_1 + t_2) + C_{2,n} \sin(t_1 + t_2)\big), \tag{4.7}$$

where

$$C_{1,n} = \int_{\frac{1}{2}}^{\frac{4}{3}} \int_{\frac{2}{3}}^{\frac{5}{4}} \cos(s_1 + s_2) G(x_n(s_1, s_2)) ds_2 ds_1, \tag{4.8}$$

and

$$C_{2,n} = \int_{\frac{1}{2}}^{\frac{4}{3}} \int_{\frac{2}{3}}^{\frac{5}{4}} \sin(s_1 + s_2) G(x_n(s_1, s_2)) ds_2 ds_1. \tag{4.9}$$

Defining

$$H_{1,n}(s_1, s_2) = \cos(s_1 + s_2) G(x_n(s_1, s_2)),\ \ H_{2,n}(s_1, s_2) = \sin(s_1 + s_2) G(x_n(s_1, s_2)), \tag{4.10}$$

and replacing $x_n(t_1, t_2)$ from (4.7) in $C_{1,n}$ and $C_{2,n}$, the following nonlinear system is obtained:

$$\begin{cases} C_{1,n} = \int_{\frac{1}{2}}^{\frac{4}{3}} \int_{\frac{2}{3}}^{\frac{5}{4}} H_{1,n}(s_1, s_2) = \int_{\frac{1}{2}}^{\frac{4}{3}} \int_{\frac{2}{3}}^{\frac{5}{4}} \cos(s_1 + s_2) G(\frac{n}{2n+1} \lambda_n (C_{1,n} \cos(s_1 + s_2) + C_{2,n} \sin(s_1 + s_2))) ds_2 ds_1 \\ C_{2,n} = \int_{\frac{1}{2}}^{\frac{4}{3}} \int_{\frac{2}{3}}^{\frac{5}{4}} H_{2,n}(s_1, s_2) = \int_{\frac{1}{2}}^{\frac{4}{3}} \int_{\frac{2}{3}}^{\frac{5}{4}} \sin(s_1 + s_2) G(\frac{n}{2n+1} \lambda_n (C_{1,n} \cos(s_1 + s_2) + C_{2,n} \sin(s_1 + s_2))) ds_2 ds_1. \end{cases} \tag{4.11}$$

The integrals in (4.11) can not be solved exactly and we use the simple two dimensional trapezoidal method to approximate them. First, we recall the formula of the simple two dimensional trapezoidal method

$$\int_a^b \int_c^d H(s_1, s_2) ds_2 ds_1 \simeq (\frac{b-a}{2})(\frac{d-c}{2})\big[H(b, d) + H(b, c) + H(a, d) + H(a, c)\big]. \tag{4.12}$$

Therefore the system (4.11) can be approximated as follows:

$$\begin{cases} C_{1,n} = (\frac{\frac{4}{3} - \frac{1}{2}}{2})(\frac{\frac{5}{4} - \frac{2}{3}}{2})\big(H_{1,n}(\frac{4}{3}, \frac{5}{4}) + H_{1,n}(\frac{4}{3}, \frac{2}{3}) + H_{1,n}(\frac{1}{2}, \frac{5}{4}) + H_{1,n}(\frac{1}{2}, \frac{2}{3})\big) \\ C_{2,n} = (\frac{\frac{4}{3} - \frac{1}{2}}{2})(\frac{\frac{5}{4} - \frac{2}{3}}{2})\big(H_{2,n}(\frac{4}{3}, \frac{5}{4}) + H_{2,n}(\frac{4}{3}, \frac{2}{3}) + H_{2,n}(\frac{1}{2}, \frac{5}{4}) + H_{2,n}(\frac{1}{2}, \frac{2}{3})\big). \end{cases} \tag{4.13}$$

The nonlinear system (4.13) is a system with two equations and three unknowns $C_{1,n}$, $C_{2,n}$ and λ_n. Now, we choose a fix value for λ_n and solve this system by Newton-Raphson method to find $C_{1,n}$ and $C_{2,n}$ [17].

Reviewing the procedure so far, first we must choose a big value for n and derive the equation (4.5), then by assuming a fix value for λ_n and solving the system (4.13) we find $C_{1,n}$ and $C_{2,n}$.

To make the procedure more clear, we choose $n = 10$, which is not so big. By assuming different values for λ_{10} and solving the system (4.15), we find out that for

$|\lambda_{10}| \geq 5.54$, this system has nontrivial solutions. Now, we let $\lambda_{10} = 9$ and solve the system (4.13) which results in $C_{1,10} = -0.6882143395$ and $C_{2,10} = 0.2192127099$. Hence, by using (4.7) we get

$$x_{10}(t_1, t_2) \simeq -0.2949490027 \cos(t_1 + t_2) + 0.9394830424 \sin(t_1 + t_2). \tag{4.14}$$

In order to find the relation of other functions, *i.e.*, $x_9(t_1, t_2), \ldots, x_1(t_1, t_2)$, consider the following relation derived from (4.2)

$$x_{n-1}(t_1, t_2) = \frac{1}{1-\delta_{n-1}} \big(\delta_n x_n(t_1, t_2) + \frac{n-1}{2n-1} \lambda_{n-1} \int_{\frac{1}{2}}^{\frac{4}{3}} \int_{\frac{2}{3}}^{\frac{5}{4}} F_{n-1}(t_1, t_2, s_1, s_2) ds_2 ds_1\big),\ n = 10, 9, \ldots, 2. \tag{4.15}$$

Using the relations (4.8) and (4.9), equation (4.15) can be written as follows:

$$x_{n-1}(t_1, t_2) = \frac{1}{1-\delta_{n-1}} \big(\delta_n x_n(t_1, t_2) + \frac{n-1}{2n-1} \lambda_{n-1} (C_{1,n-1} \cos(t_1+t_2) + C_{2,n-1} \sin(t_1+t_2))\big). \tag{4.16}$$

By substituting (4.16) in $C_{1,n-1}$ and $C_{2,n-1}$, one obtains a nonlinear system similar to (4.13), in which the values of $C_{1,n-1}$ and $C_{2,n-1}$ are solved for fixed values of $\lambda_{n-1} = 10$ for $n = 10, 9, \ldots, 2$. Hence we choose $\lambda_{n-1} = 10$ for $n = 10, 9, \ldots, 2$ and presented them in the table (1).

Table 1.

n	$C_{1,n}$	$C_{2,n}$
10	-0.06882143395	0.2192127099
9	-0.06857378254	0.2184238810
8	-0.06807788018	0.2168443135
7	-0.06743565394	0.2147986695
6	-0.06657190157	0.2120474118
5	-0.06534822955	0.2120474118
4	-0.06348080126	0.2022015188
3	-0.06028165222	0.1920114647
2	-0.05354673859	0.1705591557
1	-0.03058609698	0.09742402651

On the other hand, to check how good are the approximate values of $C_{1,n-1}$ and $C_{2,n-1}$, in table (2), we have considered bigger values for n and computed $C_{1,i}$ and $C_{2,i}$, for each n and some i.
The results in this table reveal that, for big values of n, $C_{1,i}$ and $C_{2,i}$, for $i = 1, 2, \ldots, n$ are approximately the same, which concludes that $x_i(t_1, t_2)$ for $i = 1, 2, \ldots, n$ for different values of n, are approximately the same.

Table 2.

n	$C_{1,100}$	$C_{2,100}$	$C_{1,10}$	$C_{2,10}$	$C_{1,1}$	$C_{2,1}$
100	-0.07208672981	0.2296134573	0.06896900796	0.2196827684	-0.03058609698	0.09742402651
200	-0.07210081244	0.2296583139	0.06896900796	0.2196827684	-0.03058609698	0.09742402651
300	-0.07210081244	0.2296583139	0.06896900796	0.2196827684	-0.03058609698	0.09742402651

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