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DESIGN AND ANALYSIS OF AN EFFICIENT MULTI STEP ITERATIVE SCHEME FOR SYSTEMS OF NONLINEAR EQUATIONS

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Abstract. In this paper, an efficient sixth order multi step method is proposed in Banach spaces and its local convergence is studied in order to approximate a locally unique solution of a nonlinear or a system of nonlinear equations. Based on Lipschitz's continuity condition, new convergence analysis provides radius of convergence, bounds on error and estimates on the uniqueness of solution under assumptions on the first derivative only. Hence, the method can be applied to a wider class of functions that expands its applicability. Numerical examples are presented to illustrate the theoretical results. In the end, the basins of attraction are given to show the convergence behavior in complex plane.

1. Introduction

One of the most captivating topics in theory of numerical analysis is the construction of fixed point methods to solve nonlinear equations and system of nonlinear equations. This interest arises from many applications of nonlinear equations in various disciplines of engineering and applied sciences. The paramount significance of this subject has resulted in development of plethora of numerical methods. Besides the exceptional cases, the most frequently used methods are of iterative nature (see [1–5]). In this paper, our objective is to develop an iterative method of higher convergence order for finding an approximate solution x^* of the equation

$$H(x) = 0, \tag{1.1}$$

where $H : \Omega \subseteq X_1 \to X_2$, X_1 and X_2 are Banach spaces and Ω is an open convex subset of X_1. The solution x^* can be obtained as a fixed point of some function $\psi : \Omega \subseteq X_1 \to X_2$ by means of the fixed point iteration

$$x_{k+1} = \psi(x_k), \quad k = 0, 1, 2, \ldots.$$

One of the most popular iterative method for solving nonlinear equations is the quadratically convergent Newton's method

$$x_{k+1} = x_k - H'(x_k)^{-1}H(x_k), \quad k = 0, 1, 2, \ldots \tag{1.2}$$

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where x_0 is initial point and $H'(x_k)^{-1} \in \mathcal{L}(X_2, X_1)$, where $\mathcal{L}(X_2, X_1)$ is set of bounded linear operators from X_2 into X_1. Numerous modified methods have been proposed in literature to attain the higher order of convergence, see, for example [6–22] and references therein.

Inspired by the ongoing work in this direction, we here propose a novel iterative scheme with sixth order of convergence. The scheme of present contribution consists of three steps of which the first two steps are the generalizations of Jarratt's two step fourth order method [23] for solving scalar equation $h(x) = 0$, which is given as

$$\begin{aligned} y_k &= x_k - \frac{2}{3}\frac{h(x_k)}{h'(x_k)}, \\ x_{k+1} &= x_k - \frac{5}{8}\frac{h(x_k)}{h'(x_k)} - \frac{3}{8}\frac{h(x_k)h'(x_k)}{h'(y_k)^2}. \end{aligned} \tag{1.3}$$

and the third step is a Newton-like step.

The rest of the paper is organized as follows. In Section 2, we develop the scheme and analyze its sixth order convergence. The local convergence analysis under Lipschitz continuity condition on first order Fréchet derivative is also carried out. In Section 3, numerical examples are provided to verify the theoretical results obtained so far. The performance regarding convergence properties of the proposed method is compared with existing ones. Basins of attraction are displayed to check the steadiness of the proposed method and compared with other similar existing methods in literature. Concluding remarks are reported in Section 4.

2. Development of the method and Convergence analysis

We consider the two-step Jarratt's scheme (1.3) for solving the nonlinear system $H(x) = 0$, thereby writing it in generalized form as

$$\begin{aligned} y_k &= x_k - \frac{2}{3}\Gamma_k H(x_k), \\ z_k &= x_k - \frac{5}{8}\Gamma_k H(x_k) - \frac{3}{8}H'(y_k)^{-1}H'(x_k)H'(y_k)^{-1}H(x_k), \end{aligned} \tag{2.1a}$$

where $\Gamma_k = H'(x_k)^{-1}$, $k \in \mathbb{N}$.
Now based on the two-step scheme (2.1a), let us write third step to obtain the approximation x_{k+1} to a solution of $H(x) = 0$ in the following way:

$$x_{k+1} = z_k - (aI + b\Gamma_k H'(y_k))\,\Gamma_k H(z_k), \tag{2.1b}$$

where I is identity operator on X_1 and a and b are real numbers. It is clear that the third step is a Newton-like step. So we call the scheme (2.1b) along with (2.1a) as a Newton-Jarratt scheme.
We analyze the convergence of proposed Newton-Jarratt scheme in the following section.

2.1. **Convergence order.**

Theorem 2.1. *Let the function $H : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be sufficiently differentiable in a convex set Ω containing the zero x^* of $H(x)$. Suppose that $H'(x)$ is continuous and nonsingular in x^*. If an initial approximation x_0 is sufficiently close to x^*,* *then the local convergence order of Newton-Jarratt scheme is six, provided* $a = \frac{5}{2}$ *and* $b = -\frac{3}{2}$.

Proof. Developing $H(x_k)$ in the neighborhood of x^* and assuming that $\Gamma = H'(x^*)^{-1}$ exists, we write

$$H(x_k) = H'(x^*)\left[e_k + A_2e_k^2 + A_3e_k^3 + A_4e_k^4 + A_5e_k^5 + A_6e_k^6 + O(e_k^7)\right], \qquad (2.2)$$

where $e_k = x_k - x^*$, $A_i = \frac{1}{i!}\Gamma H^{(i)}(x^*)$, $H^{(i)}(x^*) \in \mathcal{L}(\mathbb{R}^n\times, \cdot \overset{i}{\cdot\cdot}, \times\mathbb{R}^n, \mathbb{R}^n)$, $\Gamma \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ and
$(e_k)^i = (e_k, e_k, \overset{i-times}{\cdots\cdots}, e_k)$ with $e_k \in \mathbb{R}^n$, $i = 2, 3, \ldots$
Also,

$$H'(x_k) = H'(x^*)\left[I + 2A_2e_k + 3A_3e_k^2 + 4A_4e_k^3 + 5A_5e_k^4 + 6A_6e_k^5 + O(e_k^6)\right]. \quad (2.3)$$

Inversion of $H'(x_k)$ yields,

$$\Gamma_k = H'(x_k)^{-1} = \left[I + B_1e_k + B_2e_k^2 + B_3e_k^3 + B_4e_k^4 + B_5e_k^5 + O(e_k^6)\right]\Gamma, \qquad (2.4)$$

where

$$\begin{aligned}
B_1 &= -2A_2,\ B_2 = 4A_2^2 - 3A_3,\ B_3 = -(8A_2^3 - 6A_2A_3 - 6A_3A_2 + 4A_4),\\
B_4 &= 8A_2A_4 + 9A_3^2 + 8A_4A_2 - 12A_2^2A_3 - 12A_2A_3A_2 - 12A_3A_2^2 + 16A_2^4 - 5A_5,\\
B_5 &= 10A_2A_5 + 12A_3A_4 + 12A_4A_3 + 10A_5A_2 - 16A_2^2A_4 - 18A_2A_3^2 - 16A_2A_4A_2\\
&\quad -18A_3A_2A_3 - 18A_3^2A_2 - 16A_4A_2^2 + 24A_2^3A_3 + 24A_2^2A_3A_2 + 24A_2A_3A_2^2\\
&\quad +24A_3A_2^3 - 32A_2^5 - 6A_6.
\end{aligned}$$

Post multiplication of (2.4) by $H(x_k)$ and simplifying, we obtain

$$\Gamma_kH(x_k) = e_k + C_1e_k^2 + C_2e_k^3 + C_3e_k^4 + C_4e_k^5 + C_5e_k^6 + O(e_k^7), \qquad (2.5)$$

where

$$\begin{aligned}
C_1 &= -A_2,\ C_2 = 2(A_2^2 - A_3),\ C_3 = -4A_2^3 + 4A_2A_3 + 3A_3A_2 - 3A_4,\\
C_4 &= 6A_2A_4 + 6A_3^2 + 4A_4A_2 - 8A_2^2A_3 - 6A_2A_3A_2 - 6A_3A_2^2 + 8A_2^4 - 4A_5,\\
C_5 &= 8A_2A_5 + 9A_3A_4 + 8A_4A_3 + 5A_5A_2 - 12A_2^2A_4 - 12A_2A_3^2 - 8A_2A_4A_2\\
&\quad -12A_3A_2A_3 - 9A_3^2A_2 - 8A_4A_2^2 + 16A_2^3A_3 + 12A_2^2A_3A_2 + 12A_2A_3A_2^2\\
&\quad +12A_3A_2^3 - 16A_2^5 - 5A_6.
\end{aligned}$$

Taking $\tilde{e}_k = y_k - x^*$ and using (2.5), we obtain

$$\begin{aligned}
\tilde{e}_k = y_k - x^* \quad &= \quad \frac{e_k}{3} + \frac{2A_2e_k^2}{3} - \frac{4}{3}\left(A_2^2 - A_3\right)e_k^3 + \Big(- \frac{8}{3}A_2A_3 - 2A_3A_2\\
&\quad + \frac{8}{3}A_2^3 + 2A_4\Big)e_k^4 + O(e_k^5). \qquad (2.6)
\end{aligned}$$

Expanding $H'(y_k)$ about x^* and using above result, we have

$$\begin{aligned}
H'(y_k) &= H'(x^*)\left[I + 2A_2\tilde{e}_k + 3A_3\tilde{e}_k^2 + 4A_4\tilde{e}_k^3 + O(\tilde{e}_k^4)\right]\\
&= H'(x^*)\Big[I + \frac{2A_2e_k}{3} + \frac{(4A_2^2 + A_3)e_k^2}{3} + \frac{4}{27}\Big(- 18A_2^3 + 18A_2A_3 + 9A_3A_2\\
&\qquad + A_4\Big)e_k^3 + O(e_k^4)\Big]. \qquad (2.7)
\end{aligned}$$

Inverse of $H'(y_k)$ can be obtained in the same way as in (2.4). Thus we have

$$\begin{aligned} H'(y_k)^{-1} &= \Bigg[I - \frac{2A_2e_k}{3} - \left(\frac{8}{9}A_2^2 + \frac{A_3}{3}\right)e_k^2 - \frac{2}{27}\Bigg(33A_2A_3 + 15A_3A_2 - 56A_2^3 \\ &\quad +2A_4\Bigg)e_k^3 + \frac{1}{81}\big(- 316A_2A_4 - 207A_3^2 - 64A_4A_2 + 600A_2^2A_3 \\ &\quad +528A_2A_3A_2 + 204A_3A_2^2 - 704A_2^4 - 5A_5\big)e_k^4\Bigg]\Gamma. \end{aligned} \tag{2.8}$$

Then from (2.2), (2.4) and (2.8) , it follows that

$$\begin{aligned} H'(y_k)^{-1}H'(x_k)H'(y_k)^{-1}H(x_k) &= e_k + \frac{5A_2e_k^2}{3} - \frac{10}{3}\left(A_2^2 - A_3\right)e_k^3 + \frac{1}{27}\big(76A_2^3 + 21A_3A_2 \\ &\quad - 186A_2A_3 - 78A_3A_2 + 127A_4\big)e_k^4 + O(e_k^5). \end{aligned} \tag{2.9}$$

Substituting values from (2.5) and (2.9) in second step of (2.1a), we obtain

$$\hat{e}_k = z_k - x^* = \frac{1}{72}\left(104A_2^3 - 78A_3A_2 + 6A_2A_3 + 8A_4\right)e_k^4 + O(e_k^5).$$

Using Taylor's series of $H(z_k)$ about x^*, we have

$$H(z_k) = H'(x^*)\big[\hat{e}_k + A_2\hat{e}_k^2 + O(\hat{e}_k)^3\big]. \tag{2.10}$$

Further, from (2.4) and (2.7), we obtain

$$\big(aI + bH'(x_k)^{-1}H'(y_k)\big) = (a+b)I - \frac{4}{3}A_2be_k + \left(4A_2^2b - \frac{8A_3b}{3}\right)e_k^2 + O(e_k^3). \tag{2.11}$$

Employing Eqs. (2.4), (2.10) and (2.11) in (2.1b) and simplifying, we get

$$\begin{aligned} e_{k+1} &= (1-a-b)\hat{e}_k + \frac{2}{3}(3a+5b)A_2e_k\hat{e}_k + \frac{1}{3}\big(- 12aA_2^2 + 9aA_3 - 32bA_2^2 \\ &\quad +17bA_3\big)e_k^2\hat{e}_k + O(e_k^7). \end{aligned} \tag{2.12}$$

For the method to be of order six, we must have $a = \frac{5}{2}$ and $b = -\frac{3}{2}$. With these values, the above equation yields

$$e_{k+1} = (6A_2^2 - A_3)e_k^2\hat{e}_k + O(e_k^7). \tag{2.13}$$

Combining (2.10) and (2.13),

$$\begin{aligned} e_{k+1} &= \frac{1}{72}\Bigg(- 8A_3A_4 + 48A_2^2A_4 - 6A_3A_2A_3 + 57A_3^2A_2 + 36A_2^3A_3 - 342A_2^2A_3A_2 \\ &\quad -104A_3A_2^3 + 624A_2^5 + 21A_3A_2A_3 - 126A_2^3A_3\Bigg)e_k^6 + O(e_k^7). \end{aligned} \tag{2.14}$$

The error equation (2.14) yields the sixth order convergence. This completes the proof of Theorem 2.1. □

Thus, the presented method in final form is given as

$$\begin{aligned} \mathrm{y}_k &= \mathrm{x}_k - \frac{2}{3}\Gamma_k\mathrm{H}(\mathrm{x}_k), \\ \mathrm{z}_k &= \mathrm{x}_k - \frac{5}{8}\Gamma_k\mathrm{H}(\mathrm{x}_k) - \frac{3}{8}\mathrm{H}'(\mathrm{y}_k)^{-1}\mathrm{H}'(\mathrm{x}_k)\mathrm{H}'(\mathrm{y}_k)^{-1}\mathrm{H}(\mathrm{x}_k), \\ \mathrm{x}_{k+1} &= \mathrm{z}_k - \left(\frac{5}{2}I - \frac{3}{2}\Gamma_k\mathrm{H}'(\mathrm{y}_k)\right)\Gamma_k\mathrm{H}(\mathrm{z}_k). \end{aligned} \tag{2.15}$$

2.2. **Local Convergence.** In this section, we analyze the local convergence of method (2.15) in Banach space setting. We shall provide the convergence radius, computable error bounds $\|x_k - x^*\|$ and the uniqueness ball for the solution x^* of (1.1). The computation of radii of convergence balls is significant as they yield the degree of difficulty in finding out the initial approximation x_0.

Let $H : \Omega \subseteq X_1 \to X_2$ be a nonlinear Fréchet differentiable operator in an open convex subset Ω of X_1, where X_1 and X_2 are Banach spaces. Let $B(\kappa, \xi)$ and $\overline{B}(\kappa, \xi)$ denote the open and closed balls in X_1 respectively, with center $\kappa \in X_1$ and of radius $\xi > 0$. Let $\mathcal{L}(X_1, X_2)$ be the space of bounded linear operators from X_1 into X_2. For the study of local convergence of method (2.15), we introduce some scalar functions and parameters. Let the parameters $K_0 > 0$ and $K > 0$ be such that $K_0 \leq K$. Define the function G_1 on the interval $[0, \frac{1}{K_0})$ by

$$G_1(t) = \frac{1}{6(1 - K_0 t)} \left(3Kt + 2(1 + K_0 t)\right), \tag{2.16}$$

and the parameter

$$R_1 = \frac{4}{3K + 8K_0} < \frac{1}{K_0}.$$

Notice that $G_1(R_1) = 1$ and for each $t \in [0, R_1)$, we have that

$$0 \leq G_1(t) < 1.$$

Define the functions G_2 and J_2 on $\left[0, \frac{1}{K_0}\right)$ by

$$G_2(t) = \frac{1}{8(1 - K_0 t)} \left(4K + (3P(t)^2 t + 6P(t))(1 + K_0 t)\right) t, \tag{2.17}$$

and

$$J_2(t) = G_2(t) - 1, \tag{2.18}$$

where

$$P(t) = \frac{K_0(1 + G_1(t))}{1 - K_0 G_1(t) t}. \tag{2.19}$$

Now, we have that $J_2(0) < 0$ and $J_2(t) \to +\infty$ as $t \to \frac{1}{K_0}^-$. Then using intermediate value theorem, the equation $J_2 = 0$ has at least one solution in the interval $(0, \frac{1}{K_0})$ and let R_2 be the smallest one. Then, $\forall\ t \in [0, R_2)$,

$$0 \leq G_2(t) < 1.$$

Finally, the functions G_3 and J_3 are defined on interval $\left[0, \frac{1}{K_0}\right)$ as

$$G_3(t) = \left[1 + \left(1 + \frac{3K_0(1 + G_1(t))t}{2(1 - K_0 t)}\right) \frac{1 + K_0 G_2(t) t}{1 - K_0 t}\right] G_2(t), \tag{2.20}$$

and

$$J_3(t) = G_3(t) - 1. \tag{2.21}$$

Now, $J_3(0) < 0$ and $J_3(t) \to +\infty$ as $t \to \frac{1}{K_0}^-$. Let R_3 be the smallest solution of equation $J_3(t) = 0$ in the interval $\left(0, \frac{1}{K_0}\right)$. Then, $\forall\ t \in [0, R_3)$,

$$0 \leq G_3(t) < 1.$$

Define the convergence radius R by

$$R = \min\{R_i\}, \quad i = 1, 2, 3. \tag{2.22}$$

Then, $\forall\ t \in [0, R)$

$$0 \le G_1(t) < 1, \tag{2.23}$$

$$0 \le G_2(t) < 1, \tag{2.24}$$

and

$$0 \le G_3(t) < 1. \tag{2.25}$$

Furthermore, let the following conditions hold for a nonlinear Fréchet differentiable operator $H : \Omega \subseteq X_1 \to X_2$:

($\mathcal{C}_1$)**:** There exists $x^* \in \Omega$ such that $H(x^*) = 0$ and $H'(x^*)^{-1} \in \mathcal{L}(X_2, X_1)$;
($\mathcal{C}_2$)**:** $\|H'(x^*)^{-1}(H'(x) - H'(x^*))\| \le K_0\|x - x^*\|$, $\forall\ x \in \Omega$;
($\mathcal{C}_3$)**:** $\|H'(x^*)^{-1}(H'(x) - H'(y))\| \le K\|x - y\|$, $\forall\ x, y \in \Omega$;

In several studies [24–27] a fourth assumption considered is

$$\|H'(x^*)^{-1}H'(x)\| \le M, \quad \forall \quad x \in B\left(x^*, \frac{1}{K_0}\right), \tag{2.26}$$

for real M. In this paper, the condition (2.26) is dropped and entire process is deduced to find the radius of convergence ball without using the constant M. We use the subsequent results to avoid this additional condition.

Lemma 2.2. *If H satisfies Condition ($\mathcal{C}_2$) and $\overline{B}(x^*, R) \subseteq \Omega$, then for all $x \in B(x^*, R)$, the following results hold:*

$$\|H'(x^*)^{-1}H'(x)\| \le 1 + K_0\|x - x^*\| \tag{2.27}$$

and

$$\|H'(x^*)^{-1}H(x)\| \le (1 + K_0\|x - x^*\|)\|x - x^*\|. \tag{2.28}$$

Proof. The proof is obvious. □

Next, using the above mentioned notations, we present the local convergence of the method (2.15).

Theorem 2.3. *Let $H : \Omega \subseteq X_1 \to X_2$ be a nonlinear Fréchet differentiable operator, where Ω is an open convex set and X_1, X_2 are Banach spaces. Suppose that $K_0 > 0$, $K > 0$, $K_0 \le K$ and H obeys ($\mathcal{C}_1$)-($\mathcal{C}_3$). Let $x^* \in \Omega$ and*

$$\overline{B}(x^*, R) \subseteq \Omega, \tag{2.29}$$

where R is defined by (2.22). Then, starting from $x_0 \in B(x^*, R) \setminus \{x^*\}$, the sequence $\{x_k\}$ generated by method (2.15) is well defined, remains in $\overline{B}(x^*, R)$ $\forall k = 0, 1, 2, \ldots$ and converges to x^*. Furthermore, for all $k \ge 0$, the the following estimates hold:

$$\|H'(x_k)^{-1}H'(x^*)\| \le \frac{1}{1 - K_0\|x_k - x^*\|}, \tag{2.30}$$

$$\|y_k - x^*\| \le G_1(\|x_k - x^*\|)\|x_k - x^*\| < \|x_k - x^*\| < R, \tag{2.31}$$

$$\|H'(y_k)^{-1}(H'(y_k) - H'(x_k))\| \le P(\|x_k - x^*\|)\|x_k - x^*\|, \tag{2.32}$$

$$\|z_k - x^*\| \le G_2(\|x_k - x^*\|)\|x_k - x^*\| < \|x_k - x^*\| < R, \tag{2.33}$$

$$\|x_{k+1} - x^*\| \le G_3(\|x_k - x^*\|)\|x_k - x^*\| < \|x_k - x^*\| < R, \tag{2.34}$$

where the functions G_1, G_2, P and G_3 are given by (2.16), (2.17), (2.19) and (2.20) respectively. Furthermore, for $\Lambda \in [R, \frac{2}{K_0})$, the limit point x^* is the unique solution of $H(x) = 0$ in $\overline{B}(x^*, \Lambda) \cap \Omega$.

Proof. We shall show that the estimates (2.30)-(2.34) hold using mathematical induction. By using (2.22), $(\mathcal{C}_2)$ and the hypothesis $x_0 \in B(x^*, R) \setminus \{x^*\}$, we find

$$\begin{aligned}\|H'(x^*)^{-1}(H'(x_0) - H'(x^*))\| &\leq K_0\|x_0 - x^*\| \\ &\leq K_0R < 1. \end{aligned} \tag{2.35}$$

Then, from Banach lemma on invertible operators [6, 28], $H'(x_0)^{-1} \in \mathcal{L}(X_2, X_1)$ and

$$\|H'(x_0)^{-1}H'(x^*)\| \leq \frac{1}{1 - K_0\|x_0 - x^*\|} < \frac{1}{1 - K_0R}. \tag{2.36}$$

Hence, (2.30) holds for $k = 0$. Therefore, from the first substep of method (2.15) for $k = 0$, it follows that y_0 is well defined. Again, using the first substep of method (2.15) for $k = 0$ and $(\mathcal{C}_1)$, we get the following identity

$$\begin{aligned} y_0 - x^* =& x_0 - x^* - H'(x_0)^{-1}H(x_0) + \frac{1}{3}H'(x_0)^{-1}H(x_0) \\ =& -H'(x_0)^{-1}H'(x^*)\int_0^1 H'(x^*)^{-1}(H'(x^* + t(x_0 - x^*)) - H'(x_0))(x_0 - x^*)dt \\ &+ \frac{1}{3}H'(x_0)^{-1}H'(x^*)H'(x^*)^{-1}H(x_0). \end{aligned} \tag{2.37}$$

We also have that

$$H(x_0) = H(x_0) - H(x^*) = \int_0^1 H'(x^* + t(x_0 - x^*))(x_0 - x^*)dt. \tag{2.38}$$

Using (2.28) and (2.38), we find that

$$\begin{aligned}\|H'(x^*)^{-1}H(x_0)\| \leq& \|\int_0^1 H'(x^*)^{-1}H'(x^* + t(x_0 - x^*))\|dt\|x_0 - x^*\| \\ \leq& (1 + K_0\|x_0 - x^*\|)\|x_0 - x^*\|. \end{aligned} \tag{2.39}$$

Denote $e_0 = \|x_0 - x^*\|$. Using (2.22), (2.23), $(\mathcal{C}_3)$, (2.27), (2.36)-(2.39), we obtain that

$$\begin{aligned}\|y_0 - x^*\| \leq& \|H'(x_0)^{-1}H'(x^*)\|\|\int_0^1 H'(x^*)^{-1}(H'(x^* + t(x_0 - x^*)) - H'(x_0))\|dt\|x_0 - x^*\| \\ &+ \frac{1}{3}\|H'(x_0)^{-1}H'(x^*)\|\|H'(x^*)^{-1}H(x_0)\| \\ \leq& \frac{Ke_0^2}{2(1 - K_0e_0)} + \frac{(1 + K_0e_0)e_0}{3(1 - K_0e_0)} \\ =& G_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < R, \end{aligned}$$

which proves (2.31) for $k = 0$ and $y_0 \in B(x^*, R)$.

Using (2.22), (2.23), $(\mathcal{C}_2)$ and (2.31)(for $k = 0$), we find that

$$\begin{aligned}\|H'(x^*)^{-1}(H'(y_0) - H'(x^*))\| \leq& K_0\|y_0 - x^*\| \\ \leq& K_0G_1(e_0)e_0 \\ \leq& K_0\|x_0 - x^*\| < K_0R < 1. \end{aligned}$$

Then, from Banach lemma of invertible operators, $H'(y_0)^{-1} \in \mathcal{L}(X_2, X_1)$ and

$$\|H'(y_0)^{-1}H'(x^*)\| \leq \frac{1}{1 - K_0 G_1(\|x_0 - x^*\|)\|x_0 - x^*\|}. \tag{2.40}$$

Hence, from the second substep of method (2.15) for $k = 0$, z_0 is well defined. Also using $(\mathcal{C}_2)$, (2.31)(for $k = 0$) and (2.40), we obtain

$$\begin{aligned}
\|H'(y_0)^{-1}(H'(y_0) - H'(x_0))\| \leq & \|H'(y_0)^{-1}H'(x^*)\| \Big(\|H'(x^*)^{-1}(H'(y_0) - H'(x^*))\| \\
& + \|H'(x^*)^{-1}(H'(x_0) - H'(x^*))\| \Big) \\
\leq & \frac{K_0\|y_0 - x^*\| + K_0\|x_0 - x^*\|}{1 - K_0 G_1(\|x_0 - x^*\|)\|x_0 - x^*\|} \\
\leq & \frac{K_0(1 + G_1(e_0))e_0}{1 - K_0 G_1(e_0)e_0} \\
= & P(\|x_0 - x^*\|)\|x_0 - x^*\|.
\end{aligned}$$

Hence, (2.32) holds for $k = 0$. The second substep of method (2.15) for $k = 0$ can be written as

$$\begin{aligned}
z_0 - x^* = & x_0 - x^* - H'(x_0)^{-1}H'(x_0) \\
& + \frac{3}{8}\left(I - H'(y_0)^{-1}H'(x_0)H'(y_0)^{-1}H'(x_0)\right) H'(x_0)^{-1}H(x_0) \\
= & x_0 - x^* - H'(x_0)^{-1}H'(x_0) \\
& - \frac{3}{8}\Big(H'(y_0)^{-1}(H'(x_0) - H'(y_0))H'(y_0)^{-1}(H'(x_0) - H'(y_0)) \\
& + 2H'(y_0)^{-1}(H'(x_0) - H'(y_0))\Big) H'(x_0)^{-1}H'(x^*)H'(x^*)^{-1}H(x_0).
\end{aligned} \tag{2.41}$$

Using (2.22), (2.24), $(\mathcal{C}_3)$, (2.32)(for $k = 0$), (2.36), (2.39) and (2.41), we get in turn that

$$\begin{aligned}
\|z_0 - x^*\| \leq & \|x_0 - x^* - H'(x_0)^{-1}H'(x_0)\| + \frac{3}{8}\Big(\|H'(y_0)^{-1}(H'(y_0) - H'(x_0))\|^2 \\
& + 2\|H'(y_0)^{-1}(H'(y_0) - H'(x_0))\| \Big) \|H'(x_0)^{-1}H'(x^*)\|\|H'(x^*)^{-1}H(x_0)\| \\
\leq & \frac{Ke_0^2}{2(1 - K_0 e_0)} + \frac{3}{8}\Big(P(e_0)^2 e_0 + 2P(e_0) \Big) \frac{(1 + K_0 e_0)e_0^2}{1 - K_0 e_0} \\
= & G_2(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < R.
\end{aligned}$$

Thus we arrive at the estimate (2.33) for $k=0$. Hence, by third substep of method (2.15) for $k=0$, x_1 is well defined. From this substep, we also have

$$\begin{aligned} x_1 - x^* =& z_0 - x^* - \left(\frac{5}{2}I - \frac{3}{2}H'(x_0)^{-1}H'(y_0)\right)H'(x_0)^{-1}H(z_0) \\ =& z_0 - x^* - \left(I - \frac{3}{2}H'(x_0)^{-1}(H'(y_0) - H'(x_0))\right)H'(x_0)^{-1}H(z_0) \\ =& z_0 - x^* - \bigg(I + \frac{3}{2}H'(x_0)^{-1}H'(x^*)(H'(x^*)^{-1}(H'(x_0) - H'(x^*)) \\ & - H'(x^*)^{-1}(H'(y_0) - H'(x^*)))\bigg)H'(x_0)^{-1}H'(x^*)H'(x^*)^{-1}H(z_0). \end{aligned} \tag{2.42}$$

Also, taking $x_0 = z_0$ in (2.39) and using (2.33), we get that

$$\begin{aligned} \|H'(x^*)^{-1}H(z_0)\| \leq& (1 + K_0\|z_0 - x^*\|)\|z_0 - x^*\| \\ \leq& (1 + K_0G_2(\|x_0 - x^*\|)\|x_0 - x^*\|)G_2(\|x_0 - x^*\|)\|x_0 - x^*\|. \end{aligned} \tag{2.43}$$

Then, using (2.22), (2.25), ($\mathcal{C}_2$), (2.31)(for $k=0$), (2.33)(for $k=0$), (2.36), (2.42) and (2.43), we obtain that

$$\begin{aligned} \|x_1 - x^*\| \leq& \|z_0 - x^*\| + \bigg(1 + \frac{3}{2}\|H'(x_0)^{-1}H'(x^*)\|\|H'(x^*)^{-1}(H'(x_0) - H'(x^*))\| \\ & + \|H'(x^*)^{-1}(H'(y_0) - H'(x^*))\|\bigg)\|H'(x_0)^{-1}H'(x^*)\|\|H'(x^*)^{-1}H(z_0)\| \\ \leq& G_2(e_0)e_0 + \left(1 + \frac{3K_0(e_0 + G_1(e_0)e_0)}{2(1 - K_0e_0)}\right)\frac{(1 + K_0G_2(e_0)e_0)G_2(e_0)e_0}{1 - K_0e_0} \\ =& G_3(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < R, \end{aligned}$$

which shows (2.34) for $k=0$ and $x_1 \in B(x^*, R)$. We arrive at the estimates (2.30)-(2.34) by simply replacing x_k, y_k, z_k and x_{k+1} in place of x_0, y_0, z_0 and x_1 respectively in the preceding estimates. Using the estimate (2.34), $\|x_{k+1} - x^*\| \leq G_3(R)\|x_k - x^*\| < R$, we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in B(x^*, R)$. Now, to show the uniqueness of the solution x^*, assume that there exists another solution $t^* \in \overline{B}(x^*, \Lambda) \cap \Omega$ of $H(x) = 0$. Consider $Q = \int_0^1 H'(t^* + t(x^* - t^*))dt$. In view of ($\mathcal{C}_2$), we find that

$$\begin{aligned} \|H'(x^*)^{-1}(Q - H'(x^*))\| \leq& \int_0^1 K_0\|t^* + t(x^* - t^*) - x^*\|dt \\ \leq& \frac{K_0}{2}\|x^* - t^*\| \\ \leq& \frac{K_0\Lambda}{2} < 1. \end{aligned}$$

It follows from Banach lemma that $H^{-1} \in \mathcal{L}(X_2, X_1)$. Then, using the identity $0 = H(x^*) - H(t^*) = Q(x^* - t^*)$, it is concluded that $x^* = t^*$. This completes the proof. $\square$

3. Numerical results and discussion

Here, we shall illustrate the theoretical results which are proved in Section 2. We consider the following examples:

Example 3.1. *Let* $X_1 = X_2 = \mathbb{R}$, $\Omega = \left[\frac{-5}{2}, 2\right]$. *Consider the function [29]* H *defined on* Ω *by*

$$H(x) = \begin{cases} x^3 \log(\pi^2 x^2) + x^5 \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Its first Fréchet derivative is given by

$$H'(x) = 2x^2 - x^3 \cos\left(\frac{1}{x}\right) + 3x^2 \log(\pi^2 x^2) + 5x^4 \sin\left(\frac{1}{x}\right).$$

Then, we have $K_0 = K = \frac{2}{2\pi+1}(80 + 16\pi + (11 + 12\log 2)\pi^2)$. *Then, the parameters obtained are displayed in Table 1.*

Parameters for Method(2.15)
$R_1 = 0.0041263$
$R_2 = 0.0028416$
$R_3 = 0.0016766$
$R = 0.0016766$

Table 1. Numerical results for Example 3.1

Thus the convergence of the method (2.15) to $x^* = \frac{1}{\pi}$ *is guaranteed, provided* $x_0 \in B(x^*, R)$.

Example 3.2. *Let* $X_1 = X_2 = \mathbb{R}^2$, $\Omega = \overline{B}(0, 1)$, $x^* = (0, 0)^T$. *Consider the following function (see [30]) defined on* Ω *for* $w = (x, y)^T$*:*

$$H(w) = \left(\sin x, \frac{1}{3}(e^y + 2y - 1)\right)^T.$$

The first Fréchet derivative is given by

$$H'(w) = \begin{pmatrix} \cos x & 0 \\ 0 & \frac{1}{3}(e^y + 2) \end{pmatrix}$$

Then, we have $K_0 = K = 1$. *The parameters for the considered method are displayed in Table 2.*

Parameters for Method(2.15)
$R_1 = 0.3636364$
$R_2 = 0.2504262$
$R_3 = 0.1477508$
$R = 0.1477508$

Table 2. Numerical results for Example 3.2

Example 3.3. *Let $X_1 = X_2 = \mathbb{R}^3$, $\Omega = \overline{B}(0,1)$. Consider the following function, treated in [31], defined on Ω for $u = (x, y, z)^T$:*

$$H(u) = \left(e^x - 1, \frac{e-1}{2}y^2 + y, z\right)^T.$$

Notice that $x^* = (0,0,0)^T$ is a solution of $H(x) = 0$. The first Fréchet derivative is given by

$$H'(u) = \begin{pmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then, we have $K_0 = e - 1$, $K = e$. Using the definitions of parameters R_1, R_2, R_3 and R, we obtain the values as shown in Table 3.

Parameters for Method(2.15)
$R_1 = 0.1826392$
$R_2 = 0.1354046$
$R_3 = 0.0788791$
$R = 0.0788791$

Table 3. Numerical results for Example 3.3

Example 3.4. *Let $X_1 = X_2 = C[0,1]$, the space of all continuous functions defined on $[0,1]$ and $\Omega = \overline{B}(0,1)$. Consider a nonlinear Hammerstein type integral equation arising in practical problems of chemistry and electromagnetic fluid dynamics [32].*

$$H(x)(s) = x(s) - 5\int_0^1 stx(t)^3 dt,$$

where H is defined on Ω, $x(s) \in C[0,1]$ equipped with sup *norm, defined as $\|u\| = \max_{t\in[0,1]} |u(t)|$. It can be noted that*

$$H'(x)y(s) = y(s) - 15\int_0^1 stx(t)^2 y(t) dt.$$

For $x^* = 0$, we find that $K_0 = 7.5$, $K = 15$. Then, the parameters obtained are displayed in Table 4.

Parameters for Method(2.15)
$R_1 = 0.0380952$
$R_2 = 0.0295581$
$R_3 = 0.0170690$
$R = 0.0170690$

Table 4. Numerical results for Example 3.4

Example 3.5. *Finally, we consider the proposed sixth order method M_6 (2.15) to solve systems of nonlinear equations in $\mathbb{R}^n$ and compare the performance with existing sixth order methods. For example, we consider sixth order method by Cordero*

et al. [34], sixth order generalized Jarratt's method by Sharma and Arora [35] and sixth order methods by Soleymani et al. [36], Esmaeili and Ahmadi [37] and Sharma et al. [38]. The above mentioned methods are given as follows:
Method by Cordero et al. *(CM_6):*

$$\begin{aligned}
y_k &= x_k - \frac{1}{2}H'(x_k)^{-1}H(x_k),\\
z_k &= \frac{1}{3}(4y_k - x_k),\\
u_k &= y_k + \Big(H'(x_k) - 3H'(z_k)\Big)^{-1}H(x_k),\\
x_{k+1} &= u_k + 2\Big(H'(x_k) - 3H'(z_k)\Big)^{-1}H(u_k).
\end{aligned}$$

Sharma-Arora method *(SAM_6):*

$$\begin{aligned}
y_k &= x_k - \frac{2}{3}H'(x_k)^{-1}H(x_k),\\
z_k &= x_k - \left[\frac{23}{8}I - \Big(3I - \frac{9}{8}H'(x_k)^{-1}H'(y_k)\Big)H'(x_k)^{-1}H'(y_k)\right]H'(x_k)^{-1}H(x_k),\\
x_{k+1} &= z_k - \frac{1}{2}\Big(5I - 3H'(x_k)^{-1}H'(y_k)\Big)H'(x_k)^{-1}H(z_k).
\end{aligned}$$

Method by Soleymani et al. *(SM_6):*

$$\begin{aligned}
y_k &= x_k - \frac{2}{3}H'(x_k)^{-1}H(x_k),\\
z_k &= x_k - \frac{1}{2}\Big(3H'(y_k) - H'(x_k)\Big)^{-1}\Big(3H'(y_k) + H'(x_k)\Big)H'(x_k)^{-1}H(x_k),\\
x_{k+1} &= z_k - \left[\left(\frac{1}{2}\Big(3H'(y_k) - H'(x_k)\Big)^{-1}\Big(3H'(y_k) + H'(x_k)\Big)\right)^2\right]H'(x_k)^{-1}H(z_k).
\end{aligned}$$

Method by Esmaeili et al. *(EM_6):*

$$\begin{aligned}
y_k &= x_{(k)} - H'(x_{(k)})^{-1}H(x_{(k)}),\\
z_k &= y_k + \frac{1}{3}\left(H'(x_k)^{-1} + 2\Big(H'(x_k) - 3H'(y_k)^{-1}\Big)\right)H(x_k),\\
x_{k+1} &= z_k + \frac{1}{3}\left(-H'(x_k)^{-1} + 4\Big(H'(x_k) - 3H'(y_k)\Big)^{-1}\right)H(z_k).
\end{aligned}$$

Method by Sharma et al. *(SSM_6):*

$$\begin{aligned}
y_k &= x_k - H'(x_k)^{-1}H(x_k),\\
z_k &= x_k - \left[\frac{3}{2}I - \frac{1}{2}H'(x_k)^{-1}H'(y_k)\right]H'(x_k)^{-1}H(x_k),\\
x_{k+1} &= z_k - \left[\frac{7}{2}I + \Big(-4I + \frac{3}{2}H'(x_k)^{-1}H'(y_k)\Big)H'(x_k)^{-1}H'(y_k)\right]H'(x_k)^{-1}H(z_k).
\end{aligned}$$

All computations are carried out using multiple-precision arithmetic with 4096 digits in the programming package MATHEMATICA [39] in the processor with specifications Intel(R) Core(TM) i5-8250U CPU @1.60 GHz. Numerical results displayed in Table 5 include (i) the number of iterations (k) required to converge to the solution satisfying the condition $||x_{k+1} - x_k|| + ||H(x_k)|| < 10^{-100}$ (ii) Computational order of convergence (ρ_k) taking into consideration the last three approximations in the iterative process and to confirm the theoretical order of convergence (iii) The time consumed (CPU-Time)in execution of a program, which is measured by *the command "TimeUsed[]". The computational order of convergence (ρ_k) is computed using the well-known formula (see [40, 41])*

$$\rho_k = \frac{\log(||H(x_k)||/||H(x_{(k-1)})||)}{\log(||H(x_{(k-1)})||/||H(x_{(k-2)})||)},$$

(a) *Let us consider the system of thirty five nonlinear equations (selected from [42]):*

$$\begin{cases} x_i x_{i+1} - e^{-x_i} - e^{-x_{i+1}} = 0, & 1 \le i \le n-1, \\ x_n x_1 - e^{-x_n} - e^{-x_1} = 0. \end{cases}$$

By selecting $x_0 = \{1.2, 1.2, \cdot\cdot\cdot\cdot\cdot, 1.2\}^T$ as initial approximation, the solution of this problem obtained is,

$x^* = \{0.901201031729666145..., 0.901201031729666145..., \cdots, 0.901201031729666145...\}^T$.

(b) *Considering the following system of equations for $n = 99$ (see [43]):*

$$\begin{cases} x_i x_{i+1} - 1 = 0, & 1 \le i \le n-1, \\ x_n x_1 - 1 = 0. \end{cases}$$

Setting the initial approximation $x_0 = \{-4, -4, \cdots, -4\}^T$ leads to the solution: $x^* = \{-1, -1, \cdots, -1\}^T$.

(c) *Further, we solve a system of nonlinear equations which arise while solving the following nonlinear partial differential equation, (see [44])*

$$u_{xx} + u_{yy} = u^2, \quad (x, y) \in [0, 1] \times [0, 1]$$

with boundary conditions

$$\begin{aligned} u(x, 0) &= 2x^2 - x + 1, \quad u(x, 1) = 2, \\ u(0, y) &= 2y^2 - y + 1, \quad u(1, y) = 2. \end{aligned}$$

The solution of a nonlinear partial differential equation can be found using finite difference discretization thereby reducing it to a system of nonlinear equations. Let $u = u(x, y)$ be the exact solution of this poisson equation.

Let $w_{i,j} = u(x_i, y_j)$ be its approximate solution at the grid points of the mesh. Let M and N be the number of steps in x and y directions and h and k be the respective step size.

If we discretize the problem by using the central divided differences i.e.
$u_{xx}(x_i, y_j) = (w_{i+1,j} - 2w_{i,j} + w_{i-1,j})/h^2$ and $u_{yy}(x_i, y_j) = (w_{i,j+1} - 2w_{i,j} + w_{i,j-1})/k^2$,
we get the following system of nonlinear equations:

$$w_{i+1,j} - 4w_{i,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1} - h^2 w_{i,j}^2 = 0,$$
$$i = 1, 2, \ldots, M, \; j = 1, 2, \ldots, N.$$

We here consider $M = 11$ and $N = 11$ and transform the problem of solving a PDE to a nonlinear system of 100 equations in 100 unknowns using boundary conditions. The error in replacing u_{xx} by the finite difference approximation is of the order $O(h^2)$. Since k=h, the error in replacing u_{yy} by the finite difference approximation is also of the order $O(h^2)$. Hence the error in solving Poisson equation by finite difference method is of the order $O(h^2)$ (see [33]).

For the sake of brevity, we have renamed the unknowns as:

$$x_1 = w_{1,1}, \; x_2 = w_{1,2}, \; \ldots x_{10} = w_{1,10},$$

$$x_{11} = w_{2,1}, \; x_{12} = w_{2,2}, \; \ldots x_{20} = w_{2,10},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_{91} = w_{10,1},\ x_{92} = w_{10,2},\ \ldots x_{100} = w_{10,10}.$$

We have taken the initial guess as $x_0 = \{1, 1, \cdots, 1\}^T$ *towards the approximate solution of the problem given by*

$$r = \{0.925418, 0.928755, \cdots, 1.9493\}^T.$$

The approximate solution found has also been plotted in Fig. 1.

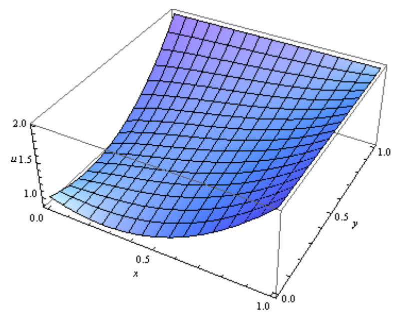


FIGURE 1. Approximate solution of Poisson's equation.

(d) *Lastly, we consider the following system of equations (see [45]):*

$$\begin{cases} x_i \sin x_{i+1} - 1 = 0, & 1 \leq i \leq n-1, \\ x_n \sin x_1 - 1 = 0. \end{cases}$$

Taking $n = 999$ *and initial guess* $x_0 = \{-1, -1, \cdots, -1\}^T$ *for this problem, the required solution is:*

$$x^* = \{-1.114157140871930087..., -1.114157140871930087..., \cdots, -1.114157140871930087...\}^T.$$

From the numerical results shown in the Table 5, it can be observed that like that of existing methods, the proposed Newton-Jarratt method shows consistent convergence behavior. From the calculation of the computational order of convergence displayed in the third column of Table 5, it is also verified that order of convergence of new method is preserved in all numerical examples.

Methods	k	$\rho_k \pm \triangle\rho_k$	*CPU-Time*
(a)			
M_6	2	$6 \pm 0.06 \times 10^{-1}$	1.281
CM_6	2	$6 \pm 7.17 \times 10^{-3}$	1.13
SAM_6	2	$6 \pm 5.96 \times 10^{-2}$	1.079
SM_6	2	6 ± 0.23	1.271
EM_6	2	6 ± 0.026	1.211
SSM_6	2	$6 \pm 8.36 \times 10^{-2}$	1.06
(b)			
M_6	5	$6 \pm 3.78 \times 10^{-34}$	4.196
CM_6	5	$6 \pm 3.34 \times 10^{-43}$	6.056
SAM_6	5	$6 \pm 5.54 \times 10^{-27}$	5.484
SM_6	5	$6 \pm 4.62 \times 10^{-47}$	6.059

EM_6	5	$6 \pm 2.31 \times 10^{-36}$	3.651
SSM_6	3	$6 \pm 4.82 \times 10^{-26}$	4.742
(c)			
M_6	3	$6 \pm 0.08 \times 10^{-1}$	0.187
CM_6	3	$6 \pm 0.09 \times 10^{-1}$	0.350
SAM_6	3	$6 \pm 0.01 \times 10^{-1}$	0.273
SM_6	3	$6 \pm 0.08 \times 10^{-1}$	0.234
EM_6	3	$6 \pm 0.05 \times 10^{-1}$	0.226
SSM_6	3	$6 \pm 0.09 \times 10^{-2}$	0.218
(d)			
M_6	4	$6 \pm 3.28 \times 10^{-52}$	714.27
CM_6	4	$6 \pm 3.04 \times 10^{-52}$	714.286
SAM_6	4	$6 \pm 4.50 \times 10^{-52}$	707.925
SM_6	4	$6 \pm 2.99 \times 10^{-49}$	780.81
EM_6	4	$6 \pm 1.84 \times 10^{-50}$	783.78
SSM_6	4	$6 \pm 1.83 \times 10^{-51}$	706.494

TABLE 5. Comparison of the performances of methods

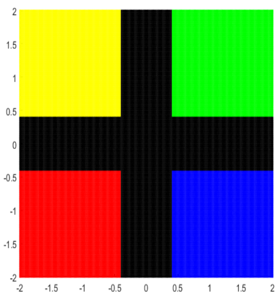
3.1. **Basins of attraction.** Basins of attraction allows us to assess those initial points which converge to the concerned root of a polynomial when an iterative method is applied. This helps us to visualize which points are good choices for initial points and which are not. One can find the basic definitions related to basins of attraction associated with iterative methods in [46, 47]. Here, we analyze the basins of attraction of the methods in previous sections on the following polynomial system (see[46])

$$\begin{cases} x_1^2 - 1 = 0, \\ x_2^2 - 1 = 0, \end{cases}$$

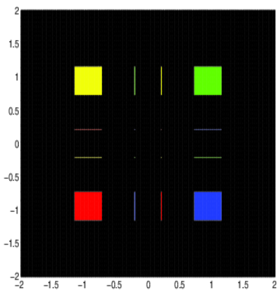
with roots $\{\{1,1\}^T, \{1,-1\}^T, \{-1,1\}^T, \{-1,-1\}^T\}$. To generate basins of attraction associated with the roots of system of nonlinear equations, we take a square $[-2,2] \times [-2,2]$ of 1024×1024 points, containing all roots of concerned nonlinear system of equations. We apply the iterative method starting in every point in the square. Starting from the point, a color is assigned to each point according to the root to which the corresponding orbit of the iterative method converges. We mark with black, the points for which the the corresponding orbit does not reach any root of the polynomial, with tolerance 10^{-3} in a maximum of 25 iterations. In Figs. 2a-2f, it can be observed that for the given test problem, all the roots of the polynomial system have their respective basins of attraction with different colors. Also the Julia set can be seen as black lines of unstable behavior. It can further be observed that the methods SAM$_6$ Fig. 2c and M$_6$ Fig. 2a take the lead while the rest are not as good.

4. Conclusions

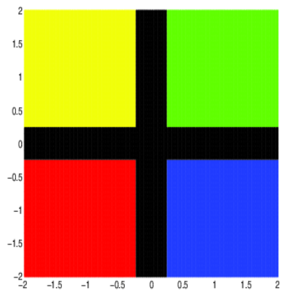
In this contribution, an efficient new sixth-order iterative method is constructed and its local convergence is established under Lipschitz continuity condition on



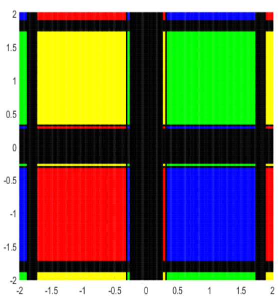
(A) M_6



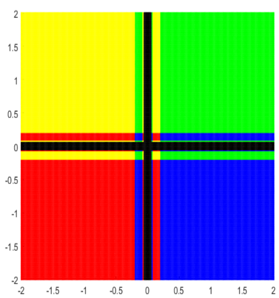
(B) CM_6



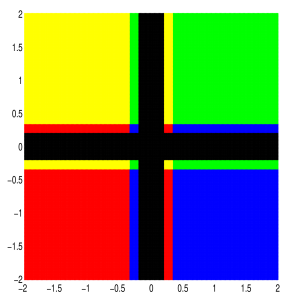
(C) SAM_6



(D) SM_6



(E) EM_6



(F) SSM_6

FIGURE 2. Basins of attraction for system of equations $x_1^2 - 1 = 0, x_2^2 - 1 = 0$ for various other sixth order methods.

first derivative in Banach spaces. The hypotheses that we set here allows us to solve even those nonlinear equations which can not be solved by other iterative methods involving second or higher order derivatives. Different nonlinear equations, including integral equation of Hammerstein type, have been solved and the radii of convergence balls defining the existence and uniqueness domains are obtained.

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