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AN ANALYSIS ON THE PERIODIC SOLUTIONS OF AN n-TH ORDER NON-LINEAR DIFFERENTIAL EQUATION

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Abstract. This paper deals with existence of w-periodic solutions of a non-linear n-th order differential equation with variable delays. Some sufficient conditions related to w-periodicity of solutions were obtained by using coincidence degree theory. In a particular case, an application showing the accuracy of our results was given.

1. Introduction

Consider the nonlinear n-th order functional differential equation

$$z^{(n)}(t) + A(t, z^{(n-1)}(t)) + \sum_{i=1}^{2} B_i(t, z(t - \eta_i(t))) = G(t), \tag{1.1}$$

where $\eta_1, \eta_2, p, G \in C(R, R)$, $B_i \in C(R^2, R)$ and η_1, η_2 and E are w- periodic functions, $B_i(t + w, z) = B_i(t, z)$ and $A(t + w, z) = A(t, z)$,$(i = 1, 2)$, $A(t, 0) = 0$, $2 \leq n < \infty$, n is an integer number. Let $n = 2$,

$$A(z(t)) = A(t, z(t))$$

and

$$B(z(t - \eta(t))) = B_1(t, z(t - \eta_1(t))) + B_2(t, z(t - \eta_2(t))).$$

Then, Eq. (1.1) reduces to

$$z''(t) + B(t, z(t - \eta(t))) + A(z(t)) = G(t), \tag{1.2}$$

which is known as the delayed Rayleigh equation. It is well known that the British mathematical physicist Lord Rayleigh, a Nobel Prize Laureate in Physics in 1904, introduced an equation of the form

$$z''(t) + f(z'(t)) + az(t) = 0,$$

to model the oscillations of a clarinet reed; for details (see [10]). Hence, we can say Eq (1.1) as a high-order Rayleigh equation. Nonlinear differential equations correspond to the mathematical formulation of many physical problems. Since there is no method to solve them, the examination of such equations is usually limited to a variety of very specific situations, and one must resort to a variety of approximation

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methods. The existence of periodic solutions of nonlinear Rayleigh equations has been extensively studied by many authors in the past two decades. For example, Li and Huang [6] used the coincidence degree theory to establish new results on the existence and uniqueness of T periodic of the second order Rayleigh differential equation without delay. Legun peng et al. [9] consider the second order differential equation with two variable delays and obtained some conditions for existence of periodic solutions by using coincidence degree theory. Zhao and Liu [13] consider the n-th order differential equation with one variable delay obtained sufficient conditions for T-periodic solution. Zhao and Tang [15] considered the second order equation with a deviating argument and obtained some new results for periodic solutions. For some the other works on the existence of periodic solutions and the existence almost and pseudo periodic solutions of various differential equations, see , also, [19-23] and the references therein. However, as far as we know, the existence and uniqueness of periodic solutions of Equation (1.1) has not been investigated by applying the coincidence degree theory. The main purpose of this article is to create sufficient conditions for the existence and uniqueness of periodic solutions in Equation (1.1) by applying the coincidence degree theory. The results of this paper are new and complement the results previously known in the literature. An illustrative example is given to show applicability of the results of this paper.

2. Preliminaries

We use the following notations throughout this article.

Let

$U = \left\{ z \,\middle|\, z \in C^{n-1}(\mathbb{R},\mathbb{R}), z(t+w) = z(t) \right\}$ and $V = \{ z \,|\, z \in C(\mathbb{R},\mathbb{R}), z(t+w) = z(t) \}$ be two Banach spaces with the following norms

$$\|F\|_U = \|F\| = \sum_{j=0}^{n-1} \left|F^j\right|_\infty, |F|_k = \left(\int_0^T |F(t)|^k dt \right)^{\frac{1}{k}}$$

and $\|F\|_V = |F|_\infty = \max_{t\in[0,T]} |F(t)|$. Define a linear operator $H : D(H) \subset U \to V$ by setting $D(H) = \left\{ z \,\middle|\, z \in U, z^{(n)} \in C(\mathbb{R},\mathbb{R}) \right\}$.

Let

$$Hz = z^{(n)}, \; z \in D(H). \tag{2.1}$$

We define a nonlinear operator $E : U \to V$ by

$$Ez = G(t) - \left[A(t, z^{(n-1)}(t)) + \sum_{i=1}^{2} B_i(t, z(t-\eta_i(t))) \right]. \tag{2.2}$$

It is easily to see that $\mathrm{Ker}H = \mathbb{R}$, and

$$\mathrm{Im}\, H = \left\{ z \,\middle|\, z \in Y, \int_0^w z(s)ds = 0 \right\}.$$

Thus, H is a Fredholm operator with index zero. Define the continuous projectors $P_1 : U \to \mathrm{Ker}H$ and $P_2 : V \to V/\,\mathrm{Im}\, H$ by setting $P_1 z(t) = \frac{1}{w}\int_0^w z(s)ds = P_2 z(t)$,

where P_1 and P_2 are continuous functions. Hence, $\mathrm{Im}\, P_1 = \mathrm{Ker}H$ and $\mathrm{Ker}P_2 = \mathrm{Im}\, H$. The function $H_{P_1}^{-1} : \mathrm{Im}\, H \to D(H) \cap \mathrm{Ker}P_1$ is the inverse of $H|_{D(H)\cap \mathrm{Ker}P_1}$ and one observes that $H_{P_1}^{-1}$ is a compact operator. Hence, E is H-compact on $\bar{\Omega}$, where $\Omega \subseteq U$ is an open bounded set. Given Eq. (2.1) and Eq. (2.2), we get

$$Hz = \lambda Ez, \tag{2.3}$$

and

$$z^{(n)}(t) + \lambda \left[A\left(t, z^{(n-1)}(t)\right) + \sum_{i=1}^{2} B_i(t, z(t-\eta_i(t))) \right] = \lambda G(t), \tag{2.4}$$

where $\lambda \in (0,1)$.

Lemma 2.1. *([4]). Let U and V be two Banach spaces, $H : D(H) \subset U \to V$ be a Fredholm operator with index zero. $\Omega \subset U$ is an open bounded set, and $E : U \to V$ is H-compact on $\bar{\Omega}$.*

Let the conditions (i) $Hz \neq \lambda u, \forall z \in \partial\Omega \cap D(H), \lambda \in (0,1)$, (ii) $Ez \notin \operatorname{Im} H, \forall z \in \partial\Omega \cap KerH$ and (iii) $\deg\{P_2E, \Omega \cap KerH, 0\} \neq 0$ hold. Then Eq. (2.4) has at least one solution on $D(H) \cap \bar{\Omega}$.

Lemma 2.2. *([17]) If $z \in C^2(\mathbb{R})$, $z(t+w) = z(t)$, then*

$$|z'(t)|_2 \leq \frac{w}{2\pi} |z''(t)|_2, \tag{2.5}$$

Lemma 2.3.

(B_{11}): $\dfrac{(B_i(t,z_2) - B_i(t,z_1))}{(z_2 - z_1)} > 0;$

(B_{12}): $\dfrac{(B_i(t,z_2) - B_i(t,z_1))}{(z_2 - z_1)} < 0, \forall i = 1,2, z_i \in \mathbb{R}, \forall t \in \mathbb{R} and\ z_1 \neq z_2;$

(B_{21}): $z\left(B_2(t,z_2) + B_1(t,z_1)\right) - G(t)) > 0;$

(B_{22}): $z\left(B_2(t,z_2) + B_1(t,z_1) - G(t)\right) < 0, \forall t \in \mathbb{R}, |z| \geq k > 0, k \in R.$

If $z(t) = z(t+w)$ is a solution of Eq. (2.4), then

$$|z|_\infty \leq k + \sqrt{w}|z'|_2 \tag{2.6}$$

Proof. Let $z(t+w) = z(t)$ and z(t) is a solution of Eq. (2.2). Set

$$\frac{d^{n-2}}{dt^{n-2}} z(t_{**}) = \min_{t \in R} \frac{d^{n-2}}{dt^{n-2}} z(t),$$
$$\frac{d^{n-2}}{dt^{n-2}} z(t_*) = \max_{t \in R} \frac{d^{n-2}}{dt^{n-2}} z(t),$$

where $-\infty < t_*, t_{**} < \infty$. Then, we can write

$$\frac{d^n}{dt^n} z(t_*) \leq 0, \quad \frac{d^{n-1}}{dt^{n-1}} z(t_*) = 0, \tag{2.7}$$
$$\frac{d^n}{dt^n} z(t_{**}) \geq 0 \text{ and } \frac{d^{n-1}}{dt^{n-1}} z(t_{**}) = 0.$$

Because of

$$A(t,0) = 0$$

and Eq. (2.4), (2.15) implies that

$$(G(t_*) - B_1(t_*, z(t_* - \eta_1(t_*))) - B_2(t_*, z(t_* - \eta_2(t_*)))) = \frac{z^{(n)}(t_*)}{\lambda} \geq 0, \tag{2.8}$$

$$G(t_{**}) - B_1(t_{**}, z(t_{**} - \eta_1(t_{**}))) - B_2(t_{**}, z(t_{**} - \eta_2(t_{**}))) = \frac{z^{(n)}(t_{**})}{\lambda} \leq 0. \tag{2.9}$$

From the continuity of the function $G(t) - B_1(t, z(t - \eta_2(t))) + B_2(t, z(t - \eta_2(t)))$, (2.6) and (2.7), we can find a constant $\bar{t} \in R$ such that

$$G(\bar{t}) - B_1(\bar{t}, z(\bar{t} - \eta_1(\bar{t}))) + B_2(\bar{t}, z(\bar{t} - \eta_2(\bar{t}))) = 0. \tag{2.10}$$

Claim. If $z(t) = z(t + w)$ is a solution of Eq. (2.4), then there exists a constant $\bar{\bar{t}} \in R$ such that

$$\left|u(\bar{\bar{t}})\right| \leq k. \tag{2.11}$$

Assume that the inequality (2.11) does not hold. Then

$$k < |z(t)| \text{ for all } t \in R. \tag{2.12}$$

Using (B_{21}) and (2.8), we show that the following inequalities hold:

$$k < z(\bar{t} - \eta_1(\bar{t})) < z(\bar{t} - \eta_2(\bar{t})); \tag{2.13}$$

$$k < z(\bar{t} - \eta_2(\bar{t})) < z(\bar{t} - \eta_1(\bar{t})); \tag{2.14}$$

$$-k > z(\bar{t} - \eta_2(\bar{t})) > z(\bar{t} - \eta_1(\bar{t})); \tag{2.15}$$

$$-k > z(\bar{t} - \eta_1(\bar{t})) > z(\bar{t} - \eta_2(\bar{t})). \tag{2.16}$$

Suppose that (2.11) holds. By means of the conditions (B_{11}), (B_{12}), (B_{21}) and (B_{22}), we take into consideration the following four separate cases:

Case (1). If (B_{21}) and (B_{11}) are satisfied, then form (2.11), we obtain

$$\begin{aligned} 0 &< B_1(\bar{t}_1, z(\bar{t} - \eta_2(\bar{t}))) + B_2(\bar{t}, z(\bar{t} - \eta_2(\bar{t}))) - G(\bar{t}) \\ &< B_1(\bar{t}, z(\bar{t} - \eta_1(\bar{t}))) + B_2(\bar{t}, z(\bar{t} - \eta_2(\bar{t}))) - G(\bar{t}), \end{aligned}$$

which contradicts to (2.8). So (2.9) is true.

Case (2). If (B_{21}) and (B_{12}) hold, then by (2.11), we obtain

$$\begin{aligned} 0 &< B_1(\bar{t}, z(\bar{t} - \eta_1(\bar{t}))) + B_2(\bar{t}, z(\bar{t} - \eta_1(\bar{t}))) - G(\bar{t}) \\ &< B_1(\bar{t}, z(\bar{t} - \sigma_1(\bar{t}))) + B_2(\bar{t}, z(\bar{t} - \sigma_2(\bar{t}))) - G(\bar{t}), \end{aligned}$$

which contradicts (2.8). So (2.9) is true.

Case (3). If (B_{22}) and (B_{11}) hold, then by (2.11), we obtain

$$\begin{aligned} 0 &> B_1(\bar{t}, z(\bar{t} - \eta_1(\bar{t}))) + B_2(\bar{t}, z(\bar{t} - \eta_1(\bar{t}))) - G(\bar{t}) \\ &> B_1(\bar{t}, z(\bar{t} - \eta_1(\bar{t}))) + B_2(\bar{t}, z(\bar{t} - \eta_2(\bar{t}))) - G(\bar{t}), \end{aligned}$$

which contradicts to (2.10). So Eq. (2.9) is true.

Case (4). If (B_{22}) and (B_{12}) are hold, then by (2.11), we obtain

$$\begin{aligned} 0 &> B_1(\bar{t}, z(\bar{t} - \eta_2(\bar{t}))) + B_2(\bar{t}, z(\bar{t} - \eta_2(\bar{t}))) - G(\bar{t}) \\ &> B_1(\bar{t}, z(\bar{t} - \eta_1(\bar{t}))) + B_2(\bar{t}, z(\bar{t} - \eta_2(\bar{t}))) - G(\bar{t}), \end{aligned}$$

which contradicts to (2.8). So (2.9) is true. Suppose that (2.12) (or other options as (2.13) and (2.14)) holds, the accuracy of (2.9) can be demonstrated by using the same methods as in the Cases (1)-(4).Thus, the proof of our claim is over.

Let $\bar{\bar{t}} = cw + t_0$, where $0 \leq t_0 \leq w$, and c is an integer number. Then

$$|z(t)| = \left| \int_{t_0}^{t} z'(s)ds + z(t_0) \right| \leq k + \int_0^w |z'(s)|\, ds.$$

Hence, we obtain

$$|z|_\infty = \max_{t \in [0,w]} |z(t)| \leq k + \sqrt{w}|z'|_2.$$

So the proof is over. $\square$

Lemma 2.4. *Assume that conditions* (B_{11}) *and* $B_{12})$ *hold and the functions* $B_i(t,x)$ *are monotone and satisfy the Lipschitz condition with constants* $b_i (i = 1, 2)$ *in* x. *Then Eq. (1.1) has at the most one* w *- periodic solution provided that*

$$K = w^n \left(\frac{K_1}{(2\pi)^n} + \frac{2}{(2\pi)^{n-1}} (b_1 + b_2) \right) < 1.$$

Proof. Let $\alpha(t) = z_1(t) - z_2(t)$, where $z_1(t + w) = z_1(t)$ and $z_2(t + w) = z_2(t)$ are two distinct solutions of Eq. (1.1). Then, from Eq. (1.1), we have

$$\alpha^{(n)}(t) + A(t, z_1^{(n-1)}(t)) - A(t, z_2^{(n-1)}(t)) \\ + \sum_{i=1}^{2} (B_i(t, z_1(t - \eta_i(t))) - z_2(t, z_2(t - \eta_i(t)))) = 0. \tag{2.17}$$

Set

$$\min_{t\in R} \frac{d^{n-2}}{dt^{n-2}}\alpha(t) = \frac{d^{n-2}}{dt^{n-2}}\alpha(\tau_2),\ \max_{t\in R} \frac{d^{n-2}}{dt^{n-2}}\alpha(t) = \frac{d^{n-2}}{dt^{n-2}}\alpha(\tau_1),$$

where $\tau_1, \tau_2 \in \mathbb{R}$. From this point, we derive

$$\frac{d^n}{dt^n}\alpha(\tau_2) \geq 0,\ \frac{d^{n-1}}{dt^{n-1}}\alpha(\tau_2) = 0,$$

and

$$\frac{d^{n-1}}{dt^{n-1}}\alpha(\tau_1) = 0,\ \frac{d^n}{dt^n}\alpha(\tau_1) \leq 0.$$

In view of (2.15), it follows that

$$\sum_{i=1}^{2} (B_i(\tau_1, z_1(\tau_1 - \eta_i(\tau_1))) - B_i(\tau_1, z_2(\tau_1 - \eta_i(\tau_1)))) > 0,$$

$$\sum_{i=1}^{2} (B_i(\tau_2, z_1(\tau_2 - \eta_i(\tau_2))) - z_i(\tau_2, z_2(\tau_2 - \eta_i(\tau_2)))) < 0.$$

Because of the continuity of $B_i(t, z_1(t - \eta_i(t))) - B_i(t, z_2(t - \eta_i(t)))$ on R and the existence of the above last two inequalities, there exists a constant $\kappa \in R$ such that

$$B_1(\kappa, z_1(\kappa - \eta_1(\kappa))) - B_i(\kappa, z_2(\kappa - \eta_1(\kappa))) + B_2(\kappa, z_1(\kappa - \eta_2(\kappa))) - B_2(\kappa, z_2(\kappa - \eta_2(\kappa))) = 0 \tag{2.18}$$

Let $\kappa - \eta_i(\kappa) = nw + \varpi$, where $\varpi \in [0, w]$ and $n \in Z$. Then, (2.16) and $K < 1$ imply that

$$\alpha(\kappa) = z_1(\varpi) - z_2(\varpi) = z_1(\kappa - \eta_i(\kappa)) - z_2(\kappa - \eta_i(\kappa)) = 0. \tag{2.19}$$

Therefore, it follows that

$$|\alpha(t)| = \left| \int_{\varpi}^{t} \alpha'(s)ds + \alpha(\varpi) \right| \leq \int_{\varpi}^{t} |\alpha'(s)|\, ds, t \in [0, w],$$

$$|\alpha|_\infty \leq \sqrt{w}|\alpha'|_2. \tag{2.20}$$

Now assume that $K < 1$ holds. Multiplying (2.17) with $\frac{d^n}{dt^n}\alpha(t)$ and then integrating the obtained inequality from 0 to w, we have

$$\left(\left| \frac{d^n}{dt^n}\alpha(t) \right|_2 \right)^2 = \int_0^w \left| \frac{d^n}{dt^n}\alpha(t) \right|^2 dt \tag{2.21}$$

$$= \int_0^w (A(t, \frac{d^n-1}{dt^{n-1}} z_1(t)) - A(t, \frac{d^{n-1}}{dt^{n-1}} z_2(t))) \frac{d^n}{dt^n} \alpha(t) dt$$

$$- \int_0^w \sum_{i=1}^2 (B_i(t, z_1(t-\eta_i(t))) - \sum_{i=1}^2 B_i(t, z_2(t-\eta_i(t)))) \frac{d^n}{dt^n} \alpha(t) dt$$

$$\leq K_1 \int_0^w \left| \frac{d^n-1}{dt^{n-1}} z_1(t) - \frac{d^{n-1}}{dt^{n-1}} z_2(t) \right| \left| \frac{d^n}{dt^n} \alpha(t) \right| dt + b_1 \int_0^w |z_1(t-\eta_1(t)) - z_2(t-\eta_1(t))|$$

$$\times \frac{d^n}{dt^n} \alpha(t) dt + b_2 \int_0^w |z_1(t-\eta_2(t)) - z_2(t-\eta_2(t))| \left| \frac{d^n}{dt^n} \alpha(t) \right| dt.$$

Hence, Lemma 2.2, (2.18), the Schwarz inequality and (2.19) imply

$$\left(\left| \frac{d^n}{dt^n} \alpha(t) \right|_2 \right)^2 \leq K_1 \left(\int_0^w |\alpha(t)|^2 \right)^{1/2} \left(\int_0^w \left| \frac{d^n}{dt^n} \alpha(t) \right|^2 \right)^{1/2} + (b_1+b_2) |\alpha|_\infty \sqrt{w} \left| \frac{d^n}{dt^n} \alpha(t) \right|_2$$

$$= K_1 |\alpha(t)|_2 \left| \frac{d^n}{dt^n} \alpha(t) \right|_2 + (b_1+b_2) \frac{w^n}{(2\pi)^{n-1}} \left| \frac{d^n}{dt^n} \alpha(t) \right|_2^2 \leq K \left| \frac{d^n}{dt^n} \alpha(t) \right|_2^2, \quad (2.22)$$

where $K = \frac{w^n}{(2\pi)^n} (K_1 + 2\pi (b_1 + b_2))$.

Since $K < 1$ and $\alpha(t), \frac{d}{dt}\alpha(t), ..., \frac{d^n}{dt^n}\alpha(t)$ are continuous w-periodic functions, then in view of (2.17) and (2.19), we have

$$\frac{d}{dt}\alpha(t) \equiv ... \equiv \frac{d^n}{dt^n}\alpha(t) \equiv \alpha(t) \equiv 0, \; for\, all\, t \in R.$$

Thus, Eq. (1.1) has at most one w-periodic solution. The proof is finished. □

3. Main Results

Theorem 3.1. *If conditions (B_{11}), B_{12} and $K < 1$ hold, then Eq. (1.1) has a unique w-periodic solution.*

Proof. By $K < 1$ and Lemma 2.4, we know that Eq. (1.1) has at most one w - periodic solution. Therefore, it is sufficient to show that there exists at least one solution of Eq. (1.1). We claim that all solutions of (2.4) are bounded. Let $z(t+w) = z(t)$ be a solution of Eq. (2.4). Multiplying (2.4) with $z^n(t)$ and then integrating the obtained result on the interval $[0, w]$, using Lemma 2.2, Lemma 2.3

and the Schwarz inequality, we can derive

$$\begin{aligned}
&\left|\frac{d^n}{dt^n}z(t)\right|_2^2=\int_0^w\left|\frac{d^n}{dt^n}z(t)\right|^2dt\\
&-\lambda\int_0^w A(t,\frac{d^{n-1}}{dt^{n-1}}z(t))\frac{d^n}{dt^n}z(t)dt\\
&-\lambda\int_0^w\sum_{i=1}^2(B_i(t,z(t-\eta_i(t))\frac{d^n}{dt^n}z(t)dt+\lambda\int_0^w G(t)\frac{d^n}{dt^n}z(t)dt \qquad (3.1)\\
&\le\int_0^w\left|A(t,\frac{d^{n-1}}{dt^{n-1}}z(t))-A(t,0)\right|\left|\frac{d^n}{dt^n}z(t)\right|dt\\
&+\int_0^w\sum_{i=1}^2|B_i(t,z(t-\eta_i(t)-B_i(t,0)|\left|\frac{d^n}{dt^n}z(t)\right|dt\\
&+\int_0^w|B_i(t,0)|\left|\frac{d^n}{dt^n}z(t)\right|dt+\int_0^w|G(t)|\left|\frac{d^n}{dt^n}z(t)\right|dt
\end{aligned}$$

$$\begin{aligned}
&\le K_1\left|z^{(n-1)}(t)\right|_2\left|\frac{d^n}{dt^n}z(t)\right|_2+b_1\int_0^w|z(t-\eta_1(t))|\left|\frac{d^n}{dt^n}z(t)\right|dt\\
&+b_2\int_0^w|z(t-\eta_2(t))|\left|\frac{d^n}{dt^n}z(t)\right|dt+\int_0^w\sum_{i=1}^2|B_i(t,0)|\left|\frac{d^n}{dt^n}z(t)\right|dt+\int_0^w|G(t)|\left|\frac{d^n}{dt^n}z(t)\right|dt\\
&\le K_1\frac{w}{2\pi}(\left|\frac{d^n}{dt^n}z(t)\right|_2)^2+(b_1+b_2)\sqrt{w}|z|_\infty\left|\frac{d^n}{dt^n}z(t)\right|_2\\
&+\left[\max\left\{\sum_{i=1}^2|B_i(t,0)|:0\le t\le w\right\}+|G|_\infty\right]\left|\frac{d^n}{dt^n}z(t)\right|_2\\
&\le(K_1\frac{w}{2\pi}+2\pi(b_1+b_2)\left(\frac{w}{2\pi}\right)^n)(\left|\frac{d^n}{dt^n}z(t)\right|_2)^2\\
&+\left[(b_1+b_2)k+\max\left\{\sum_{i=1}^2|B_i(t,0)|:0\le t\le w\right\}+|p|_\infty\right]\left|\frac{d^n}{dt^n}z(t)\right|_2.
\end{aligned}$$

Since $K<1$, then it follows from (3.1) that there exists a positive constant C_1 such that

$$\left|\frac{d^j}{dt^j}z\right|_2\le\left(\frac{w}{2\pi}\right)^{n-j}\left|\frac{d^n}{dt^n}z\right|_2<C_1, j=1,2,...,n. \qquad (3.2)$$

We know that, $\frac{d^j}{dt^j}z(0)=\frac{d^j}{dt^j}z(w)$. Then, we can find a constant $\xi_j\in[0,w]$ such that

$$\frac{d^{j+1}}{dt^{j+1}}z(\xi_j)=0,$$

and

$$\left|\frac{d^{j+1}}{dt^{j+1}}z(t)\right|=\left|\frac{d^{j+2}}{dt^{j+2}}z(\xi_j)+\int_{\xi_j}^t\frac{d^{j+1}}{dt^{j+1}}z(s)ds\right|\le\left|\int_0^w\frac{d^{j+2}}{dt^{j+2}}z(s)ds\right|\le\sqrt{w}\left|\frac{d^{j+2}}{dt^{j+2}}z\right|_2. \qquad (3.3)$$

By noting (2.4), (2.22) and (3.1), there exists a positive constant C_2 such that

$$\left|\frac{d^j}{dt^j}z\right|_\infty \leq k+\sqrt{w}\left|\frac{d^{j+1}}{dt^{j+1}}z\right|_2 \leq C_2, \tag{3.4}$$

which implies that, for all possible w-periodic solutions $z(t)$ of Eq. (2.2), there is a constant M_1 such that

$$\|z\| = \sum_{j=0}^{n-1}\left|\frac{d^j}{dt^j}z\right|_\infty < M_1,$$

where $M_1 > 0$, which is independent of λ. If $z \in \Omega_1 = \{\, z|\, z \in \mathrm{Ker}H \cap U, \text{and} \quad Ez \in \mathrm{Im}\, H\}$, then there exists a constant M_2 such that

$$z(t) = M_2, \mathit{and} \int_0^w [B_1(t, M_2) + B_2(t, M_2) - G(t)]\, dt = 0. \tag{3.5}$$

Thus,

$$|z(t)| = |M_2| < k, \text{ for all } z(t) \in \Omega_1. \tag{3.6}$$

Let $C = M_1 + M_2 + k + 1$. Set

$$\Omega = \left\{ z|\, z \in U, \|z\| = \sum_{j=0}^{n-1}\left|z^{(j)}\right|_\infty < C \right\}.$$

By Eq. (2.1) and Eq. (2.2), we see that E is H-compact $\overline{\Omega}$. From (3.3), (2.21) and $C > \max\{M_1 + M_2, k\}$, it follows that the conditions (i) and (ii) in Lemma 2.1 hold. Let us define a continuous function $H(z,\mu)$ by setting

$$H(z,\mu) = (1-\mu)z - \mu.\frac{1}{w}\int_0^w [B_1(t,z) + B_2(t,z) - G(t)]; \mu \in [0,1].$$

If conditions (B_{11}) and (B_{12}) hold, then

$$zH(z,\mu) \neq 0 \text{ for all } z \in \partial\Omega \cap \mathrm{Ker}H.$$

Thus, by the homotopy invariance theorem, we get

$$\deg\{P_2H, \Omega\cap KerH, 0\} = \deg\left\{-\frac{1}{w}\int_0^w [B_1(t,z) + B_2(t,z) - G(t)]\, dt, \Omega \cap KerH, 0\right\} \neq 0.$$

This completes the proof. □

Theorem 3.2. *If conditions* (B_{12}) *and* (B_{22}) *and* $K < 1$ *hold, then Eq. (1.1) has a unique* w*-periodic solution.*

Proof. The proof of this theorem is similar to that of Theorem 3.1. We omit the details of the proof. □

4. Example

Consider the nonlinear Rayleigh equation with two variable delays:

$$z'' + \cos^2(\frac{t}{2})\frac{z'}{16} + \frac{\sin z'}{16} + B_1(t, z(t-\sin t)) + B_2(t, z(t-\cos t)) = \frac{1}{16}\sin^2 t, \tag{4.1}$$

which has a unique 2π- periodic solution. If we compare to Eq. (4.1) with Eq. (1.1), we have the following relations:

$$A(t,z') = \cos^2(\frac{t}{2})\frac{z'}{16} + \frac{\sin z'}{16},$$

$$B_1(t,z) = -\frac{1}{144\pi^2}z \text{ for all } t \in R, z \in R,$$

$$B_2(t,z) = -z^3 \text{ for all } t \in R, z \leq 0$$

and

$$B_2(t,z) = -\frac{1}{144\pi^2}z \text{ for all } t \in R, z > 0.$$

Then, we can write

$$B_1(t,z) + B_2(t,z) - G(t) = -\frac{1}{72\pi^2}z - \frac{1}{16}\cos^2 2t \geq -\frac{1}{16\pi^2}z - \frac{1}{16}$$

for all $t \in R, z > 0$. So, it is clear that all the conditions of Theorem 3.1 are satisfied. Therefore, by Theorem 3.1, Eq. (4.1) has a unique 2π-periodic solution.

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