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HIGHER ORDER STRONGLY EXPONENTIALLY BICONVEX FUNCTIONS AND BIVARIATIONAL INEQUALITIES

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Abstract. In this paper, some new classes of the higher order strongly exponentially biconvex functions are introduced. New relationships among various concepts of higher order strongly exponentially biconvex functions are established. New parallelogram laws are obtained as novel applications of higher order exponentially biconvex functions. It is shown that the optimality conditions of differentiable higher order strongly exponentially biconvex functions can be characterized by exponential bivariational inequalities. Some iterative methods for solving exponential bivariational inequalities using the auxiliary principle technique. Convergence analysis of the proposed methods are investigated using pseudo-monotonicity, which is weaker condition than monotonicity. Some special cases are discussed, which can be obtained as application of the results. As special cases, one can obtain various new and known results from our results. Results obtained in this paper can be viewed as refinement and improvement of previously known results.

1. Introduction

In recent years, several extensions and generalizations have been considered for classical convexity. Polyak [48] introduced the concept of Strongly convex functions, which play an important part in the optimization theory, variational inequalities and related areas. Karmardian[20] used the strongly convex functions to discuss the unique existence of a solution of the nonlinear complementarity problems. Zu and Marcotte [53] investigated the the convergence analysis for iterative methods for solving variational inequalities and equilibrium problems using the strongly convex functions. Awan et al [9, 10] have derived Hermite-Hadamard type inequalities, which provide upper and lower estimate for the integrand. For applications and properties of the strongly convex functions, see, for example, [1, 2, 3, 12, 16, 17, 18, 19, 23, 24, 25, 33, 35, 36, 37, 38, 44, 45, 47, 48, 53] and the references therein. Noor and Noor [19] introduced the higher order strongly convex functions and studied their basic properties. They have also shown that the parallelogram laws for uniformly Banach spaces can be obtained as novel applications of these results. Lin and Fukushima [21], Mako and Pales [23] and Olbrys

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[45] have considered higher order strongly convex functions and discussed their applications in mathematical programming with equilibriums, variational inequalities and optimization programming.

To be more precise, a function F on the convex and closed set $K \subseteq H$ is said to be a higher order strongly convex, if there exists a constant $\mu \geq 0$ such that

$$F((1-t)u+tv) \leq (1-t)F(u)+tF(v)-\mu\varphi(t)\|v-u\|^p, \quad \forall u,v \in K, t \in [0.1], p \geq 1, \tag{1.1}$$

where

$$\varphi(t) = t(1-t). \tag{1.2}$$

If $p = 2$, then higher order strongly convex functions become strongly convex functions with the same $\varphi(t)$ as defined in (1.2). That is,

$$F((1-t)u+tv) \leq (1-t)F(u)+tF(v)-\mu t(1-t)\|v-u\|^2, \quad \forall u,v \in K, t \in [0.1]. \tag{1.3}$$

We have noticed that the function $\varphi(t)$ in (1.2) should be

$$\varphi(t) = \{t^p(1-t)+t(1-t(^p\}.$$

Characterizations of the higher order strongly convex functions discussed in Lin and Fukushima [21], Alabdali et al [2], Mako et al [23] and Olbrys [45] are not correct. These facts and observations inspired Mohsen et al [24] and Noor et al [33, 36, 37, 38, 41, 42] to consider higher order strongly convex functions and their variant forms.

It is known that the algorithmically convex functions can be used to derive more accurate inequalities. Closely related to the log-convex functions, we have the concept of exponentially convex(concave) functions. The origin of exponentially convex functions can be traced back to Bernstein [12]. Avriel [7, 8] introduced and studied the concept of r-convex functions, where as the (r,p)-convex functions were studied by Antczak [6]. For further properties of the r-convex functions, see [6, 18, 46, 52] and the references therein. Exponentially convex functions have important applications in information theory, big data analysis, machine learning and statistic. See [4, 18, 46, 52] and the references therein. Noor and Noor [30, 31, 32, 33, 34, 35, 41] considered the concept of exponentially convex functions and discussed the basic properties. It is worth mentioning that these exponentially convex functions [30, 31, 32, 33] are distinctly different from the exponentially convex functions considered and studied by Bernstein [12] and Awan et al.[9]. They have shown that the exponentially functions enjoy the same interesting properties which convex functions have.

It is well known that the minimum u of differentiable convex function f on the convex set $K \subseteq H$ is equivalent to finding $u \in K$ such that

$$\langle F'(u), v-u\rangle \geq 0, \quad \forall v \in K. \tag{1.4}$$

The inequality of the type (1.4 is known as the variational inequality, which was introduced and studied by Stampcchia [50] in the field of potential theory. Variational inequalities can be viewed as a novel and important generalization of the variational principles. The origin of which can be traced back to Euler, Lagrange and Bernoulli brothers. It has been shown that a wide class of unrelated problems, which arise in various filed of mathematical and engineering sciences can be studied in the general framework of variational inequalities. For the recent developments and other aspects of variational inequalities, see [17, 22, 26, 27, 28, 29, 37, 38, 39, 40, 41, 42, 43, 50, 53] and the references therein.

Noor [26] and Noor et al. [39, 40, 42] have introduced the biconvex sets and biconvex functions, which can be viewed as refinement of the convex functions. Motivated by the research work going in this field, we introduce and consider some new classes of nonconvex functions with respect to an arbitrary bifunction, which is called the higher order exponentially biconvex functions. Several new concepts of monotonicity are introduced. We establish the relationship between these classes and derive some new results under some mild conditions. We obtain the higher order strongly convex functions as a special case, which can be viewed as novel applications of the higher order strongly exponentially biconvex functions. New parallelogram type laws are derived as novel applications of the higher order strongly exponentially affine biconvex, which can be used to characterize the uniform Banach spaces. It is shown that the optimality conditions for differentiable higher order strongly exponentially biconvex are characterized by exponentially bivariational inequalities, Auxiliary principle technique is used to suggest some iterative methods for exponential bivariational inequalities. Convergence analysis is also discussed using the pseud-monotonicity, which is weaker condition than monotonicity. Our method of proof is very simple as compared with other techniques and dose not involved projection method and its variant forms. As special cases, one can obtain various new and refined versions of known results. It is expected that the ideas and techniques of this paper may stimulate further research in this field.

2. Preliminary Results

Let K_β be a nonempty closed set in a real Hilbert space H. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the inner product and norm, respectively. Let $F : K_\beta \to H$ be a continuous function and $\beta(.-.) : K_\beta \times K_\beta \to H$ be an arbitrary continuous bifunction.

For the sake of completeness and for the convenience of the readers, we recall the following known results [16, 25, 30, 31, 39, 40, 41, 42, 47].

Definition 2.1. *A set K in H is said to be a convex set in the Hilbert space H, if*

$$u + \lambda(v - u) \in K, \qquad \forall u, v \in K, \lambda \in [0, 1].$$

Definition 2.2. *A function F is said to be convex, if*

$$F((1-\lambda)u + tv) \leq (1-\lambda)F(u) + \lambda F(v), \quad \forall u, v \in K, \quad \lambda \in [0, 1]. \tag{2.1}$$

Definition 2.3. [30, 31] *A function F is said to be exponentially convex function, if*

$$e^{F((1-\lambda)u+\lambda v)} \leq (1-\lambda)e^{F(u)} + \lambda e^{F(v)}, \quad \forall u, v \in K, \quad \lambda \in [0, 1].$$

We remark that Definition 2.3 can be rewritten in the following equivalent way, which is due to Antczak [6].

Definition 2.4. *A function F is said to be exponentially convex function, if*

$$F((1-\lambda)u+\lambda v) \leq \log[(1-\lambda)e^{F(a)}+\lambda e^{F(b)}], \quad \forall a, b \in K, \quad \lambda \in [0,1]. \tag{2.2}$$

It is obvious that two concepts are equivalent. This equivalent formulation has been used to discuss various aspects of the exponentially convex functions. It is worth mentioning that one can also deduce the concept of exponentially convex functions from r-convex functions, which were considered by Avriel [7, 8] and Bernstein [12].

For the applications of the exponentially convex functions in the mathematical programming and information theory, see Antczak [6], Alirezaei and Mathar [4] and Pal et al. [46]. For the applications of the exponentially concave functions in the communication and information theory, we have the following example.

Example 2.1. [4] *The error function*

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\lambda^2} d\lambda,$$

becomes an exponentially concave function in the form $erf(\sqrt{x}), \quad x \geq 0$, which describes the bit/symbol error probability of communication systems depending on the square root of the underlying signal-to-noise ratio. This shows that the exponentially concave functions can play important part in communication theory and information theory.

We now introduce some new classes of exponentially biconvex functions and their variant forms.

Definition 2.5. [26, 39, 42]. *The set K_β in H is said to be biconvex set with respect to an arbitrary bifunction $\beta(\cdot - \cdot)$, if*

$$u+\lambda\beta(v-u) \in K_\beta, \qquad \forall u, v \in K_\beta, \lambda \in [0,1].$$

The biconvex set K_β is also called β-connected set. If $\beta(v-u) = v-u$, then the biconvex set K_β is a convex set, but the converse is not true. For example, the set $K_\beta = R - (-\frac{1}{2}, \frac{1}{2})$ is an biconvex set with respect to β, where

$$\beta(v-u) = \begin{cases} v-u, & \text{for} \quad v>0, u>0 \quad \text{or} \quad v<0, u<0 \\ u-v, & \text{for} \quad v<0, u>0 \quad \text{or} \quad v<0, u<0. \end{cases}$$

It is clear that K_β is not a convex set.

Remark. *We would like to emphasize that, if $u+\beta(v-u) = v, \quad \forall u, v \in K_\beta$, then $\beta(v-u) = v-u$. Consequently, the β-biconvex set reduces to the convex set K. Thus, $K_\beta \subset K$. This implies that every convex set is a biconvex set, but the converse is not true.*

Definition 2.6. *The function F on the biconvex set K_β is said to be higher order strongly exponentially biconvex, if there exists a constant $\mu > 0$, such that*

$$e^{F(u+\lambda\beta(v-u))} \leq (1-\lambda)e^{F(u)}+\lambda e^{F(v)} - \mu\{\lambda^p(1-\lambda)+\lambda(1-\lambda^p)\}\|\beta(v-u)\|^p, \tag{2.3}$$
$$\forall u, v \in K_\beta, \lambda \in [0,1], p > 1.$$

Note that every higher order strongly exponentially convex function is a higher order strongly exponentially biconvex , but the converse is not true.

If $t = \frac{1}{2}$, then the function F satisfies

$$e^{F(\frac{2u+\beta(v-u)}{2})} \leq \frac{1}{2}\{e^{F(u)} + e^{F(v)}\} - \frac{1}{2^p}\mu\|\beta(v-u)\|^p, \forall u, v \in K_\beta, p > 1$$

and is called higher order strongly exponentially Jensen biconvex function.

We now discuss some special cases of the higher order strongly strongly exponentially biconvex functions:

I. If $p = 2$, then the higher order strongly exponentially biconvex function becomes:

Definition 2.7. *The function F on the biconvex set K_β is said to be strongly exponentially biconvex, if there exists a constant $\mu > 0$, such t*

$$e^{F(u+\lambda\beta(v-u))} \leq (1-\lambda)e^{F(u)} + \lambda e^{F(v)} - \mu\lambda(1-\lambda) \parallel \beta(v-u) \parallel^2, \forall u, v \in K_\beta, \lambda \in [0,1].$$

It is remarked that the strongly exponentially biconvex functions have been introduced and studied by Noor et al. [42].

II. If $\beta(v-u) = v - u$, then Definition 2.6 becomes:

Definition 2.8. *The function F on the convex set K is said to be higher order strongly exponentially convex, if there exists a constant $\mu > 0$, such that*

$$e^{F(u+\lambda(v-u))} \leq (1-\lambda)e^{F(u)} + \lambda e^{F(v)} - \mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|v-u\|^p, \quad \forall u, v \in K, \lambda \in [0,1], p > 1.$$

For the properties and applications of the higher order exponentially convex functions, see Noor and Noor [34].

III. If $e^{F(u)} = \phi(u)$, then Definition 2.8 reduces to:

Definition 2.9. *The function F on the convex set K_β is said to be higher order strongly convex, if there exists a constant $\mu > 0$, such that*

$$\phi(u + \lambda\beta(v-u)) \leq (1-\lambda)\phi(u) + \lambda\phi(v) - \mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|v-u\|^p, \forall u, v \in K_\beta, \lambda \in [0,1], p \geq 1. \quad (2.4)$$

It worth mentioning that the equation (2.4) represents the correct form of the higher order strongly functions of (1.1). Consequently, the equation (2.4) can be viewed as a significant refinement of (1.1). These facts and observations motivated Noor et al. [36, 37, 38, 41] to explore the characterizations and properties of the higher order strongly convex functions and its variant forms.

IV. If $\mu = 0$ then Definition 2.9 becomes.

Definition 2.10. *The function F on the convex set K_β is said to be exponentially biconvex, if*

$$e^{F(u+\lambda\beta(v-u))} \leq (1-\lambda)e^{F(u)} + \lambda e^{F(v)} \quad \forall u, v \in K_\beta, \lambda \in [0,1].$$

The exponentially biconvex functions were studied by Noor [26] and Noor et al. [39, 42].

We now define the exponentially biconvex functions on the interval
$K_\beta = I_\beta = [a, a+\beta(b-a)]$.

Definition 2.11. *Let $I = [a, a+\beta(b-a)]$. Then F is an exponentially biconvex function, if and only if,*

$$\begin{vmatrix} 1 & 1 & 1 \\ a & x & a+\beta(b-a) \\ e^{F(a)} & e^{F(x)} & e^{F(b)} \end{vmatrix} \geq 0; \quad a \leq x \leq a+\beta(b-a).$$

One can easily show that the following are equivalent:

(1) F is an exponentially biconvex function.
(2) $e^{F(x)} \leq e^{F(a)} + \frac{e^{F(b)}-e^{F(a)}}{\beta(b-a)}(x-a)$.
(3) $\frac{e^{F(x)}-e^{F(a)}}{x-a} \leq \frac{e^{F(b)}-e^{F(a)}}{\beta(b-a)}$.
(4) $\frac{e^{F(a)}}{(\beta(b-a))(a-x)} + \frac{e^{F(x)}}{(x-a-\beta(b-a))(a-x)} + \frac{e^{F(b)}}{\beta(b-a)(x-b)} \leq 0$,

where $x = a+\lambda\beta(b-a) \in [a, a+\beta(b-a)]$. Clearly $[a, a+\beta(b-a)] \subset [a, b]$.

In brief, for appropriate choice of the arbitrary bifunction $\beta(.-.)$, one can obtain a wide class of exponentially biconvex convex functions and their variant forms.

We remark that the function F is higher order strongly exponentially biconcave, if and only if, the function $-F$ is higher order strongly exponentially biconvex functions. This observation enables us to define the concept of affine functions.

Definition 2.12. *The function F on the biconvex set K_β is said to be higher order strongly exponentially affine biconvex, if there exists a constant $\mu > 0$, such that*

$$e^{F(u+\lambda\beta(v-u))} = (1-\lambda)e^{F(u)} + \lambda e^{F(v)} - \mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\beta(v-u)\|^p\}, \quad \forall u, v \in K_\beta, \lambda \in [0,1], p > 1.$$

If $\lambda = \frac{1}{2}$, then the function F satisfies

$$e^{F(\frac{2u+\beta(v-u)}{2})} = \frac{1}{2}\{e^{F(u)} + e^{F(v)}\} - \frac{1}{2^p}\mu\|\beta(v-u)\|^p, \forall u, v \in K_\beta, p > 1$$

and is called higher order strongly exponentially affine Jensen biconvex function.

Definition 2.13. *The function F on the biconvex set K_β is said to be higher order strongly exponentially quasi biconvex, if there exists a constant $\mu > 0$ such that*

$$e^{F(u+\lambda\beta(v-u))} \leq \max\{e^{F(u)}, e^{F(v)}\} - \mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\beta(v-u)\|^p, \quad \forall u, v \in K_\beta, \lambda \in [0,1], p > 1.$$

Definition 2.14. *The function F on the biconvex set K_β is said to be higher order strongly exponentially* log*-biconvex, if there exists a constant $\mu > 0$ such that*

$$e^{F(u+\lambda\beta(v-u))} \leq (e^{F(u)})^{1-\lambda}(e^{F(v)})^{\lambda} - \mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\beta(v-u)\|^p, \quad \forall u, v \in K_\beta, \lambda \in [0,1], p > 1,$$

where $F(\cdot) > 0$.

From the above definitions, we have

$$\begin{aligned} e^{F(u+t\beta(v-u))} &\leq (e^{F(u)})^{1-t}(e^{F(v)})^{t} \\ &\quad -\mu\{\lambda^{p}(1-\lambda)+\lambda(1-\lambda)^{p}\}\|\beta(v-u)\|^{p} \\ &\leq (1-t)e^{F(u)}+te^{F(v)} \\ &\quad -\mu\{\lambda^{p}(1-\lambda)+\lambda(1-\lambda)^{p}\}\|\beta(v-u)\|^{p} \\ &\leq \max\{e^{F(u)},e^{F(v)}\} \\ &\quad -\mu\{\lambda^{p}(1-\lambda)+\lambda(1-\lambda)^{p}\}\|\beta(v-u)\|^{p}. \end{aligned}$$

It is observed that
every higher order strongly exponentially log-biconvex function
$\Rightarrow$ higher order strongly exponentially biconvex function
$\Rightarrow$ higher order strongly exponentially quasi-biconvex function.
However, the converse is not true.

For $\lambda = 1$, Definition 2.6 and 2.12 reduce to the following condition.
Condition A.

$$e^{F(u+\beta(v-u))} \leq e^{F(v)}, \quad \forall u, v \in K_{\beta}.$$

Let $F'(u))$, be the differential of F at $u \in K_{\beta}$.

Definition 2.15. *The differentiable operator $F'(.)$ is said to be higher order strongly exponentially biconvex function with respect to an bifunction $\beta(\cdot - \cdot)$, if there exists a constant $\mu > 0$ such that*

$$e^{F(v} - e^{F(u)} \geq \langle e^{F(u)}F'(u), \beta(v-u)\rangle + \mu\|\beta(v-u)\|^{p}, \forall u, v \in K_{\beta}.$$

We also need the following assumption regarding the bifunction $\beta(\cdot - \cdot)$.
Condition C. Let the bifunction $\beta(\cdot - \cdot) : K_{\beta} \times K_{\beta} \to H$ satisfy assumptions:

$$\begin{aligned} (i).\quad \beta(\gamma\beta(v-u)) &= \gamma\beta(v-u), \quad \forall u, v \in K_{\beta}, \quad \gamma \in R^{n}. \\ (ii).\quad \beta(v-u-\lambda\beta(v-u)) &= (1-\lambda)\beta(v-u), \quad \forall u, v \in K_{\beta}. \end{aligned}$$

Remark. *Consider*

$$\beta(v-u) = \beta(v-z) + \beta(z-u), \quad \forall u, v \in K_{\beta}.$$

One can easily show that

$$\beta(v-u) = 0 \quad \Leftrightarrow \quad v = u, \quad \forall u, v \in K_{\beta}$$

and

$$\beta(v-u) + \beta(u-v),$$

This means that the bifunction $\beta(. - .)$ is skew symmetric.

3. Main Results

In this section, we consider some basic properties of higher order exponentially biconvex functions on the biconvex set K_{β}.

Theorem 3.1. *Let F be a differentiable function on the biconvex set K_β in H and Condition C hold. Then function F is higher order strongly exponentially biconvex function, if and only, if F satisfies*

$$e^{F(v)} - e^F(u) \geq \langle e^{F(u)} F'(u), \beta(v-u)) \rangle + \mu\|\beta(v-u)\|^p, \quad \forall u, v \in K_\beta. \tag{3.1}$$

Proof. Let F be a higher order strongly exponentially biconvex function. Then

$$\begin{aligned} e^{F(u+\lambda\beta(v-u))} &\leq (1-\lambda)e^{F(u)} + \lambda e^{F(v)} \\ &\quad -\mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\beta(v-u)\|^p, \quad \forall u, v \in K_\beta, \end{aligned}$$

which can be written as

$$e^{F(v)} - e^{F(u)} \geq \{\frac{e^{F(u+\lambda\beta(v-u))} - e^{F(u)}}{\lambda}\} + \mu\{\lambda^{p-1}(1-\lambda) + (1-\lambda)^p\}\|\beta(v-u)\|^p.$$

Taking the limit in the above inequality as $\lambda \to 0$, we have

$$e^{F(v)} - e^F(u) \geq \langle e^{F(u)} F'(u), \beta(v-u)) \rangle + \mu\|\beta(v-u)\|^p,$$

which is the required (3.1).

Conversely, let F be a higher order strongly exponentially biconvex function on the biconvex set K_β. Then $\forall u, v \in K_\beta, \lambda \in [0,1]$, $v_\lambda = u + \lambda\beta(v-u) \in K_\beta$, using Condition C, we have

$$\begin{aligned} & e^{F(v)} - e^{F(u+\lambda\beta(v-u))} \\ \geq\ & \langle e^{F(u+\lambda\beta(v-u))} F'(u+\lambda\beta(v-u)), \beta(v-u+\lambda\beta(v-u)) \rangle \\ & +\mu\|\beta(v-u+t\beta(v-u))\|^p \\ =\ & (1-\lambda)\langle e^{F(u+\lambda\beta(v-u))} F'(u+\lambda\beta(v-u)), \beta(v-u) \rangle \\ & +\mu(1-\lambda)^p\|\beta(v-u)\|^p. \end{aligned} \tag{3.2}$$

In a similar way, we have

$$\begin{aligned} & e^{F(u)} - e^{F(u+\lambda\beta(v-u))} \\ \geq\ & \langle e^{F(u+\lambda\beta(v-u))} F'(u+\lambda\beta(v-u)), \beta(u-u-\lambda\beta(v-u)) \rangle \\ & +\mu\|\eta(u-u-\lambda\beta(v-u))\|^p \\ =\ & -\lambda e^{F(u+\lambda\beta(v-u))} F'(u+\lambda\beta(v-u)), \beta(v-u) \rangle + \mu\lambda^p\|\beta(v-u)\|^p. \end{aligned} \tag{3.3}$$

Multiplying (5.1) by λ and (3.3) by $(1-\lambda)$ and adding the resultant, we have

$$e^{F(u+\lambda\beta(v-u))} \leq (1-\lambda)e^{F(u)} + \lambda e^{F(v)} - \{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\beta(v-u)\|^p.$$

showing that F is a higher order strongly exponentially biconvex function. □

Theorem 3.2. *Let F be a differentiable higher order strongly exponentially biconvex function on the biconvex set K_β. Then the operator $F'(.)$ is a higher order strongly exponentially β-monotone.*

Proof. Let F be a higher order strongly exponentially biconvex function on the biconvex set K_β. Then, form Theorem 3.1, we have

$$e^{F(v)} - e^{F(u)} \geq \langle e^{F(u+\lambda\beta(v-u))} F'(u), \beta(v-u) \rangle + \mu\|\beta(v-u)\|^p, \quad \forall u, v \in K_\beta. \tag{3.4}$$

Changing the role of u and v in (3.4), we have

$$e^{F(u)} - e^{F(v)} \geq \langle e^{F(v)} F'(v), \beta(u-v) \rangle + \mu\|\beta(u-v)\|^p \quad \forall u, v \in K_\eta. \tag{3.5}$$

Adding (3.4) and (3.5), we have

$$\begin{aligned}\langle e^{F(u)}F'(u),\beta(v-u))\rangle \quad + \quad & \langle e^{F(v)}F'(v),\beta(u-v)\rangle \\ & \leq -\mu\{\|\beta(v-u)\|^p+\|\beta(u-v)\|^p\}, \qquad (3.6)\end{aligned}$$

which shows that $F'(.)$ is higher order strongly exponentially β-monotone.

□

Theorem 3.3. *Let the differential $F'(.)$ be higher order strongly exponentially β-monotone. If the Condition C holds, then*

$$e^{F(v)}-e^{F(u)} \geq \langle e^{F(u)}F'(u),\beta(v-u)\rangle+\mu\frac{1}{p}\|\beta(v-u)\|^p, \quad \forall u,v\in K_\beta. \qquad (3.7)$$

Proof. Let F' be higher order strongly exponentially β-monotone. From (3.6), we have

$$\begin{aligned}\langle e^{F(v)}F'(v),\beta(u-v)\rangle \quad \leq \quad & -\langle e^{F(u)}F'(u),\beta(v-u))\rangle \\ & -\mu\{\|\beta(v-u)\|^p+\|\beta(u-v)\|^p\}, \qquad (3.8)\end{aligned}$$

Since K_β is a biconvex set, $\forall u,v\in K_\beta$, $t\in[0,1]$ $v_\lambda=u+\lambda\beta(v-u)\in K_\beta$. Taking $v=v_\lambda$ in (3.8), using Condition C, we have

$$\begin{aligned} & \langle e^{F(v_\lambda)}F'(v_\lambda),\beta(u-u-t\beta(v-u))\rangle \\ \leq \quad & -\langle e^{F(u)}F'(u),\beta(u+\lambda\beta(v-u)-u)\rangle \\ & -\mu\{\|\eta(u+\lambda\beta(v-u),u))\|^p+\|\beta(u-u-\lambda\beta(v-u))\|^p\} \\ = \quad & -\lambda\langle e^{F(u)}F'(u),\beta(v-u)\rangle-2\lambda^p\mu\|\beta(v-u)\|^p,\end{aligned}$$

which implies that

$$\langle e^{F(v_\lambda)}F'(v_\lambda),\beta(v-u)\rangle \geq \langle e^{F(u)}F'(u),\beta(v-u)+2\mu\lambda^{p-1}\|\beta(v-u)\|^p. \qquad (3.9)$$

Let

$$\varphi(\lambda)=e^{F(u+\lambda\beta(v-u))}. \qquad (3.10)$$

Then

$$\varphi(0)=e^{F(u)}, \quad \varphi(1)=e^{F(u+\lambda\beta(v-u))}. \qquad (3.11)$$

From (3.9), we have

$$\begin{aligned}\varphi'(t) \quad = \quad & \langle e^{F(u+\lambda\beta(v-u))}F'(u+\lambda\beta(v-u)),\beta(v-u)\rangle \\ \geq \quad & \langle e^{F(u)}F'(u),\beta(v-u)\rangle+2\mu\lambda^{p-1}\|\beta(v-u)\|^p. \qquad (3.12)\end{aligned}$$

Integrating (3.12) between 0 and 1, we have

$$\varphi(1)-\varphi(0) \geq \langle e^{F(u)}F'(u),\eta(v,u)\rangle+\mu\frac{1}{p}\|\beta(v-u)\|^p.$$

that is,

$$e^{F(u+\beta(v,u))}-e^{F(u)} \geq \langle e^{F(u)}F'(u),\beta(v-u)\rangle+\mu\frac{1}{p}\|\beta(v-u)\|^p..$$

By using Condition A, we have

$$e^{F(v)}-e^{F(u)} \geq \langle e^{F(u)}F'(u),\beta(v-u)\rangle+\mu\frac{1}{p}\|\beta(v-u)\|^p.$$

the required result. □

Theorem 3.4. *Let F' be exponentially higher order strongly relaxed β-pseudomonotone and Condition A and C hold. Then F is a higher order strongly exponentially β-pseudo biconvex function.*

Proof. Let F' be a higher order strongly exponentially relaxed β-pseudomonotone. Then

$$\langle e^{F(u)}F'(u), \beta(v-u)\rangle + \alpha\|\beta(v-u)\|^p \geq 0, \quad \forall u, v \in K_\beta,$$

implies that

$$-\langle e^{F(v)}F'(v), \beta(u-v)\rangle \geq \alpha\|\beta(v-u)\|^p. \tag{3.13}$$

Since K_β is a biconvex set, $\forall u, v \in K_\beta$, $\lambda \in [0,1]$,
$v_\lambda = u + \lambda\beta(v-u) \in K_\beta$. Taking $v = v_\lambda$ in (3.13), using Condition C, we have

$$-\langle e^{F(u+\lambda\beta(v-u))}F'(u+\lambda\beta(v-u)), \beta(u-v)\rangle \geq \lambda\alpha\|\beta(v-u)\|^p. \tag{3.14}$$

Let

$$\varphi(t) = e^{F(u+\lambda\beta(v-u))}, \quad \forall u, v \in K_\beta, \lambda \in [0,1].$$

Then, using (3.14), we have

$$\varphi'(\lambda) = \langle e^{F(u+\lambda\beta(v-u))}F'(u+\lambda\beta(v-u)), \beta(u-v)\rangle \geq \lambda\alpha\|\beta(v-u)\|^p.$$

Integrating the above relation between 0 to 1, we have

$$\varphi(1) - \varphi(0) \geq \frac{\alpha}{2}\|\beta(v-u\|^p),$$

that is,

$$e^{F(u+\beta(v-u))} - e^{F(u)} \geq \frac{\alpha}{2}\|\beta(v-u)\|^p,$$

which implies, using Condition A, that

$$e^{F(v)} - e^{F(u)} \geq \frac{\alpha}{2}\|\beta(v-u)\|^p,$$

showing that F is a higher order strongly exponentially β-pseudo biconvex function. □

As special cases of Theorem 3.4, we have the following:

Theorem 3.5. *Let the differentiable $F'(u)$ of a function $F(u)$ on the biconvex set K_η be higher order strongly exponentially β-pseudomonotone and let Conditions A and C hold. Then F is exponentially pseudo β-biconvex function.*

Theorem 3.6. *Let the differential $F'(u)$ of a function $F(u)$ on the biconvex set K_β be higher order strongly exponentially η-pseudomonotone and Conditions A and C hold. Then F is higher order strongly exponentially pseudo β-biconvex function.*

Theorem 3.7. *Let the differential $F'(u)$ of a function $F(u)$ on the biconvex set K_β be higher order strongly exponentially β-pseudomonotone and let Conditions A and C hold. Then F is higher order strongly exponentially pseudo β-biconvex function.*

Theorem 3.8. *Let the differential $F'(u)$ of a function $F(u)$ on the biconvex set be higher order strongly exponentially β-pseudomonotone and Conditions A and C hold. Then F is higher order strongly exponentially pseudo biconvex function.*

Theorem 3.9. *Let the differential* $e^{F(u)}F'(u)$ *of a differentiable higher order strongly biconvex function* $F(u)$ *be Lipschitz continuous on the biconvex set* K_β *with a constant* $\beta > 0$. *Then*

$$e^{F(u+\beta(v-u))} - e^{F(u)} \leq \langle e^{F(u)}F'(u), \beta(v-u)\rangle + \frac{\beta}{2}\|\beta(v-u)\|^2, \quad u, v \in K_\beta.$$

Definition 3.1. *The function* F *is said to be higher order sharply strongly exponentially pseudo biconvex, if there exists a constant* $\mu > 0$ *such that*

$$\begin{aligned} & \langle e^{F(u)}F'(u), \beta(v-u)\rangle \geq 0 \\ & \Rightarrow \\ e^{F(v)} \quad \geq \quad & e^{F(v+\lambda\beta(v-u))} + \mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\beta(v-u)\|^p, \forall u, v \in K_\beta, \lambda \in [0,1]. \end{aligned}$$

Theorem 3.10. *Let* F *be a higher order strongly sharply exponentially pseudo biconvex function on* K_β *with a constant* $\mu > 0$. *Then*

$$-\langle e^{F(u)}F'(v), \beta(v-u)\rangle \geq \mu\|\beta(v-u)\|p, \quad \forall u, v \in K_\beta.$$

Proof. Let F be a higher order strongly sharply exponentially pseudo biconvex function on the biconvex set K_β. Then

$$e^{F(v)} \geq e^{F(v+\lambda\beta(v-u))} + \mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\beta(v-u)\|^p, \quad \forall u, v \in K_\beta, \lambda \in [0,1].$$

from which we have

$$\{\frac{e^{F(v+\lambda\beta(v-u))} - e^{F(v)}}{\lambda}\} + \mu\{\lambda^{p-1}(1-\lambda) + (1-\lambda)^p\}\|\beta(v-u)\|^p \leq 0.$$

Taking limit in the above inequality, as $t \to 0$, we have

$$-\langle e^{F(u)}F'(v), \beta(v-u)\rangle \geq \mu\|\beta(v-u)\|^p,$$

the required result. □

Definition 3.2. *A function* F *is said to be a higher order strongly exponentially pseudo biconvex function, if there exists a strictly positive bifunction* $B(.-.)$, *such that*

$$\begin{aligned} e^{F(v)} \quad & < \quad e^{F(u)} \\ & \Rightarrow \\ e^{F(u+\lambda\beta(v-u))} \quad & < \quad e^{F(u)} + \lambda(\lambda-1)B(v,u), \forall u, v \in K_\beta, \lambda \in [0,1]. \end{aligned}$$

Theorem 3.11. *If the function* F *is higher order strongly exponentially pseudo biconvex function such that* $e^{F(v)} < e^{F(u)}$, *then the function* F *is higher order strongly exponentially generalized pseudo biconvex.*

Proof. Since $e^{F(v)} < e^{F(u)}$ and F is higher order strongly exponentially pseudo biconvex function, then $\forall u, v \in K_\beta, \quad \lambda \in [0,1]$, we have

$$\begin{aligned} e^{F(u+\lambda\beta(v-u))} \quad & \leq \quad e^{F(u)} + \lambda(e^{F(v)} - e^{F(u)}) \\ & \quad -\mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\beta(v-u))\|^p \\ & < \quad e^{F(u)} + \lambda(1-\lambda)(e^{F(v)} - e^{F(u)}) \\ & \quad -\mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\beta(v-u)\|^p \\ & = \quad e^{F(u)} + \lambda(\lambda-1)(e^{F(u)} - e^{F(v)}) \\ & \quad -\mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\beta(v-u)\|^p \\ & < \quad e^{F(u)} + \lambda(\lambda-1)B(u,v) - \mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\beta(v-u)\|^p, \end{aligned}$$

where $B(u,v) = e^{F(u)} - e^{F(v)} > 0$. This shows that F is a higher order strongly exponentially pseudo biconvex function □

It is well known that each strongly convex functions is of the form $- \pm \|.\|^2$, where f is a convex function. We now establish a similar result for the higher order strongly exponentially biconvex functions.

Theorem 3.12. *Let f be a higher order strongly exponentially affine generalized biconvex function. Then F is a higher order strongly exponentially biconvex function, if and only if, $\zeta = F - f$ is a exponentially biconvex function.*

Proof. Let f be higher order strongly exponentially affine biconvex function. Then

$$\begin{aligned} e^{f(u+\lambda\beta(v-u))} &= (1-\lambda)e^{f(u)} + \lambda e^{f(v)} \\ &\quad -\mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\beta(v-u)\|^p. \end{aligned} \tag{3.15}$$

From the higher order strongly exponentially generalized biconvexity of F, we have

$$\begin{aligned} e^{F(u+\lambda\beta(v-u))} &\leq (1-\lambda)e^{F(u)} + te^{F(v)} \\ &\quad -\mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\beta(v-u)\|^p. \end{aligned} \tag{3.16}$$

From (3.15) and (3.16), we have

$$e^{F(u+\lambda\beta(v-u))} - e^{f(u+\lambda\beta(v-u))} \leq (1-\lambda)(e^{F(u)} - e^{f(u)}) + \lambda(e^{F(v)} - e^{f(v)}), \tag{3.17}$$

from which it follows that

$$\begin{aligned} e^{\zeta(u+\lambda\beta(v-u))} &= e^{F((u+\lambda\beta(v-u)))} - e^{f((1-\lambda)u+\lambda v)} \\ &\leq (1-\lambda)e^{F(u)} + \lambda e^{F(v)} - (1-\lambda)e^{f(u)} - \lambda e^{f(v)} \\ &= (1-\lambda)(e^{F(u)} - e^{f(u)}) + \lambda(e^{F(v)} - e^{f(v)}), \end{aligned}$$

which show that $\zeta = F - f$ is an exponentially biconvex function.
The inverse implication is obvious. □

We would like to remark that one can show that F is a higher order strongly exponentially biconvex function, if and only if, F is strongly exponentially affine biconvex function essentially using the technique of Adamek [1] and Noor et al. [37, 38, 41].

Remark. *It is worth mentioning that the higher order strongly exponentially biconvex is also higher order Wright strongly exponentially biconvex functions. From the definition 2.6, we have*

$$\begin{aligned} &e^{F(u+\lambda\beta(v-u))} + e^{F(v+\lambda\beta(u-v)} \\ &\leq e^F(u) + e^F(v) - 2\mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\beta(v-u)\|^p, \forall u,v \in K_\beta, \lambda \in [0,1], \end{aligned}$$

which is called higher order strongly Wright exponentially biconvex function. It is an open problem to investigate the properties and its applications in various branches of pure and applied sciences.

4. Parallelogram Laws

In this section, we derive new parallelogram laws for uniformly Banach spaces as a novel application of higher order strongly exponentially biconvex functions. From definition 2.12, we have

$$\|e^{F(u+\lambda\beta(v-u))}\|^p = (1-\lambda)e^{F(u)} + \lambda e^{F(v)}$$
$$-\mu\{\lambda^p(1-\lambda)+(\lambda)(1-\lambda)^p\}\|\beta(v-u)\|^p, \ \forall u,v \in K_{\beta,}\lambda \in [0,1]. \quad (4.1)$$

Taking $\lambda = \frac{1}{2}$ in (4.1), we have

$$\|e^{F(u+\frac{1}{2}\beta(v-u))}\|^p + \mu\frac{1}{2^p}\|\beta(v-u)\|^p = \frac{1}{2}\{e^{F(u)} + e^{F(v)}\}, \forall u,v \in K_\beta. \quad (4.2)$$

If $e^{F(u)}$ is homogeneous, then equation (4.2) can be written as

$$\|e^{F(u+\beta(v-u))}\|^p + \mu\|\beta(v-u)\|^p = 2^{p-1}\{e^{F(u)} + e^{F(v)}\}, \forall u,v \in K_\beta. \quad (4.3)$$

which is known as the exponentially parallelogram-like laws for the Banach spaces involving higher order strongly exponentially biconvex functions.

We now discuss some special cases of the exponentially parallelogram-like laws.

(I). If $\beta(v,u) = v-u, \quad \lambda = \frac{1}{2}$ and the function F is homogeneous, then (4.3) reduces to the parallelogram-like law as:

$$\|e^{F(v+u)}\|^p + \mu\|v-u\|^p = 2^{p-1}\{e^{F(u)} + e^{F(v)}\}, \quad (4.4)$$

which is known as the exponentially parallelogram-like law for the uniform Banach spaces involving the exponentially biconvex functions.
(II). If $e^{F(u)} = \|u\|^p$, then (4.4) collapses to:

$$\|v+u\|^p + \mu\|v-u\|^p = 2^{p-1}\{\|u\|^p + \|v\|^p\|\}, \quad \forall u,v \in K_\beta, \quad p>1, \quad (4.5)$$

which is known as the parallelogram law for the uniform Banach spaces involving convex functions. Xu [51] obtained these characterizations of p-uniform convexity and q-uniform smoothness of a Banach space via the functionals $\|.\|^p$ and $\|.\|^q$, respectively. Bynum [13] and Chen et al [14, 15] have studied the properties and applications of the parallelogram laws for the Banach spaces. For the applications of the parallelogram laws in Banach spaces in prediction theory and applied sciences, see [13, 14, 15] and the references therein.

(III). If $p = 2$, then parallelogram law (4.5) reduces to

$$\|v+u\|^2 + \mu\|v-u|^2 = 2\{\|u\|^2 + \|v\|^2\}. \quad (4.6)$$

which is used to characterize the inner product spaces.
For suitable and appropriate choice of the function $\beta(.-.)$ and parameter p, one can obtain a wide class of parallelogram laws, which can be used to characterize various classes of inner products spaces.

5. Exponentially bivariational inequalities

We now discuss the optimality condition for the differentiable strongly exponentially biconvex functions, which is the main motivation of our next result.

Theorem 5.1. *Let F be a differentiable higher order strongly exponentially biconvex function with modulus $\mu > 0$. If $u \in K_\beta$ is the minimum of the function F, then*

$$e^{F(v)} - e^{F(u)} \geq \mu \|\beta(v-u)\|^p, \quad \forall u, v \in K_\beta. \tag{5.1}$$

Proof. Let $u \in K_\beta$ be a minimum of the function F. Then

$$F(u) \leq F(v), \forall v \in K_\beta$$

from which, we have

$$e^{F(u)} \leq e^{F(v)}, \forall v \in K_\beta. \tag{5.2}$$

Since K_β is an biconvex set, so, $\forall u, v \in K_\beta, \quad \lambda \in [0,1]$,

$$v_\lambda = u + \lambda\beta(v-u) \in K_\beta.$$

Taking $v = v_\lambda$ in (5.2), we have

$$0 \leq \lim_{\lambda \to 0} \{ \frac{e^{F(u+\lambda\beta(v-u))} - e^{F(u)}}{\lambda} \} = \langle e^{F(u)} F'(u), \beta(v-u) \rangle. \tag{5.3}$$

Since F is a differentiable higher order strongly exponentially biconvex function, so

$$\begin{aligned} e^{F(u+\lambda\beta(v-u))} \quad \leq \quad & e^{F(u)} + \lambda(e^{F(v)} - e^{F(u)}) \\ & -\mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\beta(v-u)\|^p, \quad u, v \in K_\beta, \lambda \in [0,1], \end{aligned}$$

from which, using (5.3), we have

$$\begin{aligned} e^{F(v)} - e^{F(u)} \quad & \geq \quad \lim_{\lambda \to 0} \{ \frac{e^{F(u+\lambda\beta(v-u))} - e^{F(u)}}{\lambda} \} \\ & \quad + \mu\{\lambda^{p-1}(1-\lambda) + (1-\lambda)^p\}\|\beta(v-u)\|^p. \\ & = \quad \langle e^{F(u)} F'(u), \beta(v-u) \rangle + \mu\|\beta(v-u)\|^p) \\ & \geq \quad \mu\|\beta(v-u)\|^p, \end{aligned}$$

the required result (5.1). □

Remark. *If $u \in K_\beta$ satisfy the inequality*

$$\langle e^{F(u)} F'(u), v-u \rangle + \mu\|\beta(v-u)\|^p \geq 0, \quad \forall u, v \in K_\beta, \tag{5.4}$$

then $u \in K_\beta$ is the minimum of the function F.

We would like to emphasize that the minimum $u \in K_\beta$ of the exponentially biconvex functions can be characterized by inequality

$$\langle e^{F(u)} F'(u), \beta(v-u) \rangle + \mu\|\beta(v-u)\|^p \geq 0, \forall v \in K_\beta, \tag{5.5}$$

which is called the higher order exponentially bivariational inequality and appears to be a new one. It is an interesting problem to study the existence of a unique solution of the exponentially variational inequality (5.5) and its applications.

We now consider a higher order strongly exponentially bivariational inequality problem.

For given an operator T and bifunction $\beta(. - .)$ we consider the problem of finding $u \in K_\beta$ such that

$$\langle e^{Tu}, \beta(v-u)\rangle + \mu\|\beta(v-u)\|^p \geq 0, \forall v \in K_\beta, p > 1, \tag{5.6}$$

where μ is a constant. The inequality of the type (5.6) is called the higher order strongly exponentially bivariational inequality.

(I). If $\mu = 0$, then (5.6) is equivalent to finding $u \in K_\beta$, such that

$$\langle e^{Tu}, \beta(v-u)\rangle \geq 0, \forall v \in K_\beta, \tag{5.7}$$

which is known as the exponentially bivariational inequality. For recent developments in bivariational inequalities, see Noor [23, 24, 25, 26, 27, 40] and the references therein.

(II). If $p = 1$, then problem (5.6) reduces to: For a given operator T and $\beta(, -,)$, we consider the problem of finding $u \in K_\beta$ for a constant μ such that

$$\langle e^{Tu}, \beta(v-u))\rangle + \mu\|\beta(v-u)\| \geq 0, \forall v \in K_\beta, \tag{5.8}$$

which is called the approximate exponentially bivariational inequality.

(III). If $p = 2$, then (5.6) is equivalent to finding $u \in K_\beta$, such that

$$\langle e^{Tu}, \beta(v-u)\rangle + \mu\|\beta(v-u)\|^2 \geq 0, \forall v \in K_\beta,$$

which is called the strongly exponential bivariational inequality.

For suitable and appropriate choice of arbitrary functions T, bifunction $\beta(, -,)$, the parameter μ and p, one can obtain several new and known classes of exponentially bivariational inequalities, see [16, 18, 23, 24, 25, 26, 27, 28, 31, 40].

We now consider some iterative methods for solving the problem (5.6). We remark that the projection method and its variant forms can be used to study the higher order strongly exponentially bivariational inequalities (5.6)due to its inherent structure. To overcome this drawback, we consider the auxiliary principle technique, which is mainly due to Glowinski et al[17] and Lions and Stampacchia [22] as developed by Noor [29]. We use this technique to suggest some iterative methods for solving the higher order strongly exponentially bivariational inequalities (5.6).

For given $u \in K_\beta$ satisfying (5.6), consider the problem of finding $w \in K_\beta$, such that

$$\langle \rho e^{Tw}, \beta(v-w)\rangle + \langle w - u + \alpha(u-u), v - w\rangle + \nu\rho\|\beta(v-w)\|^p \geq 0, \quad \forall v \in K_\beta, p > 1, \tag{5.9}$$

where $\rho > 0$, $\quad \alpha$ are parameters. The problem (5.9) is called the auxiliary higher order strongly exponentially bivariational inequality. It is clear that the relation (5.9) defines a mapping connecting the problems (5.6) and (5.9). We not that, if $w(u) = u$, then w is a solution of problem (5.6). This simple observation enables to suggest an iterative method for solving (5.6).

Algorithm 5.2. . *For given $u_0 \in K_\beta$, find the approximate solution u_{n+1} by the scheme*

$$\langle \rho e^{Tu_{n+1}}, \beta(v - u_{n+1})\rangle + \langle u_{n+1} - u_n + \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle$$
$$+\nu\rho\|\beta(v - u_{n+1})\|^p \geq 0, \quad \forall v \in K_\beta, p > 1. \tag{5.10}$$

The Algorithm 5.2 is known as the implicit method. Such type of methods have been studied extensively for various classes of variational inequalities. See [16, 17] and the reference therein.
If $\nu = 0$, then Algorithm 5.2 reduces to:

Algorithm 5.3. *For given $u_0 \in K_\beta$, find the approximate solution u_{n+1} by the scheme*

$$\langle \rho e^{Tu_{n+1}}, \beta(v - u_{n+1})\rangle + \langle u_{n+1} - u_n + \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle \geq 0, \forall v \in K_\beta,$$

which appears to be new ones even for solving the exponentially bivariational inequalities.

In order to study the convergence analysis of Algorithm 5.2, we need the following.

Definition 5.1. *The operator e is said to be pseudo β-monotone with respect to $\mu\|\beta(v-u)\|^p$, if*

$$\langle \rho e^{Tu}, \beta(v - u)\rangle + \mu\|\beta(v - u)\|^p \geq 0, \forall v \in K_\beta, p > 1,$$
$$\Longrightarrow$$
$$\langle -\rho e^{Tv}, \beta(v - u)\rangle - \mu\|\beta(u - v)\|^p \geq 0, \forall v \in K_\beta, p > 1.$$

If $\mu = 0$, then Definition 5.1 reduces to:

Definition 5.2. *The operator T is said to be pseudo β-monotone, if*

$$\langle \rho e^{Tu}, \beta(v - u)\rangle \geq 0, \forall v \in K_\beta$$
$$\Longrightarrow$$
$$\langle -\rho e^{Tv}, \beta(v - u)\rangle \geq 0, \forall v \in K_\beta.$$

This appears to be a new one.
We now study the convergence analysis of Algorithm 5.2 for the case $\alpha = 0$.

Theorem 5.4. *Let $u \in K_\beta$ be a solution of (5.6) and u_{n+1} be the approximate solution obtained from Algorithm 5.2. If e^T is a pseudo β- monotone operator, then*

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - \|u_{n+1} - u_n\|^2. \tag{5.11}$$

Proof. Let $u \in K_\beta$ be a solution of (5.6), then

$$\langle \rho e^{Tu}, \beta(v - u)\rangle + \mu\|\beta(v - u)\|^p, \forall v \in K_\beta,$$

implies that

$$\langle -\rho e^{Tv}, \beta(u - v)\rangle - \mu\|\beta(u - v)\|^p, \forall v \in K_\beta, \tag{5.12}$$

Now taking $v = u_{n+1}$ in (5.12), we have

$$\langle \rho e^{Tu_{n+1}}, \beta(u_{n+1} - u)\rangle - \mu\|\beta(u_{n+1} - u)\|^p \geq 0. \tag{5.13}$$

Taking $v = u$ in (5.10), we have

$$\langle \rho e^{Tu_{n+1}}, \beta(u - u_{n+1})\rangle + \langle u_{n+1} - u_n, v - u_{n+1}\rangle$$
$$+\nu\rho\|\beta(u - u_{n+1})\|^p \geq 0. \quad \forall v \in K_\beta, p > 1. \tag{5.14}$$

Combining (5.13) and (5.14), we have

$$\langle u_{n+1} - u_n, u_{n+1} - u\rangle \geq 0.$$

Using the inequality

$$2\langle a, b\rangle = \|a + b\|^2 - \|a\|^2 - \|b\|^2, \forall a, b \in H,$$

we obtain

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - \|u_{n+1} - u_n\|^2,$$

the required result (5.11). □

Theorem 5.5. *Let the operator e^T be a pseudo β-monotone. If u_{n+1} be the approximate solution obtained from Algorithm 5.2 and $u \in K_\beta$ is the exact solution (5.6), then*

$$\lim_{n\to\infty} u_n = u.$$

Proof. Let $u \in K_\beta$ be a solution of (5.6). Then, from (5.11), it follows that the sequence $\{\|u - u_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. From (5.11), we have

$$\sum_{n=0}^{\infty} \|u_{n+1} - u_n\|^2 \leq \|u_0 - u\|^2,$$

from which, it follows that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{5.15}$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence u_n converge to $\hat{u} \in H$. Replacing u_n by u_{n_j} in (5.10), taking the limit $n_j \to 0$ and from (5.15), we have

$$\langle e^{T\hat{u}}, \beta(v - \hat{u})\rangle + \mu\|\beta(v - \hat{u})\|^p, \quad \forall v \in K_\beta, p > 1.$$

This implies that $\hat{u} \in K_\beta$ satisfies (5.6) and

$$\|u_{n+1} - u_n\|^2 \leq \|u_n - \hat{u}\|^2.$$

Thus it follows from the above inequality that the sequence u_n has exactly one cluster point $\hat{u}$ and

$$\lim_{n\to\infty} u_n = \hat{u}.$$

□

In order to implement the implicit Algorithm 5.2, one uses the predictor-corrector technique. Consequently, Algorithm 5.2 for the case $\alpha = 0$, is equivalent to the following two-step iterative method for solving the higher order exponential bivariational inequality (5.6).

Algorithm 5.6. *For a given $u_0 \in K_\beta$, find the approximate solution u_{n+1} by the schemes*

$$\langle \rho e^{Tu_n}, \beta(v-y_n)\rangle + \langle y_n - u_n, v-y_n\rangle + \rho\mu\|\beta(v-y_n)\|^p \geq 0, \forall v \in K_\beta, p>1$$
$$\langle \rho e^{Ty_n}, \beta(u_n-y_n)\rangle + \langle u_n - y_n, v-y_n\rangle + \rho\mu\|\beta(v-u_n)\|^p \geq 0, \forall v \in K_\beta, p>1.$$

Algorithm 5.6 is called the predictor-corrector iterative method and appears to be a new one.

Using the auxiliary principle technique, we now suggest an other iterative method for solving the higher order strongly exponentially bivariational inequalities and related optimization problems.

For a given $u \in K_\beta$ satisfying (5.6), consider the problem of finding $w \in H : w \in K$, such that

$$\langle \rho e^{Tu}, \beta(v-w)\rangle + \langle w-u, v-w\rangle + \rho\nu\|\beta(v-w)\|^p \geq 0, \forall v \in K_\beta, p>1, \quad (5.16)$$

where $\rho > 0$ is a parameter. The problem (5.16) is called the auxiliary higher order strongly exponentially bivariational inequality. It is clear that the relation (5.16) defines a mapping connecting the problems (5.6) and (5.16). We note that, if $w(u) = u$, then w is a solution of problem (5.6). This simple observation enables to suggest an iterative method for solving (5.6).

Algorithm 5.7. *For given $u_0 \in K_\beta$, find the approximate solution u_{n+1} by the scheme*

$$\langle \rho e^{Tu_n}, \beta(v-u_{n+1})\rangle + \langle u_{n+1} - u_n, v-u_{n+1}\rangle + \nu\rho\|\beta(v-u_{n+1})\|^p \geq 0, \quad \forall v \in K_\beta,$$

which is an explicit algorithm.

Remark. *It is worth mentioning that the auxiliary principle technique can be used efficiently to suggest a wide class of iterative methods for solving higher order strongly exponentially bivariational inequalities. We have only given some glimpses of the higher order strongly exponentially bivariational inequalities. It is an interesting problem to explore the applications of such type variational inequalities in various fields of pure and applied sciences.*

Conclusion

In this paper, we have introduced and studied a new class of biconvex functions with respect to any arbitrary bifunction. which is called the higher order strongly exponentially biconvex function. It is shown that several new classes of strongly biconvex and convex functions can be obtained as special cases of the higher order strongly exponentially biconvex functions. Some basic properties of these functions are explored. Exponentially parallelogram laws are derived as novel applications of the higher order exponentially biconvex functions, which can be used to characterize some uniform Banach spaces. Optimality conditions are characterized by some new classes of exponentially bivariational inequalities. The auxiliary principle technique is used to suggest and analyze several iterative methods for solving exponentially bivariational inequalities. Convergence analysis of the proposed iterative methods is considered under some suitable weaker conditions. Several known and new results are obtained as applications of the results. It is interesting and challenging problem to implement these iterative schemes for solving bivariational inequalities. The ideas and techniques of this paper may motivate further research.

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