

UNIFORM CONVERGENCE THEOREMS MOTIVATED BY DINI'S THEOREM FOR A SEQUENCE OF FUNCTIONS

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ABSTRACT. Being motivated by Dini's Theorem, the uniform convergence of a sequence of functions is characterized by that of another decreasing sequence of functions. This result allows us to describe the uniform convergence of a sequence of measurable functions on a closed interval in terms of that of a decreasing sequence of Lebesgue integrable functions. For a sequence of continuous functions on a compact set, its uniform convergence is further characterized by the pointwise convergence of another decreasing sequence of continuous functions. It also turns out that the uniform convergence of a sequence of Riemann integrable functions on a closed interval can be determined by that of another decreasing sequence of Riemann integrable functions. As applications, characterizations for a uniformly convergent series of functions have been presented and one fixed point theorem has been established.

1. INTRODUCTION

Uniform convergence of a sequence of functions is an important concept in studying the continuity, differentiability, and integrability of its limit function. It is known that a sequence $\{f_n\}$ of real-valued functions converges uniformly to a function f on a set A if and only if $\lim_{n \rightarrow +\infty} M_n = 0$, where

$$M_n := \sup\{|f_n(x) - f(x)| : x \in A\} \quad \text{for } n \in N.$$

Dini's Theorem [3, 7.13 Theorem, p.150] states that a pointwise convergent sequence $\{f_n\}$ of functions is also uniformly convergent on A if the following conditions are satisfied:

- (D1) A is a compact set in a metric space (X, d) ;
- (D2) each f_n is continuous on A for $n \in N$;
- (D3) $\{f_n\}$ is decreasing: $0 \leq f_{n+1}(x) \leq f_n(x)$ for all $n \in N$ and all $x \in A$.

The theorem provides a sufficient condition for $f_n \rightarrow 0$ uniformly. Even though a uniformly convergent sequence of functions is not necessary to satisfy all conditions (D1) – (D3), Dini's Theorem motivates us to discuss the following questions:

- Is the uniform convergence of a sequence of functions on A determined by that of another sequence of functions satisfying condition (D3)?

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- Can we characterize the uniform convergence of a sequence of continuous functions on a compact set in terms of another pointwise convergent sequence of functions satisfying conditions (D2) – (D3)?
- What will happen to a uniformly convergent sequence of measurable functions? Especially to a uniformly convergent sequence of Riemann integrable functions.

To give positive answers to them, we organize the rest of the paper as below: In section 2, we prove that for a uniformly convergent sequence on a set there exists a uniformly convergent sequence dominating it and satisfying condition (D3). For a sequence of measurable functions on a closed interval, its uniform convergence is completely determined by that of another decreasing sequence of Lebesgue integrable functions. Section 3 is devoted to characterize the uniform convergence of a sequence of continuous functions on a compact set by the pointwise convergence of another decreasing sequence of continuous functions. Using the result we show that the uniform convergence of a sequence of Riemann integrable functions on a closed interval can be characterized by that of a decreasing sequence of Riemann integrable functions. In Section 4, we discuss the uniform convergence of a series of functions and present some sufficient conditions for it. One fixed point theorem is also established.

2. UNIFORM CONVERGENCE OF A SEQUENCE OF FUNCTIONS ON A SET

In this section we begin with characterizing the uniform convergence of a sequence of functions defined on a set in terms of that of another sequence of functions satisfying (D3).

Theorem 2.1. *Let $\{f_n\}$ be a sequence of real-valued functions defined on a set A in a metric space (X, d) . Then the following statements are equivalent:*

- (i) $f_n \rightarrow 0$ uniformly on A .
- (ii) There exists a sequence $\{g_n\}$ of functions on A such that $g_n \rightarrow 0$ uniformly, $|f_n(x)| \leq g_n(x)$, and $g_{n+1}(x) \leq g_n(x)$ for all $n \in N$ and all $x \in A$.
- (iii) There exist a sequence $\{g_n\}$ of functions on A and a sequence $\{M_n\}$ of real numbers such that $\lim_{n \rightarrow +\infty} M_n = 0$, $|f_n(x)| \leq g_n(x) \leq M_n$, $g_{n+1}(x) \leq g_n(x)$, and $M_{n+1} \leq M_n$ for all $n \in N$ and all $x \in A$.
- (iv) There exists a sequence $\{M_n\}$ of real numbers such that $\lim_{n \rightarrow +\infty} M_n = 0$, $|f_n(x)| \leq M_n$, and $M_{n+1} \leq M_n$ for all $n \in N$ and all $x \in A$.

Proof. Since the implications (iii) \Rightarrow (iv) \Rightarrow (i) are obvious, it suffices to show (i) \Rightarrow (ii) \Rightarrow (iii).

(i) \Rightarrow (ii): Let (i) be true. Then for each $\epsilon > 0$ there exists $n_\epsilon \in N$ such that

$$|f_n(x)| \leq \epsilon \quad \text{for all } n \geq n_\epsilon \text{ and all } x \in A.$$

For each $n \in N$, taking

$$g_n(x) := \sup\{|f_m(x)| : n \leq m \in N\} \quad \text{for } x \in A,$$

we have

$$|f_n(x)| \leq g_n(x) \text{ and } g_{n+1}(x) \leq g_n(x) \quad \text{for all } n \in N \text{ and all } x \in A.$$

In addition, $0 \leq g_n(x) \leq \epsilon$ for all $n \geq n_\epsilon$ and all $x \in A$. Thus (ii) follows.

To prove (ii) \Rightarrow (iii), let $\{g_n\}$ be the sequence in (ii). For each $n \in N$, define

$$M_n := \sup\{|g_m(x)| : n \leq m \in N, x \in A\}.$$

Then

$$|f_n(x)| \leq g_n(x) \leq M_n \quad \text{and} \quad M_{n+1} \leq M_n \quad \text{for all } n \in N \text{ and all } x \in A.$$

For each $\epsilon > 0$, by the uniform convergence $g_n \rightarrow 0$, there exists $n_\epsilon \in N$ such that

$$|g_n(x)| \leq \epsilon \quad \text{for all } n \geq n_\epsilon \text{ and all } x \in A.$$

From this we obtain $M_n \leq \epsilon$ for all $n \geq n_\epsilon$ and hence $\lim_{n \rightarrow +\infty} M_n = 0$. So (iii) is valid. \square

Remark. Theorem 2.1 shows that a sequence $\{f_n\}$ uniformly converging to 0 on A must be dominated by a decreasing sequence $\{M_n\}$, which satisfies condition (D2).

In Theorem 2.1, if $\{f_n\}$ is a sequence of measurable real-valued functions on a closed interval $[a, b]$, then, by [3, 11.16 and 11.17 Theorems, p. 311], each g_n defined in the proof of Theorem 2.1 is measurable. In addition, if $f_n \rightarrow 0$ uniformly on $[a, b]$, then g_n satisfies $g_n(x) \leq M_1$ for all $x \in [a, b]$ and hence, by [3, 11.32 Theorem, p. 321], it is Lebesgue integrable on $[a, b]$. In this case, the following result follows immediately from Theorem 2.1.

Corollary 2.2. *Let $\{f_n\}$ be a sequence of measurable real-valued functions on a closed interval $[a, b]$. Then the following statements are equivalent:*

- (i) $f_n \rightarrow 0$ uniformly on $[a, b]$.
- (ii) *There exists a sequence $\{g_n\}$ of Lebesgue integrable functions on $[a, b]$ such that $g_n \rightarrow 0$ uniformly on $[a, b]$, $|f_n(x)| \leq g_n(x)$, and $g_{n+1}(x) \leq g_n(x)$ for all $n \in N$ and all $x \in [a, b]$.*

3. UNIFORM CONVERGENCE OF A SEQUENCE OF CONTINUOUS FUNCTIONS ON A COMPACT SET

To characterize the uniform convergence of a sequence of continuous functions on a compact set, it is useful to study one property of a uniformly convergent sequence of functions on a set A . For a uniformly convergent sequence $\{f_n\}$ on A , under certain conditions, it can be extended to a uniformly convergent sequence on the closure \bar{A} of A .

Lemma 3.1. *Suppose that $f_n \rightarrow f$ uniformly on a set A in a metric space (X, d) and that for each limit point x of A there exists $\lim_{t \rightarrow x} f_n(t)$, denoted by $f_n(x)$, for each $n \in N$. Then $\{f_n\}$ is uniformly convergent on \bar{A} .*

Proof. For any given $\epsilon > 0$, by the uniform convergence of $\{f_n\}$, there exists $n_\epsilon \in N$ such that

$$|f_n(t) - f_m(t)| \leq \epsilon \quad \text{for all } m, n \geq n_\epsilon \text{ and all } t \in A.$$

For each $x \in \bar{A} \setminus A$, by taking $t \rightarrow x$ in the above inequality, we have

$$|f_n(x) - f_m(x)| \leq \epsilon \quad \text{for all } m, n \geq n_\epsilon.$$

This implies that $\{f_n(x)\}$ is Cauchy and hence converges to a point, denoted by $f(x)$. Letting $m \rightarrow +\infty$ in the above two inequalities gives

$$|f_n(t) - f(t)| \leq \epsilon \quad \text{for all } n \geq n_\epsilon \text{ and } t \in \bar{A}.$$

This shows that $f_n \rightarrow f$ uniformly on \bar{A} . \square

As Dini's Theorem [3, 7.13 Theorem] states, a pointwise convergent decreasing sequence $\{g_n\}$ of nonnegative continuous functions on a compact set A is uniformly convergent. Our main result confirms that for each uniformly convergent sequence of continuous functions on a compact set there must exist a dominating sequence $\{g_n\}$ as in Dini's Theorem.

Theorem 3.2. *Let $\{f_n\}$ be a sequence of real-valued continuous functions on a compact set \bar{A} in a metric space (X, d) . Then the following statements are equivalent:*

- (i) $f_n \rightarrow 0$ uniformly on A .
- (ii) $f_n \rightarrow 0$ uniformly on \bar{A} .
- (iii) There exists a sequence $\{g_n\}$ of continuous functions on \bar{A} such that, for each $x \in \bar{A}$, $\lim_{n \rightarrow +\infty} g_n(x) = 0$, $|f_n(x)| \leq g_n(x)$, and $g_{n+1}(x) \leq g_n(x)$ for all $n \in N$.
- (iv) There exists a sequence $\{g_n\}$ of continuous functions on \bar{A} such that $g_n \rightarrow 0$ uniformly on \bar{A} , $|f_n(x)| \leq g_n(x)$, and $g_{n+1}(x) \leq g_n(x)$ for all $n \in N$ and all $x \in \bar{A}$.
- (v) There exists a sequence $\{g_n\}$ of continuous functions on A such that $g_n \rightarrow 0$ uniformly, $|f_n(x)| \leq g_n(x)$, and $g_{n+1}(x) \leq g_n(x)$ for all $n \in N$ and all $x \in A$.
- (vi) There exist a sequence $\{g_n\}$ of continuous functions on A and a sequence $\{M_n\}$ of real numbers such that $\lim_{n \rightarrow +\infty} M_n = 0$, $|f_n(x)| \leq g_n(x) \leq M_n$, $g_{n+1}(x) \leq g_n(x)$, and $M_{n+1} \leq M_n$ for all $n \in N$ and all $x \in A$.

Proof. Since (i) \Rightarrow (ii) is immediate from Lemma 3.1 and the implications (iv) \Rightarrow (v) and (vi) \Rightarrow (i) are obvious while (v) \Rightarrow (vi) can be obtained as (ii) \Rightarrow (iii) in Theorem 2.1, we only need to show (ii) \Rightarrow (iii) \Rightarrow (iv).

(ii) \Rightarrow (iii): Let (ii) be true. Then for each $\epsilon > 0$ there exists $n_\epsilon \in N$ such that

$$|f_n(x)| \leq \epsilon \quad \text{for all } n \geq n_\epsilon \text{ and all } x \in \bar{A}.$$

For each $n \in N$, taking

$$g_n(x) := \sup\{|f_m(x)| : n \leq m \in N\} \quad \text{for } x \in \bar{A},$$

we have $|f_n(x)| \leq g_n(x) \leq \epsilon$ and $g_{n+1}(x) \leq g_n(x)$ for all $n \geq n_\epsilon$ and all $x \in \bar{A}$. Thus $\lim_{n \rightarrow +\infty} g_n(x) = 0$ for $x \in \bar{A}$.

Next we prove that each g_n is continuous on \bar{A} . Let $c \in \bar{A}$. Then $g_n(c) \geq 0$. If $g_n(c) = 0$, then $g_m(c) = 0$ and hence $|f_m(c)| = 0$ for all $m \geq n$. As we stated above, for each $\epsilon > 0$ there exists $n_\epsilon \geq n$ such that

$$|f_m(x)| \leq \epsilon \quad \text{for all } x \in \bar{A} \text{ and all } m \geq n_\epsilon.$$

For each $n \leq m < n_\epsilon$, f_m is continuous at c , so there exists $\delta_\epsilon > 0$ such that

$$|f_m(x)| \leq |f_m(c)| + \epsilon = \epsilon \quad \text{for all } x \in \bar{A} \text{ with } d(x, c) < \delta_\epsilon \text{ and all } n \leq m < n_\epsilon.$$

Hence

$$0 \leq g_m(x) - g_m(c) \leq \epsilon \quad \text{for all } x \in \bar{A} \text{ with } d(x, c) < \delta_\epsilon \text{ and all } m \geq n.$$

This shows that g_m is continuous at c for each $n \leq m \in N$.

If $g_n(c) > 0$, then for $\epsilon_c := \frac{g_n(c)}{3}$ there exists $n_c > n$ such that

$$|f_m(x)| \leq \epsilon_c \quad \text{for all } x \in \bar{A} \text{ and all } n_c \leq m \in N.$$

From this it follows that $g_m(x) \leq \epsilon_c$ for all $x \in \bar{A}$ and all $n_c \leq m \in N$. Thus

$$g_n(c) = \max\{|f_m(c)| : n \leq m < n_c\}.$$

Denote $I(n) = \{m \in N : |f_m(c)| = g_n(c), n \leq m < n_c\}$. Then there exists $\delta_c > 0$ such that

$$2\epsilon_c = |f_m(c)| - \epsilon_c \leq |f_m(x)| \leq |f_m(c)| + \epsilon_c = 4\epsilon_c$$

for all $x \in \bar{A}$ with $d(x, c) < \delta_c$ and all $m \in I(n)$. This shows that

$$g_n(x) = \max\{|f_m(x)| : n \leq m < n_c\} \text{ for all } x \in \bar{A} \text{ with } d(x, c) < \delta_c.$$

Now, for each $\epsilon \in (0, \epsilon_c)$, by the continuity of f_m , there exists $\delta \in (0, \delta_c)$ such that

$$|f_m(c)| - \epsilon \leq |f_m(x)| \leq |f_m(c)| + \epsilon$$

for all $x \in A$ with $d(x, c) < \delta$ and all $n \leq m < n_c$. Thus

$$g_n(c) - \epsilon \leq g_n(x) \leq g_n(c) + \epsilon \quad \text{for all } x \in \bar{A} \text{ with } d(x, c) < \delta.$$

This proves that g_n is continuous at c .

(iii) \Rightarrow (iv): Suppose that (iii) is true. For each $n \in N$, by the continuity of g_n on the compact set \bar{A} , there exists $x_n \in \bar{A}$ such that

$$g_n(x_n) = \max\{g_n(x) : x \in \bar{A}\}.$$

For $x \in \bar{A}$, since $\{g_n(x)\}$ is nonincreasing,

$$g_n(x_n) \geq g_n(x_{n+1}) \geq g_{n+1}(x_{n+1}) \geq 0.$$

In addition, we must have $g_n(x_n) \rightarrow 0$ as $n \rightarrow +\infty$. Otherwise, suppose that $g_n(x_n) \not\rightarrow 0$ as $n \rightarrow +\infty$. Then there exists $\epsilon_0 > 0$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\epsilon_0 < g_{n_k}(x_{n_k}) \quad \text{for all } k \in N.$$

Since \bar{A} is compact, there exists a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ and a point $\bar{x} \in \bar{A}$ such that

$$\lim_{i \rightarrow +\infty} x_{n_{k_i}} = \bar{x}.$$

Now, since each g_{n_k} is continuous on \bar{A} ,

$$g_{n_k}(\bar{x}) = \lim_{i \rightarrow +\infty} g_{n_k}(x_{n_{k_i}}) \geq \liminf_{i \rightarrow +\infty} g_{n_{k_i}}(x_{n_{k_i}}) \geq \epsilon_0.$$

This contradicts $\lim_{k \rightarrow +\infty} g_{n_k}(\bar{x}) = 0$. Therefore $g_n(x_n) \rightarrow 0$, which implies $g_n \rightarrow 0$ uniformly on \bar{A} , and hence (iv) follows. \square

The proof of the implication (iii) \Rightarrow (iv) of Theorem 3.2 is also immediate from [2, 2.66 Dini's Theorem] or [3, 7.13 Theorem]. We show it only for a self-contained proof of the theorem.

Example 3.1. Consider the functions $f_n(x) = x^n$ on $A = (0, 1)$ for $n \in N$. Since

$$\sup\{x^m : x \in (0, 1), n \leq m \in N\} = 1 \quad \text{for all } n \in N,$$

(vi) in Theorem 3.2 is not satisfied, $\{f_n\}$ is not uniformly convergent on A .

Example 3.2. Recall that, for a Lebesgue integrable function f on $[a, b]$, the function

$$F(x) := \int_a^x f dt$$

is continuous on $[a, b]$ [3, Exercise 9, p. 333]. For a sequence $\{f_n\}$ of real-valued Lebesgue integrable functions on $[a, b]$, the uniform convergence of the sequence of

functions $\int_a^x |f_n| \rightarrow 0$ on $[a, b]$ can be characterized based on Theorem 3.2 just by replacing $|f_n(x)|$ in it with $\int_a^x |f_n|$. In particular, if also $0 \leq f_{n+1}(x) \leq f_n(x)$ for all $n \in N$ and all $x \in [a, b]$, then $\int_a^x f_n \rightarrow 0$ uniformly on $[a, b]$ iff $\int_a^x f_n \rightarrow 0$ for each $x \in [a, b]$.

Example 3.3. Note that a sequence $\{f_n\}$ converges uniformly to a function f on A if and only if $(f_n - f) \rightarrow 0$ uniformly on A . If each f_n is continuous on a compact set, then f is also continuous on it. So Theorem 3.2 can equivalently be stated by replacing f_n with $f_n - f$.

From the proof of (ii) \Rightarrow (iii) in Theorem 3.2 we see that if $f_n \rightarrow 0$ uniformly on $[a, b]$ and all functions f_n are continuous at $c \in [a, b]$, then each function g_n is continuous at $c \in [a, b]$. So, if each f_n is Riemann integrable on $[a, b]$, then f_n is continuous almost everywhere in $[a, b]$ and hence g_n is Riemann integrable since it is measurable and continuous almost everywhere in $[a, b]$, based on [3, 11.17 Theorem]. Thus the following result is valid.

Theorem 3.3. *Let $\{f_n\}$ be a sequence of real-valued Riemann integrable functions on $[a, b]$. Then the following statements are equivalent:*

- (i) $\{f_n\}$ converges uniformly to a function f on $[a, b]$.
- (ii) There exist a Riemann integrable function f on $[a, b]$ and a sequence $\{g_n\}$ of real-valued Riemann integrable functions on $[a, b]$ such that $g_n \rightarrow 0$ uniformly on $[a, b]$, $|(f_n - f)(x)| \leq g_n(x)$, and $g_{n+1}(x) \leq g_n(x)$ for all $n \in N$ and all $x \in [a, b]$.
- (iii) There exist a Riemann integrable function f on $[a, b]$, a sequence $\{g_n\}$ of real-valued Riemann integrable functions on $[a, b]$, and a sequence $\{M_n\}$ of real numbers such that $\lim_{n \rightarrow +\infty} M_n = 0$, $|(f_n - f)(x)| \leq g_n(x) \leq M_n$, $g_{n+1}(x) \leq g_n(x)$, and $M_{n+1} \leq M_n$ for all $n \in N$ and all $x \in [a, b]$.

Proof. Since the implication (ii) \Rightarrow (iii) is immediate from Theorem 2.1 and (iii) \Rightarrow (i) is trivial, it suffices to show (i) \Rightarrow (ii).

(i) \Rightarrow (ii): For each $n \in N$, let

$$D_n := \{x \in [a, b] : f_n - f \text{ is not continuous at } x\}.$$

As we know, (i) implies that the function f is Riemann integrable on $[a, b]$ (see [3, 7.16 Theorem]), so $f_n - f$ is Riemann integrable on $[a, b]$ and hence D_n has measure zero. Thus the union $\cup_{n \in N} D_n$ has measure zero.

For each $\epsilon > 0$, by the uniform convergence $f_n \rightarrow f$, there exists $n_\epsilon \in N$ such that

$$|(f_n - f)(x)| \leq \epsilon \quad \text{for all } n \geq n_\epsilon \text{ and all } x \in [a, b].$$

For each $n \in N$, taking

$$g_n(x) := \sup\{|(f_m - f)(x)| : n \leq m \in N\} \text{ for } x \in [a, b],$$

we have

$$|(f_n - f)(x)| \leq g_n(x) \text{ and } g_{n+1}(x) \leq g_n(x) \leq \epsilon \text{ for all } n \geq n_\epsilon \text{ and all } x \in [a, b].$$

From this we see that $g_n \rightarrow 0$ uniformly. By definition, the above inequalities (without ϵ) are also valid for all $n < n_\epsilon$ and $x \in [a, b]$. In addition, each g_n is continuous in $[a, b] \setminus (\cup_{n \in N} D_n)$ and hence is Riemann integrable on $[a, b]$. So we obtain (ii). \square

For a sequence $\{f_n\}$ of real-valued Riemann integrable functions on $[a, b]$, if $f_n \rightarrow 0$ uniformly on $[a, b]$, then $\int_a^x |f_n| \rightarrow 0$ uniformly on $[a, b]$. Therefore the following statements are valid.

Corollary 3.4. *Let $\{f_n\}$ be a sequence of real-valued Riemann integrable functions on $[a, b]$. If $f_n \rightarrow 0$ uniformly on $[a, b]$, then there hold*

- (i) $\int_a^x |f_n| \rightarrow 0$ uniformly on $[a, b]$;
- (ii) *there exists a sequence $\{g_n\}$ of Riemann integrable functions on $[a, b]$ such that, for each $x \in [a, b]$, $\lim_{n \rightarrow +\infty} \int_a^x g_n = 0$,*

$$\int_a^x |f_n| \leq \int_a^x g_n, \text{ and } g_{n+1}(x) \leq g_n(x) \quad \text{for all } n \in N;$$

- (iii) *there exist a sequence $\{g_n\}$ of Riemann integrable functions on $[a, b]$ and a sequence $\{M_n\}$ of real numbers such that $\lim_{n \rightarrow +\infty} M_n = 0$,*

$$\int_a^x |f_n| \leq \int_a^x g_n \leq M_n(x - a), \quad g_{n+1}(x) \leq g_n(x), \text{ and } M_{n+1} \leq M_n$$

for all $n \in N$ and all $x \in [a, b]$.

Remark. *For a measurable function f on $[a, b]$, by [3, 11.16 Theorem], $|f|$ is also measurable on $[a, b]$. For a sequence $\{f_n\}$ of measurable real-valued functions on $[a, b]$, if it is uniformly convergent to 0 on $[a, b]$, then $\int_a^x |f_n| \rightarrow 0$ uniformly on $[a, b]$. In such a case, by Corollary 2.2, we can obtain a result similar to Corollary 3.4 by replacing Riemann integrable functions g_n in it with Lebesgue integrable functions.*

4. SOME APPLICATIONS

As the limit function of a sequence of partial sums of functions, the uniform convergence of a series of functions can similarly be discussed. For a series of continuous functions on a compact set, we can apply Theorem 3.2 to derive the following result:

Theorem 4.1. *Let $\{f_n\}$ be a sequence of real-valued continuous functions on a compact set A in a metric space (X, d) . Then the following statements are equivalent:*

- (i) $\sum_{k=1}^{\infty} |f_k|$ is uniformly convergent to a function f on A .
- (ii) *There exists a continuous function f on A such that, for each $x \in A$,*

$$\sum_{k=1}^{\infty} |f_k(x)| = f(x).$$

If also there holds $|f_{n+1}(x)| \leq |f_n(x)|$ for all $n \in N$ and all $x \in A$, then (i) is equivalent to

- (iii) *For any $1 < q \in N$ and each $x \in A$ the series $\sum_{k=1}^{\infty} q^k |f_{q^k}(x)|$ converges.*

Furthermore, if there exists some $1 < q \in N$ such that

$$\lim_{n \rightarrow +\infty} \frac{|f_{q^{n+1}}(x)|}{|f_{q^n}(x)|} < \frac{1}{q},$$

then $\sum_{k=1}^{\infty} |f_k|$ is uniformly convergent on A .

Proof. Statement (i) is valid if and only if there exists a continuous function f on A such that $g_n := f - \sum_{k=1}^n |f_k| \rightarrow 0$ uniformly on A . It is easy to see that

$$0 \leq g_{n+1}(x) \leq g_n(x) \quad \text{for all } n \in N \text{ and all } x \in A.$$

Thus, by Theorem 3.2, $g_n \rightarrow 0$ uniformly on A if and only if, for each $x \in A$, $\lim_{n \rightarrow +\infty} g_n(x) = 0$, that is, (ii) is true.

Next, if also there holds $|f_{n+1}(x)| \leq |f_n(x)|$ for all $n \in N$ and all $x \in A$, then, by [4, Theorem 2], (ii) \Leftrightarrow (iii) and hence (i) \Leftrightarrow (iii).

Finally, if for some $1 < q \in N$ there holds

$$\lim_{n \rightarrow +\infty} \frac{|f_{q^{n+1}}(x)|}{|f_{q^n}(x)|} < \frac{1}{q},$$

then, by [4, Theorem 3], $\sum_{k=1}^{\infty} |f_k(x)|$ is convergent at each $x \in A$. So $\sum_{k=1}^{\infty} |f_k|$ is uniformly convergent to a function f on A . \square

Example 4.1. Consider the series

$$\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} a_n x^n \quad \text{for } x \in [-1, 1],$$

where $a_n \geq a_{n+1} > 0$ for all $n \in N$. If there exists some $1 < q \in N$ such that

$$\lim_{n \rightarrow +\infty} \frac{a_{q^{n+1}}}{a_{q^n}} < \frac{1}{q},$$

then

$$\lim_{n \rightarrow +\infty} \frac{|a_{q^{n+1}} x^{q^{n+1}}|}{|a_{q^n} x^{q^n}|} = \lim_{n \rightarrow +\infty} \frac{a_{q^{n+1}} |x|^q}{a_{q^n}} < \frac{|x|^q}{q} \leq \frac{1}{q} \quad \text{for } x \in [-1, 1],$$

it follows from Theorem 4.1 that $\sum_{n=1}^{\infty} a_n |x|^n$ is convergent for each $x \in [-1, 1]$. By Theorem 4.1, $\sum_{n=1}^{\infty} a_n |x|^n$ is uniformly convergent on $[-1, 1]$. Therefore $\sum_{n=1}^{\infty} a_n x^n$ is uniformly convergent on $[-1, 1]$.

Note that when $a_n = \frac{1}{n^p}$ for $p > 1$, the convergence of $\sum_{n=1}^{\infty} a_n x^n$ in Example 4.1 cannot be determined by the existing ratio test for a series.

Finally, as an application of Theorem 3.2, a fixed point theorem can be obtained as below.

Theorem 4.2. *Let A be a subset in a metric space (X, d) and let $f : A \rightarrow X$ be a continuous mapping. For $x \in A$, denote*

$$f^1(x) := f(x), \quad f^{n+1}(x) := f(f^n(x)), \quad \text{and} \quad f_n(x) := d(f^n(x), f^{n+1}(x))$$

provided that $f^n(x) \in A$ for $n \in N$. Then the following statements are equivalent:

- (i) *There exists $a \in A$ such that $f(a) = a$.*
- (ii) *There exists a nonempty compact set $C \subseteq A$ such that $f(C) \subseteq C$ and for each $x \in C$ there exists n_x such that $(n+1)f_{n+1}(x) \leq n f_n(x)$ for all $n \geq n_x$.*
- (iii) *There exists a nonempty compact set $C \subseteq A$ such that $f(C) \subseteq C$ and for each $x \in C$ there exists a strictly increasing nonnegative function g on N such that $\lim_{n \rightarrow +\infty} g(n) = \infty$ and for some $n_x \in N$ there holds*

$$g(n+1)f_{n+1}(x) \leq g(n)f_n(x) \quad \text{for all } n \geq n_x.$$

- (iv) *There exists a nonempty compact set $C \subseteq A$ such that $f(C) \subseteq C$ and $f_n \rightarrow 0$ uniformly on C .*

- (v) There exist a nonempty compact set $C \subseteq A$ and a nonincreasing sequence $\{g_n\}$ of continuous functions on C such that $f(C) \subseteq C$, $f_n(x) \leq g_n(x)$ for all $n \in N$, and $\lim_{n \rightarrow +\infty} g_n(x) = 0$ for each $x \in C$.
- (vi) There exist a compact set $C \subseteq A$ such that $f(C) \subseteq C$ and $\lim_{n \rightarrow +\infty} f_n(x) = 0$ for some point x in C .
- (vii) There exists a compact set $C \subseteq A$ such that $f(C) \subseteq C$ and for each $x \in C$ with $d(x, f(x)) > 0$ there exists $\bar{x} \in C$ satisfying $d(\bar{x}, f(\bar{x})) < d(x, f(x))$.

Proof. The implication (i) \Rightarrow (ii) is immediate by taking $C := \{a\}$, (ii) \Rightarrow (iii) and (v) \Rightarrow (vi) are obvious, and (iv) \Rightarrow (v) is direct from Theorem 3.2, so it suffices to show that (iii) \Rightarrow (iv), (v) \Rightarrow (vii) \Rightarrow (i), and (vi) \Rightarrow (i).

To show (iii) \Rightarrow (iv), we suppose that (iii) is true. Then, for the compact set C in (iii) and each $x \in C$, the function g in (iii) satisfies

$$f_n(x) \leq \frac{g(n_x)f_{n_x}(x)}{g(n)} \quad \text{and} \quad f_{n+1}(x) \leq f_n(x) \quad \text{for all } n \geq n_x.$$

It follows that $\lim_{n \rightarrow +\infty} f_n(x) = 0$ for $x \in C$. By Theorem 3.2, (iv) is valid.

Next, to prove the implications (v) \Rightarrow (vii) \Rightarrow (i), we suppose that (v) is true. Then, for each $x \in C$ with $d(x, f(x)) > 0$, there exists $n_x \in N$ such that

$$f_n(x) \leq g_n(x) < d(x, f(x)) \quad \text{for all } n \geq n_x.$$

Taking $n = n_x$ and $\bar{x} = f^{n_x}(x)$, we have $d(\bar{x}, f(\bar{x})) = f_{n_x}(x) < d(x, f(x))$, that is, (vii) is valid.

Now, let (vii) be true. Since the function $d(x, f(x))$ is continuous on the compact set C , it attains its minimum at a point a in C . By (vii), we must have $d(a, f(a)) = 0$, that is, (i) is true. Therefore (vii) \Rightarrow (i).

Finally, for the compact set C , $x \in C$, and the function f in (vi), we have $f^n(x) \in C$ for all $n \in N$, so there exists a subsequence $\{f^{n_k}(x)\}$ of $\{f^n(x)\}$ converging to some $a \in C$. From this we obtain

$$d(a, f(a)) = \lim_{k \rightarrow +\infty} d(f^{n_k}(x), f(f^{n_k}(x))) = \lim_{k \rightarrow +\infty} f_{n_k}(x) = 0.$$

Thus (vi) \Rightarrow (i) follows. \square

Remark. Comparing with the sufficient condition $(d(f(x), f(y)) < d(x, y))$ for $x, y \in C$ and $x \neq y$ in [1, Theorem 1.2] for the existence of a fixed point of f , the condition in (vii) of Theorem 4.2 is slightly weaker. However, the latter is not sufficient for the uniqueness of fixed points of f .

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