Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 12 Issue 2 (2021), Pages 44-53.

SYMMETRIZATION PROCEDURES FOR THE LAW OF THE ITERATED LOGARITHM

ZAHRA SHOKOOH GHAZANI

Abstract. In this paper, the general symmetrization procedures for convergence rates for the law of the iterated logarithm for independent and identically distributed random variables are established. Necessary and sufficient conditions are provided for the Hartman-Wintner law for a symmetrized version of the array.

1. Introduction

Let $\{Y, Y_m, m \geq 1\}$ be a sequence of i.i.d random variables. For such sequences of random variables, with mean zero and variance one, the law of the Iterated Logarithm states that $P(limsup(|\sum_{m=1}^{n} Y_m|)(2mloglogm)^{-\frac{1}{2}} = 1) = 1$. In this paper, some probabilities, whenever this law applicable, and convenient convergence rates are appointed. We apply one of the classical results of probability Hartman-Wintner law of the iterated logarithm. The results extend to the general situation by the standard symmetrization procedure. With the law of large numbers and the central limit theorem, the iterated logarithm law is considered the fundamental limit theorem in Probability theory. When $EY^2 = 1$ the central limit theorem claims that the distribution of Sum of random variables tends to the standard normal distribution, whiles by the law of the iterated logarithm $p(limsup(|\sum_{m=1}^{n} Y_m|)(2mloglogm)^{-\frac{1}{2}} = 1) = 1$. The following statements are related to the extended classical Hartman-Wintner law of the iterated logarithm. We start with a discussion of the equivalence of these four statements.

$$EY = 0 \quad \text{and} \quad EY^2 = 1, \tag{1.1}$$

$$\sum_{m=1}^{\infty} \frac{1}{m} P\left\{\left|\sum_{n=1}^{m} Y_n\right| > (1+\varepsilon)\sqrt{2m\log\log m}\right\} \begin{cases} < \infty, & \text{if } \varepsilon > 0 \\ = \infty, & \text{if } \varepsilon < 0, \end{cases} \tag{1.2}$$

$$\sum_{m=1}^{\infty} \frac{\log\log m}{m} P\left\{\left|\sum_{n=1}^{m} Y_n\right| > (1+\varepsilon)\sqrt{2m\log\log m}\right\} \begin{cases} < \infty, & \text{if } \varepsilon > 0 \\ = \infty, & \text{if } \varepsilon < 0, \end{cases} \tag{1.3}$$

2000 *Mathematics Subject Classification.* 60FXX, 60F15.
Key words and phrases. Law of the iterated logarithm; Hartman-Wintner law; Symmetrization procedure.
©2021 Ilirias Research Institute, Prishtinë, Kosovë.
Submitted January 26, 2021. March 17, 2021.
Communicated by Vincenzo Vespri.

$$\sum_{m=1}^{\infty} \frac{1}{mf(m)} P\left\{\left|\sum_{n=1}^{m} Y_n\right| > (1+\varepsilon)\sqrt{2m\log\log m}\right\} \begin{cases} < \infty, & \text{if } \varepsilon > 0 \\ = \infty, & \text{if } \varepsilon < 0, \end{cases} \tag{1.4}$$

by assuming $\int_1^{\infty}(af(a))^{-1}da = \infty$, and $g(a) = \int_1^{a}(tf(t))^{-1}dt$, $a \geq 1$. wherein $f(\cdot)$ be a positive nondecreasing function on $\mathbb{R}^+$.

Davis [3] extended Baum et al. [1] results and showed that (1.2) follows from (1.1). The equivalence "(1.1)$\Leftrightarrow$(1.3)", was formulated by Hartman and Wintner [11]. The implication "(1.2)$\Rightarrow$(1.1)", was proved by Gut [10]. Chen and Wang [2] showed the relations "(1.1)$\Rightarrow$(1.2)" and "(1.1)$\Rightarrow$(1.3)". Friedman et al. [4] applied this convergence rate concept to the central limit theorem. Li and Spataru [15] gave a refined result of complete convergence of the above results. Lai [9] generalized these results to the moving average processes. Jiang and Zhang [12] discussed the special rates in the law of the iterated logarithm for complete moment convergence. Li and Rosalsky [14] proved The equivalence of (1.1) and (1.4). The starting point for this study is the equivalences of (1.1) and (1.3) discuss the convergence rate of the classical law of the iterated logarithm researched by Hartman and Wintner [11]. In this paper, we extend the equivalences of (1.1) and (1.4) for the symmetrized version of independent and identically random variables.

2. Preliminaries and Lemmas

The following lemmas applied for the convergence rate are potent tools in the limit theory area that are useful for proving the Main Theorem.

Lemma 2.1. *Let $\{Y_m, m \geq 1\}$ be a sequence of random variables and $\{Y'_m, m \geq 1\}$ be an independent copy of $\{Y_m, m \geq 1\}$, then for every $s > 0$, if $Y_m \xrightarrow{P} 0$, then $E|Y_m|^s \to 0$, if and only if $E|Y_m - Y'_m|^s \to 0$.*

Proof. We first assume that $E|Y_m|^s \to 0$. By C_r-inequality $E|Y_m|^s \to 0$, implies that $E|Y_m - Y'_m|^s \to 0$. conversely, now assume $E|Y_m - Y'_m|^s \to 0$. Using the definition of expectation by integral, we divide it into two integrals and conclude

$$E|Y_m|^s = \int_0^{\infty} P\left\{|Y_m| > a^{\frac{1}{s}}\right\} da \leq \varepsilon + \int_{\varepsilon}^{\infty} P\left(|Y_m| > a^{\frac{1}{s}}\right) da,$$

for every $\varepsilon > 0$. Since

$$\sup_{a\geq\varepsilon} P\left(|Y_m| > \frac{a^{\frac{1}{s}}}{2}\right) \leq P\left\{|Y_m| > \frac{\varepsilon^{\frac{1}{s}}}{2}\right\} \to 0,$$

then we can conclude for enough large m,

$$E|Y_m|^s \leq \varepsilon + 2\int_{\varepsilon}^{\infty} P\left\{|Y_m - Y'_m| > \frac{a^{\frac{1}{s}}}{2}\right\} da \leq \varepsilon + 2^{s+1}E|Y_m - Y'_m|^s.$$

Thus, we can deduce that $E|Y_m|^s \to 0$, holds whenever $E|Y_m - Y'_m|^s \to 0$.

□

The following lemma express lévy inequalities for a sequence of symmetric random variables. These inequalities also hold under sub-linear expectation [6].

Lemma 2.2 ([9]). *Let $\{Y_m\}$ be a symmetric sequence of random variables. Let $S_m = \sum_{j=1}^{m} Y_j$. Then, for every n and $a > 0$,*

(a) $P\left\{\max\limits_{m\leq n}|S_m|>a\right\}\leq 2P\left\{|S_n|>a\right\}$,

(b) $P\left\{\max\limits_{a\leq n}|Y_j|>a\right\}\leq 2P\left\{|S_n|>a\right\}$.

Lemma 2.3. *Let $\{Y_m, m\leq n\}$ be independent positive random variables. Then, for every $a>0$,*

$$P\left\{\max_{m\leq n}Y_m>a\right\}\geq\frac{\sum_{m=1}^{n}P\{Y_m>a\}}{1+\sum_{m=1}^{n}P\{Y_m>a\}}.$$

In special case, if $P\left\{\max\limits_{m\leq n}Y_m>a\right\}\leq\frac{1}{2}$,

$$\sum_{m=1}^{n}P\{Y_m>a\}\leq 2P\left\{\max_{m\leq n}Y_m>a\right\}.$$

Proof. According to the expansion of the exponential function, we can get $1-y\leq\exp(-y)$ and $1-\exp(-y)\geq\frac{y}{1+y}$, for $y\geq 0$, and by independence, we have that

$$\begin{aligned}P\left\{\max_{m\leq n}Y_m>a\right\}&=1-\prod_{m=1}^{n}\left(1-P\{Y_m>a\}\right)\\&\geq 1-\exp\left(-\sum_{m=1}^{n}P\{Y_m>a\}\right)\\&\geq\frac{\sum_{m=1}^{n}P\{Y_m>a\}}{1+\sum_{m=1}^{n}P\{Y_m>a\}}.\end{aligned}$$

The first inequality is proofed. To establish in the particular case, now, assume $P\left\{\max\limits_{m\leq n}Y_m>a\right\}\leq\frac{1}{2}$, consider to the first inequality and put the upper bound of $\frac{1}{2}$ in it, it is concluded that $\sum\limits_{m=1}^{n}2P(Y_m>a)\leq 1+\sum\limits_{m=1}^{n}P(Y_m>a)$ then clearly $\sum\limits_{m=1}^{n}P(Y_m>a)\leq 1$. □

Lemma 2.4. *Let $\{Y_n, 1\leq n\leq n_m, m\geq 1\}$ be an array of rowwise independent random variables such that $\{n_m, m\geq 1\}$ be a sequence of positive integers . Assume $|Y_n|\leq\eta$ a.s. for some $\eta>0$, such that for every $m\geq 1$, $1\leq n\leq n_m$. If $\sum\limits_{n=1}^{n_m}Y_{mn}\xrightarrow{P}0$ then for a large enough m, we have $E\left|\sum\limits_{n=1}^{m}Y_{mn}\right|\to 0$.*

Proof. Suppose $\{Y'_{mn}, 1\leq n\leq n_m, m\geq 1\}$ be an independent copy of $\{Y_{mn}, 1\leq n\leq n_m, m\geq 1\}$. Hence it is enough to show that

$$E\left|\sum_{n=1}^{n_m}(Y_{mn}-Y'_{mn})\right|\to 0\quad\text{as}\quad m\to\infty.\tag{2.1}$$

Since $\sum\limits_{n=1}^{n_m}Y_{mn}\xrightarrow{P}0$, and by $\{Y'_{mn}, 1\leq n\leq n_m, m\geq 1\}$ be an independent copy of $\{Y_{mn}, 1\leq n\leq n_m, m\geq 1\}$, so we can deduce, $\sum\limits_{n=1}^{n_m}Y'_{mn}\xrightarrow{P}0$ holds, then deduces

$\sum\limits_{n=1}^{n_m}(Y_{mn} - Y'_{mn}) \to 0$ in Pr. and $|Y_{mn} - Y'_{mn}| \le 2\eta$ holds. Therefore (2.1) yields by Lemma 2.1. □

Lemma 2.5. *Assume $0 < V_m \nearrow \infty$, and $\{Y, Y_m, m \ge 1\}$ be a sequence of i.i.d random variables. If $V_m^{-1}\sum\limits_{n=1}^{m} Y_n \to 0$ in Pr., then*

$$E\left|V_m^{-1}\sum_{n=1}^{m} Y_n I(|Y_n| \le V_m)\right| \to 0 \quad \text{as} \quad m \to \infty.$$

Proof. Let $\{Y', Y'_m, m \ge 1\}$ be an independent copy of $\{Y, Y_m, m \ge 1\}$ and since $V_m^{-1}\sum\limits_{n=1}^{m} Y_n \to 0$ in Pr., then

$$V_m^{-1}\sum_{n=1}^{m}(Y_n - Y'_n) \to 0, \quad \text{in Pr.} \tag{2.2}$$

By Lemma 2.2 for all $a > 0$,

$$P\left\{\max_{1\le n\le m}|Y_n - Y'_n| > a\right\} \le 2P\left\{\left|\sum_{n=1}^{m}(Y_n - Y'_n)\right| > a\right\}.$$

choose $a = V_m/2$, Then (2.2) and recent inequality implies that

$$P\left\{\max_{1\le n\le m}|Y_n - Y'_n| > \frac{V_m}{2}\right\} \to 0, \quad \text{as} \quad m \to \infty. \tag{2.3}$$

By Lemma 2.3

$$\begin{aligned} mP\left\{|Y - Y'| > \frac{V_m}{2}\right\} &= \sum_{n=1}^{m} P\left\{|Y_n - Y'_n| > \frac{V_m}{2}\right\} \\ &\le 2P\left\{\max|Y_n - Y'_n| > \frac{V_m}{2}\right\}, \end{aligned} \tag{2.4}$$

then we can get

$$P\{|Y| > V_m\} \le 2P\left\{|Y - Y'| > \frac{V_m}{2}\right\}, \tag{2.5}$$

for sufficiently large m.

Therefore by (2.3), (2.4), and (2.5),

$$mP\{|Y| > V_m\} \to 0, \quad \text{as} \quad m \to \infty. \tag{2.6}$$

For every $\varepsilon > 0$,

$$P\left\{\left|\sum_{n=1}^{m} Y_n I(|Y_n| \le V_m)\right| > \varepsilon V_m\right\} \le mP\{|Y| > V_m\} + P\left\{\left|\sum_{n=1}^{m} Y_n\right| > \varepsilon V_m\right\}.$$

Then by (2.6) and since $V_m^{-1}\sum\limits_{n=1}^{m} Y_n \to 0$ in Probability the right side of the last inequality above tends to zero, we obtain can deduce

$$V_m^{-1}\sum_{n=1}^{m} Y_n I(|Y_n| \le V_m) \to 0, \quad \text{in Pr.}$$

The proof follows from Lemma 2.4. □

Lemma 2.6. *Assume $f(\cdot)$ be a nondecreasing and positive function on $(0,\infty)$ such that $\int_1^{\infty}(af(a))^{-1}da = \infty$. Let $g(a) = \int_1^{a}(tf(t))^{-1}dt$, $a \geq 1$. Suppose Y be a random variable satisfying*

$$\sum_{m=1}^{\infty}\frac{1}{f(m)}P(|Y| > \sqrt{m\log g(m)}) < \infty, \tag{2.7}$$

then for every $t > 2$,

$$\sum_{m=1}^{\infty}\frac{1}{f(m)}\frac{1}{(m\log g(m))^{\frac{m}{2}}}E|Y|^tI(|Y| \leq \sqrt{m\log g(m)}) < \infty.$$

Proof. Let $V_0 = 0$ and $V_m = \sqrt{m\log g(m)}$, $m \geq 1$. since $g(m) \nearrow$ then if $1 \leq n \leq m$ we can deduce $0 \leq \log g(n) \leq \log g(m)$ so we have $\log g(m) \nearrow$ then $\frac{V_m}{\sqrt{m}} \nearrow$. Then $\frac{V_n}{V_m} \leq \sqrt{\frac{n}{m}}$. Therefore

$$\begin{aligned}
&\sum_{m=1}^{\infty}\frac{1}{f(m)}\frac{1}{(m\log g(m))^{\frac{t}{2}}}E|Y|^tI(|Y| \leq \sqrt{m\log g(m)})\\
&= \sum_{m=1}^{\infty}\frac{1}{f(m)V_m^t}\sum_{n=1}^{m}E|Y|^tI(V_{n-1} < |Y| \leq V_n)\\
&\leq \sum_{m=1}^{\infty}\frac{1}{f(m)V_m^t}\sum_{n=1}^{m}V_n^tP\{V_{n-1} < |Y| \leq V_n\}\\
&= \sum_{n=1}^{\infty}V_n^tP\{V_{n-1} < |Y| \leq V_n\}\sum_{m=n}^{\infty}\frac{1}{f(m)V_m^t}\\
&\leq \sum_{n=1}^{\infty}n^{\frac{t}{2}}P\{V_{n-1} < |Y| \leq V_n\}\sum_{m=n}^{\infty}\frac{1}{m^{\frac{t}{2}}f(m)}\\
&\leq K\sum_{n=1}^{\infty}\frac{n}{f(n)}P\{V_{n-1} < |Y| \leq V_n\}\\
&\leq \frac{K}{f(1)} + K\sum_{n=1}^{\infty}\left\{\frac{n+1}{f(n+1)} - \frac{n}{f(n)}\right\}P(|Y| > V_n)\\
&\leq \frac{K}{f(1)} + K\sum_{n=1}^{\infty}\frac{1}{f(n)}P(|Y| > V_n) < \infty.
\end{aligned}$$

Wherein $K = \left(\frac{t}{2} - 1\right)^{-1}$. This ends the proof. □

Lemma 2.7. *Let $f(\cdot)$ and $g(\cdot)$, be as in Lemma 2.6. Then for every random variable Y, (2.7) is equivalent to*

$$\sum_{m=1}^{\infty}\frac{1}{f(m)}P\{|Y| > L\sqrt{m\log g(m)}\} < \infty, \tag{2.8}$$

for some $L > 0$.

Proof. Suppose (2.7) holds, then we prove (2.8) deduced from that for all $0 < L < 1$. Let $V_m = \sqrt{m\log g(m)}$, $m \geq 1$.Since $g(m) \nearrow$ and $\log g(m) \nearrow$ therefore $\frac{V_m}{\sqrt{m}} \nearrow$.

Then $\frac{V_m}{\sqrt{m}} \leq \frac{V_2 m}{\sqrt{2m}}$ and $V_m \leq \frac{1}{\sqrt{2}} V_{2m}$ for $m \geq 1$. We can get,

$$\frac{1}{f(2m)} P\left\{|Y| > \frac{1}{\sqrt{2}} V_{2m}\right\} \leq \frac{1}{f(m)} P\{|Y| > V_m\},$$

and

$$\begin{aligned}\frac{1}{f(2m+1)} P\left\{|Y| > \frac{1}{\sqrt{2}} V_{2m+1}\right\} &\leq \frac{1}{f(2m)} P\left\{|Y| > \frac{1}{\sqrt{2}} V_{2m}\right\} \\ &\leq \frac{1}{f(m)} P\{|Y| > V_m\},\end{aligned}$$

which implies that,

$$\begin{aligned}\sum_{m=1}^{\infty} \frac{1}{f(m)} P\left\{|Y| > \frac{1}{\sqrt{2}} V_m\right\} =& \frac{1}{f(1)} P\left\{|Y| > \frac{1}{\sqrt{2}} V_1\right\} \\ &+ \sum_{m=1}^{\infty} \frac{1}{f(2m)} P\left\{|Y| > \frac{1}{\sqrt{2}} V_{2m}\right\} \\ &+ \sum_{m=1}^{\infty} \frac{1}{f(2m+1)} P\left\{|Y| > \frac{1}{\sqrt{2}} V_{2m+1}\right\} \\ \leq& \frac{1}{f(1)} P\left\{|Y| > \frac{1}{\sqrt{2}} V_1\right\} \\ &+ 2\sum_{m=1}^{\infty} \frac{1}{f(m)} P\{|Y| > \sqrt{m \log g(m)}\} \\ <& \infty,\end{aligned}$$

Then for every integer $m \geq 1$,

$$\sum_{m=1}^{\infty} \frac{1}{f(m)} P\left\{|Y| > \frac{1}{2^{\frac{n}{2}}} V_m\right\} < \infty.$$

wherein $L = \frac{1}{2^{n/2}}$ and $V_m = \sqrt{m \log g(m)}$. The proof is completed. $\square$

3. Main Results

This section proves the main theorem by lemmas and shows some applications by providing some examples.

Theorem 3.1. *Suppose $f(\cdot)$ and $g(\cdot)$, be as in Lemma 2.6. Let $\{Y, Y_m, m \geq 1\}$, be a sequence of i.i.d random variables. Let*

$$(\sqrt{m \log g(m)})^{-1} \sum_{n=1}^{m} Y_n \to 0 \quad \text{in Pr.}$$

(i) *Assume that* (2.7) *holds and*

$$EY = 0 \quad \text{and} \quad EY^2 < \infty, \tag{3.1}$$

Then

$$\sum_{m=1}^{\infty} \frac{1}{m f(m)} P\left\{\left|\sum_{n=1}^{m} Y_n\right| > (1+\varepsilon)\sqrt{2\sigma^2 m \log g(m)}\right\} \begin{cases} < \infty, & \text{if } \varepsilon > 0 \\ = \infty, & \text{if } \varepsilon < 0. \end{cases} \tag{3.2}$$

(ii) *Conversely, let*

$$\sum_{m=1}^{\infty} \frac{1}{mf(m)} P\left\{\left|\sum_{n=1}^{m} Y_n\right| > L\sqrt{m \log g(m)}\right\} < \infty, \tag{3.3}$$

holds for some $L > 0$. *Then* (2.7) *and* (3.1) *hold.*

Proof. For every $m \geq 1$ let $U_m = \sqrt{2\sigma^2 m \log g(m)}$, $V_m = \sqrt{m \log g(m)}$, and $Y_{mn} = Y_n I(|Y_n| \leq V_m)$, $Z_{mn} = Y_{mn} - EY_{mn}$,where in $1 \leq n \leq m$, $m \geq 1$.

(i) Assume that (2.7) and (3.1) holds. We show that

$$\sum_{m=1}^{\infty} \frac{1}{mf(m)} P\left\{\left|\sum_{n=1}^{m} Y_n\right| > (1+\varepsilon)U_m\right\} < \infty, \ \forall \varepsilon > 0. \tag{3.4}$$

For every $\varepsilon > 0$, we have

$$P\left\{\left|\sum_{n=1}^{m} Y_n\right| > (1+\varepsilon)U_m\right\} \leq mP\{|Y| > V_m\} + P\left\{\left|\sum_{n=1}^{m} Y_{mn}\right| > (1+\varepsilon)U_m\right\}.$$

Then, by (2.7), to prove (3.4), it's enough we show that

$$\sum_{m=1}^{\infty} \frac{1}{mf(m)} P\left\{\left|\sum_{n=1}^{m} Y_{mn}\right| > (1+\varepsilon)U_m\right\} \infty, \ \forall \varepsilon > 0. \tag{3.5}$$

By Jensen's Inequality and Lemma 2.5

$$\frac{1}{V_m}\left|\sum_{n=1}^{m} EY_{mn}\right| \leq \frac{1}{V_m} E\left|\sum_{n=1}^{m} Y_{mn}\right| \to 0, \quad \text{as} \quad m \to \infty,$$

and

$$\frac{1}{V_m} E\left|\sum_{n=1}^{m} Z_{mn}\right| \leq \frac{2}{V_m} E\left|\sum_{n=1}^{m} Y_{mn}\right| \to 0, \quad \text{as} \quad m \to \infty.$$

Then to prove (3.5), it's enough to show that

$$\sum_{m=1}^{\infty} \frac{1}{mf(m)} P\left\{\left|\sum_{n=1}^{m} Z_{mn}\right| > 2E\left|\sum_{n=1}^{m} Z_{mn}\right| + (1+\varepsilon)U_m\right\} < \infty, \ \forall \varepsilon > 0. \tag{3.6}$$

Then for some $t > 2$ and any $\eta > 0$,

$$\begin{aligned} &P\left\{\left|\sum_{n=1}^{m} Z_{mn}\right| > 2E\left|\sum_{n=1}^{m} Z_{mn}\right| + (1+\varepsilon)U_m\right\} \\ &\leq \exp\left\{-\frac{(1+\varepsilon)^2 U_m^2}{(2+\eta)\Delta_m^2} + \frac{C}{V_m^t}\sum_{n=1}^{m} E|Z_{mn}|^t\right\}. \end{aligned} \tag{3.7}$$

Wherein $\Delta_m^2 = \sup \sum_{n=1}^{m} E(Z_{mn})^2$. Note that

$$E(Z_{mn}^2) \leq E(Y_{mn})^2 \leq EY^2 \quad 1 \leq n \leq m, \ m \geq 1.$$

Therefore $\Delta_m^2 \leq m\sigma^2$, $m \geq 1$. Choose $\eta > 0$ which be nearby enough to 0 such that $a = \frac{2(1+\varepsilon)^2}{(2+\eta)} > 1$. Then

$$\sum_{m=1}^{\infty} \frac{1}{mf(m)} \exp\left\{-\frac{(1+\varepsilon)^2 U_m^2}{(2+\eta)\Delta_m}\right\} \leq \sum_{m=1}^{\infty} \frac{1}{mf(m)} \exp\left\{-\frac{(1+\varepsilon)^2 U_m^2}{(2+\eta)\Delta_m}\right\} \tag{3.8}$$

$$\leq \sum_{m=1}^{\infty} \frac{1}{mf(m)} \exp\{-a \log g(m)\}$$

$$\leq \sum_{m=1}^{\infty} \frac{1}{mf(m)} \frac{1}{(g(m))^a} < \infty.$$

According to the integral test for convergence investigate of series and since$\int_0^{\infty} \frac{ds}{sf(s)g^a(s)} < \infty$, the convergence of the recent series resulted. By the C_r-inequality, Holder's inequality, and Lemma 2.6,

$$\sum_{m=1}^{\infty} \frac{1}{mf(m)} \frac{1}{V_m^t} \sum_{n=1}^{m} E|Z_{mn}|^t \tag{3.9}$$

$$\leq \sum_{m=1}^{\infty} \frac{1}{f(m)} \frac{1}{(m \log g(m))^{\frac{t}{2}}} E|Y|^t I(|Y| \leq \sqrt{m \log g(m)}) < \infty.$$

By (3.7), (3.8), and (3.9), (3.6) holds and hence (3.4) holds.

Now we prove

$$\sum_{m=1}^{\infty} \frac{1}{mf(m)} P\left\{\left|\sum_{n=1}^{m} Y_n\right| > (1+\varepsilon)U_m\right\} = \infty, \ \forall \varepsilon > 0. \tag{3.10}$$

By (3.1), $EY = 0$ and $EY^2 < \infty$. Then by the implication (1.1)$\Rightarrow$(1.4), for all $\varepsilon < 0$

$$\sum_{m=1}^{\infty} \frac{1}{mf(m)} P\left\{\left|\sum_{n=1}^{\infty} Y_n\right| > (1+\varepsilon)\sqrt{2EY^2 m \log g(m)}\right\} = \infty. \tag{3.11}$$

Then (3.10) holds by (3.2). Hence (3.2) follows from (3.4) and (3.10) .

(ii) Suppose that (3.3) holds for some $L > 0$. Then for every Y,

$$\sum_{m=1}^{\infty} \frac{1}{mf(m)} P\left\{\left|\sum_{n=1}^{m} Y_n\right| > L_m\right\} < \infty.$$

Then by the implication "(2.3)$\Rightarrow$(2.4)", it follows that $EY = 0$ and $EY^2 < \infty$, hence (3.1) holds.Let $\{Y', Y'_m, m \geq 1\}$ be an independent copy of $\{Y, Y_m, m \geq 1\}$. Then by the Lemma 2.5,

$$mP\{|Y| > 4LV_m\} \leq \Delta P\left\{\left|\sum_{n=1}^{m} (Y_n - Y'_n)\right| > 2LV_m\right\}$$

$$\leq 16P\left\{\left|\sum_{n=1}^{m} Y_n\right| > LV_m\right\},$$

which by (3.3) ensures that

$$\sum_{m=1}^{\infty} \frac{1}{f(m)} P(|Y| > 4LV_m) < \infty,$$

and (2.7) follows from Lemma 2.7.

□

Remark 3.2. *Let* $EY^2 < \infty$. *Then* (2.7) *holds.*

Proof. By Lemma 2.6, $f(\cdot)$ be a nondecreasing and positive function on $(0, \infty)$ and $m \geq 1$.Then

$$\begin{aligned}\sum_{m=1}^{\infty} \frac{1}{f(m)} P\left(|Y| > \sqrt{m \log g(m)}\right) &\leq \frac{1}{f(1)} \sum_{m=1}^{\infty} P\left(|Y| > \sqrt{m}\right) \\ &\leq \frac{1}{f(1)} EY^2 < \infty.\end{aligned}$$

□

The main theorem that is proved in this article is possible when Lemma 2.6 and (2.7) are correct. In the following, we explain the conditions equivalent to Lemma 2.6 and (2.7) by giving some examples.

Example 3.3. *Define* $f(a) = (\log \log a)^v$ *in Lemma 2.6, where* $v \geq 0$. *Hence as* $a \to \infty$, *equivalence* $\log g(a) \sim \log \log a$ *holds. Then* $EY^2/[\log \log |Y|]^{v+1} < \infty$ *is equivalent to* (2.7).

Example 3.4. *Define* $f(a) = (\log a)^s$ *in Lemma 2.6, where* $0 \leq s < \frac{1}{2}$. *Hence as* $a \to \infty$, *equivalence* $\log g(a) \sim (s-1) \log \log a$ *holds. Then* $Ey^2/\{(\log |Y|)^s \log \log |Y|\} < \infty$ *is equivalent to* (2.7).

Example 3.5. *In example 3.3, take* $v = 0$, *or in example 3.4, take* $s = 0$. *Then* $EY^2/ \log \log |Y| < \infty$ *is equivalent to* (2.7).

With the law of large numbers, the iterated logarithm law is considered. Weak convergence that proved in Theorem 3.1, also holds in case of almost sure convergence for symmetric random variables [5]. This strong convergence is also established for non-negative random variables [7]. An application of this strong convergence for ρ-mixing random variables is given in [8].

References

[1] Baum, Leonard E., and Melvin Katz, *Convergence rates in the law of large numbers*, Transactions of the American Mathematical Society, **120**(1) (1965) 108–123.

[2] Chen, P. Y. and Wang, D. C., *Convergence rates for probabilities of moderate deviations for moving average processes*, Acta Mathematica Sinica, English Series, **24**(4) (2008) 611–622.

[3] Davis, J. A., *Convergence rates for the law of the iterated logarithm*, The Annals of Mathematical Statistics, **39**(5) (1968) 1479–1485.

[4] Friedman, N., Melvin Katz, and L. H. Koopmans., *Convergence rates for the central limit theorem*, Proceedings of the National Academy of Sciences of the United States of America, **56**(4) (1966).

[5] Ghazani, Z. S., *Strong convergence for symmetric random variables*, Journal of Mathematical analysis, **9**(4) (2018) 1-8.

[6] Ghazani, Z. S., *Symmetrization method for laws of large numbers under sub-linear expectation*, Journal of Mathematical analysis, **10**(3) (2019) 68-75.

[7] Ghazani, Z. Shokooh., *Generalization of strong law of large numbers for nonnegative random variables*, International Journal of Applied Mathematics and Statistics, **57**(5) (2018) 128-132.

[8] Ghazani, Z. S., *Applications of the Rosenthal-type inequality for ρ-mixing random variables*, Romai Journal, **12**(2) (2017) 71-76.

[9] Gut, A., *Probability: a graduate course*, Springer Science & Business Media, **75**, 2013.

[10] Gut, A., *Convergence rates for probabilities of moderate deviations for sums of random variables with multidimensional indices*, The Annals of Probability, **8**(2) (1980) 298–313.
[11] Hartman, P. and Wintner, A., *On the law of the iterated logarithm*, American Journal of Mathematics, **63**(1) (1941) 169–176.
[12] Jiang, Ye, and Li Xin Zhang., *Precise rates in the law of iterated logarithm for the moment of iid random variables*, Acta Mathematica Sinica 3 , **22**(3) (2006) 781–792.
[13] Lai, Tze Leung., *Limit theorems for delayed sums*, The Annals of Probability, **2**(3) (1974) 432–440.
[14] Li, D. and Rosalsky, A., *A supplement to the Davis-Gut law*, Journal of mathematical analysis and applications, **330**(2) (2007) 1488–1493.
[15] Li, D., Sptaru, A., *Refinement of convergence rates for tail probabilities*, Journal of Theoretical Probability, **18**(4) (2005) 933–947.

Zahra Shokooh Ghazani
Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran
E-mail address: zah.shokooh_ghazani@iauctb.ac.ir