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NUMERICAL SOLUTION OF NONLINEAR FREDHOLM–VOLTERRA INTEGRAL EQUATION VIA AIRFOIL POLYNOMIALS WITH ERROR ANALYSIS

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Abstract. This paper is devoted to a new numerical technique for solving nonlinear Fredholm–Volterra integral equation (NFVIE). This method is based on airfoil polynomials and Clenshaw–Curtis quadrature method. Also the error analysis for the proposed method is considered. Finally, some illustrative examples and the comparison with other methods are considered to prove the validity and the capability of this novel technique.

1. Introduction

In this paper, a novel numerical technique for solving NFVIE as follows

$$\begin{aligned} u(x) \quad &= \quad f(x) + \lambda_1 \int_0^x k_1(x,t) F\big(t, u(t)\big)\, dt + \\ &\quad \lambda_2 \int_0^1 k_2(x,t) G\big(t, u(t)\big)\, dt, \quad 0 \leqslant x, t \leqslant 1, \end{aligned} \tag{1.1}$$

is proposed. The parameters λ_1, λ_2 and

$$f(x),\ k_1(x,t),\ k_2(x,t),\ F\big(t,u(t)\big),\ G\big(t,u(t)\big)$$

are known and $u(x)$ is an unknown function.

Over the last years, the integral equations have been used increasingly in different areas of applied science. Integral equations has many applications in fields including problems of mathematical physics, dynamic models in chemical reactors, control theory, and even elliptic partial differential equation with nonlinear boundary conditions [1]–[13]. It should be noticed that nonlinear mixed Volterra–Fredholm integral equations were widely investigated numerically in [14]–[21], and in [25]–[30]. They have a fundamental pattern in some sections of pure analysis and stochastic processes [24] also and in the mathematical modeling of an epidemic, physical and biological aspects [1] as well.

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Nowadays a wireless networks are widely develop in application to the new area of an industrial internet of things. Various fields such as smart home, manufacturing, agriculture, surgery etc. employee wireless networks capabilities. At the same time, the control theory plays key role of that kind of technology in which a nonlinear integral equations like NFVIE arise.

In present paper, we attempt to apply projection approximations to solve NFVIE using airfoil polynomials of the first type. The purpose of the present method is to approximate the solution of Eq. (1.1) with Clenshaw–Curtis quadrature by means of airfoil polynomials. The reason to utilize of that kind of quadrature is that it is based on Chebyshev–type grid nodes related to the fast Fourier transform which require $O(n \log n)$ operations (in case of optimized algorithm) only. Another feature of the proposed scheme is a numerical solution of a nonlinear system of algebraic equations since the nonlinear nature of the considered integral equation is observed. This circumstance strongly affects on the convergence of numerical procedure to the solution in general case (without any optimization) and may be considered as a bottleneck place of the proposed technique. Never the less it can be managed under some conditions.

The paper is organized as follows: some properties of airfoil polynomials are given in Section 2. In Section 3, airfoil polynomials are applied to approximate unknown solution and Clenshaw–Curtis quadrature along with collocation method to convert the NFVIE to a system of nonlinear equations is applied. The exact and obtained numerical solutions from the proposed method are compared. The error analysis for the method is discussed in Section 4. An issue of complexity is considered as a subsection 4.1. In Section 5, numerical examples clarify the method. Finally, in Section 6 the conclusion of present work is given.

2. Airfoil polynomials

Let's introduce briefly the airfoil polynomial $t_n(x)$ of the first type is defined by the expressions below

$$t_n(x) = \frac{\cos\left[(n+\frac{1}{2})\arccos(x)\right]}{\cos\left(\frac{1}{2}\arccos(x)\right)}, \tag{2.1}$$

and the airfoil polynomial $u_n(x)$ of the second type is defined in the following form

$$u_n(x) = \frac{\sin\left[(n+\frac{1}{2})\arccos(x)\right]}{\sin\left(\frac{1}{2}\arccos(x)\right)}. \tag{2.2}$$

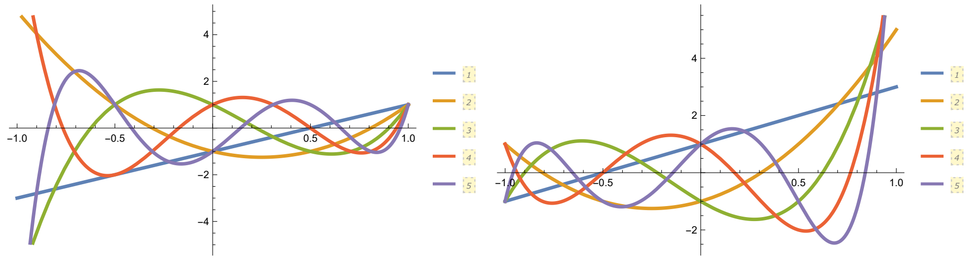
Functional space $C^0\big([-1,1]\big)$ presents the space of continuous functions defined on the interval $[-1,1]$ with the norm $\|\cdot\|_\infty$. Let's assume that the universe of our discourse be the space $X = C\big([-1,1],\mathcal{C}\big)$, and set

$$\Omega = \{x \in X : x' \in X,\, x(-1) = 0\}.$$

According to [22] the relation between the first kind and second kind airfoil polynomials and their derivatives is as follows

$$(1+x)t_i'(x) = \left(i + \frac{1}{2}\right)u_i(x) - \frac{1}{2}t_i(x), \quad i \in \mathbb{N}. \tag{2.3}$$

Plots of several polynomials of the first order (left column) and second order (right column) as well are presented in Fig. 1.



$t_n(x),\ n = \overline{1,5}$ $\quad$ $u_n(x),\ n = \overline{1,5}$

Figure 1. First five airfoil polynomials of the first and second order

2.1. **Function approximation.** By using the airfoil polynomials of the first type $t_n(x)$ to obtain an approximate solution of the problem (1.1) is in the form of

$$\varphi_n(x) \simeq \sum_{i=0}^{N} a_i t_i(x) = A^T T, \tag{2.4}$$

where $a_i = \langle f(x), t_i(x)\rangle$, in which $\langle \cdot, \cdot \rangle$ shows the inner product in $L^2\big([0,1]\big)$. For simplicity one can write the corresponding relation

$$a_i \langle \varphi(x), \varphi(x)\rangle = \langle f(x), \varphi(x)\rangle,$$

where vector

$$\langle f(x), \varphi(x)\rangle = \int_0^1 f(x)\varphi(x)\,dx,$$

has $(n+1)\times 1$ dimension and $\langle \varphi(x), \varphi(x)\rangle$, is an $(n+1)\times(n+1)$ matrix, which is called dual matrix of $\varphi(x)$. Also A, T are vectors denoted as follows

$$A = [a_0, a_1, \cdots, a_N],$$
$$T = [T_0(t), T_1(t), \cdots, T_N(t)]^T,$$

where $T_n(t)$ are airfoil polynomials of the first type of degree n are orthogonal with the weight function of the form

$$w(x) = \sqrt{\frac{1+x}{1-x}}.$$

Some more properties of the airfoil polynomials one may find in [22].

3. Applying the method for NFVIE

Consider an application of approximation (2.4) of the nonlinear Eq. (1.1). Now unknown function $u(x)$ is approximated as follows

$$u(x) = A^T T(x),$$

and it is able to substitute the above relation into Eq. (1.1) to achieve

$$\begin{aligned} A^T T(t) &= f(x) + \lambda_1 \int_0^x k_1(x,t) F\big(t, A^T T(t)\big)\, dt + \\ &\quad \lambda_2 \int_0^1 k_2(x,t) G\big(t, A^T T(t)\big)\, dt. \end{aligned} \tag{3.1}$$

Afterwards, one is able to change the intervals $[0, s_i]$ and $[0, 1]$ into $[-1, 1]$ by transformations

$$\tau_1 = \frac{2}{s_i} t - 1, \quad \tau_2 = 2t - 1,$$

and consider the set of $N+1$ collocation so–called Chebyshev points s_j, which are zeros of $t_{n+1}(x)$ polynomials and exact expression of them is as follows

$$s_j = \frac{2j-1}{2N+2}, \quad j = \overline{0, N}. \tag{3.2}$$

Let's define an auxiliary functions as it is shown below

$$H_1(x,t) = k_1(x,t) F\big(t, A^T T(t)\big), \; H_2(x,t) = k_2(x,t) G\big(t, A^T T(t)\big).$$

By using collocation points (3.2) in Eq. (3.1) one may get

$$\begin{aligned} A^T T(s_j) &= f(s_j) + \lambda_1 \frac{s_j}{2} \int_{-1}^1 H_1\left(s_j, \frac{s_j(\tau_1+1)}{2}\right) dt + \\ &\quad \frac{\lambda_2}{2} \int_{-1}^1 H_2\left(s_j, \frac{\tau_2+1}{2}\right) dt, \quad j = \overline{1, N}. \end{aligned} \tag{3.3}$$

Thus it is possible to apply Clenshaw–Curtis quadrature [23] to approximate the integrand in (3.3) as follows

$$\begin{aligned} A^T T(s_j) &= f(s_j) + {\sum_{k=0}^{N}}' w_k \left[\lambda_1 \frac{s_j}{2} H_1\left(s_j, \frac{s_j(s_j+1)}{2}\right) + \right. \\ &\quad \left. \frac{\lambda_2}{2} H_2\left(s_j, \frac{s_j+1}{2}\right)\right], \quad j = \overline{0, N}. \end{aligned} \tag{3.4}$$

Due to the computational point of view it is convenient to utilize an explicit formula for the weights in the Clenshaw–Curtis quadrature rule. For even values of N, such a formula is given

$$w_k = \frac{4}{N} {\sum_{n=0}^{N}}' \frac{1}{1-n^2} \cos\left(\frac{nk\pi}{N}\right), \tag{3.5}$$

where primed sum indicates that first and last terms are to be weighted by factor of 1/2.

Finally, one is able to transform the problem into a $(N+1) \times (N+1)$ nonlinear algebraic equations with the unknown coefficients a_n, $n = \overline{0, N}$, since Eqs. (3.4) contains $N+1$ nonlinear equations. As a result, approximate solution will be obtained by solving this nonlinear system. By using a standard iterative method such as Newton's method or any suitable method one may reach an acceptable solution.

4. Error Analysis

Theorem 4.1. *Let Clenshaw–Curtis and Gauss quadrature be applied to a function* $f(x) \in C\big([-1,1]\big)$*. If* $f(x)$*,* $f'(x)$*,* $f''(x), \cdots, f^{(k-1)}(x)$ *are absolutely continuous on* $[-1,1]$ *and* $\big\|f^{(k)}(x)\big\|_T = V < \infty$ *for some* $k \geqslant 1$*, then for all sufficiently large* $n \in \mathbb{N}$ *the following inequality is valid*

$$|I - I_n| \leq \frac{32V}{15\pi k(2n+1-k)^k}.$$

Sufficiently large n means for the Clenshaw–Curtis scheme that $n \geqslant n_k$ for some n_k that depends on k but not $f(x)$ or value V and for the Gauss quadrature $n \geqslant \dfrac{k}{2}$. Here integral

$$I \equiv I(f) = \int_{-1}^{1} f(x)\,dx$$

is approximated by sums in the following form

$$I_n = I_n(f) = \sum_{k\in\mathbb{N}} w_k f(x_k)$$

for various natural integers n, where the nodes x_k depend on n but not on the function $f(x)$ itself. The Gauss quadrature is positive and also converges for every continuous function $f(x) \in \big([-1,1]\big)$. The nodes and weights of this formula can be evaluated in $O(n^2)$ operations by solving an ordinary eigenvalue problem, as was proved in [24].

Theorem 4.2. *The difference measured value of approximate and exact solution is defined as Absolute Error (AE), which is given by:*

$$AE(x) = |u(x) - u_N(x)|, \tag{4.1}$$

where $u(x)$ *and* $u_N(x)$ *are the exact and the approximate solutions at the point* x*, respectively.*

Proposition 4.3. *If* $\big(C[J], \|\cdot\|\big)$ *is a Banach space of all continuous functions on* $J=[0,1]$*, with norm* $\|u(x)\| = \max\limits_{0\leqslant x\leqslant 1}|u(x)|$ *and the conditions below on* k_1*,* k_2 *and* F*,* G *for Eq.*(1.1) *are satisfied and we define* $k_s = k(s,t)$ *for* $x, t \in [0,1]$*, then the solution of Eq.* (1.1) *converges.*

1. $\lim\limits_{s\to\tau} \|k_s - k_\tau\| = 0, \quad \tau \in [0,1]$.
2. $M_1 = \sup|k_1(x,t)| < \infty,\ M_2 = \sup|k_2(x,t)| < \infty$.
3. $F(x,t)$*,* $G(x,t)$ *are continuous in* $x \in [0,1]$ *and Lipschitz conditions in* $t \in [-\infty, \infty]$ *are satisfied, there exist two constants* C_1*,* C_2*, for which the following inequalities are valid*

$$|F(x,t_m) - F(x,t_n)| \leq C_1|t_m - t_n|, \quad \forall t_m, t_n \in (-\infty,\infty),$$

$$|G(x,t_m) - G(x,t_n)| \leq C_2|t_m - t_n|, \quad \forall t_m, t_n \in (-\infty,\infty).$$

For more details one can refer to [25, 26].

Theorem 4.4. *The error term $\|e(x)\|$ obtained by airfoil polynomials application for solving of FVIE converges to zero with $n \to \infty$ for $0 < \alpha < 1$, where*

$$k_1 = \sup_{0<x<1} \int_0^x |k_1(x,t)|\,dt, \quad k_2 = \sup_{0<x<1} \int_0^1 |k_2(x,t)|\,dt$$

and suppose the nonlinear terms $F\big(t, u(t)\big) = F(t)$ and $G\big(t, u(t)\big) = G(t)$ and L_1, L_2 are Lipschitz constants.

Proof. Let $u_N(x)$ be the approximate solution of Eq.(1.1) and $\|e_j(x)\|$ be the error term of each solutions. Define

$$e(x) = \sum_{j\in\mathbb{N}} e_j(x). \tag{4.2}$$

Now

$$\begin{aligned} \|e_j(x)\| &= \|u(x) - u_N(x)\| \leqslant |\lambda_1| \int_0^x \|k_1(x,t)\| \big\|F\big(u(t)\big) - F\big(u_N(t)\big)\big\|\,dt + \\ &\quad |\lambda_2| \int_0^1 \|k_2(x,t)\| \big\|G\big(u(t)\big) - G\big(u_N(t)\big)\big\|\,dt. \end{aligned} \tag{4.3}$$

From Eq. (4.3), one may get

$$\begin{aligned} \|e_j(x)\| &\leqslant |\lambda_1| k_1 \big\|F\big(u(x)\big) - F\big(u_N(x)\big)\big\| + |\lambda_2| k_2 \big\|G\big(u(x)\big) - G\big(u_N(x)\big)\big\| \\ &= |\lambda_1| k_1 L_1 \|u(x) - u_N(x)\| + |\lambda_2| k_2 L_2 \|u(x) - u_N(x)\| \end{aligned}$$

then we can say

$$\|e_j(x)\| \leq \|u(x) - u_N(x)\| \big(|\lambda_1| k_1 L_1 + |\lambda_2| k_2 L_2\big)$$

where L_1, $L_2 > 0$ are Lipschitz constants. Introduce parameter α as follows

$$\alpha = |\lambda_1| k_1 L_1 + |\lambda_2| k_2 L_2, \text{ and}$$

$$(1-\alpha)\|u(x) - u_N(x)\| \leqslant 0,$$

since $\|e_j(x)\| = \|u(x) - u_N(x)\|$. Now let's choose value of α such that $0 < \alpha < 1$ and consequently $\|u(x) - u_N(x)\| \to 0$ which yields $\|e_j(x)\| \to 0$ as $n \to \infty$. Hence, form (4.2) $\|e(x)\| \to 0$ as $n \to \infty$ at the end. □

4.1. **Discussion on the complexity issue of the proposed technique.** A question of the complexity of the proposed technique seems to be an important and interesting one to discuss. First of all, due to the nonlinear nature of the considered problem it is quite complicated to provide a straightforward estimations of dependence on the number of grid nodes. Nevertheless, one may achieve the following remarks on that issue below.

Remark. *For the linear case, in case of Fredholm equation an issue of complexity was discussed and some estimations were introduced in* [31]. *Also, an optimization problem for linear integral equations of the second kind was considered and an optimal fast solvers were introduced by means of quadrature method* [32]. *The problem of complexity in compact operator form was considered in Hilbert space also* [33].

Remark. *For the nonlinear case* (1.1) *the key role plays an iterative algorithm of nonlinear system numerical solution. Since that one may find some details on complexity issue on Newton's method as far as a modern version of it in* [34].

5. Numerical examples

In this part we consider 3 illustrative examples of abilities of NFVIE approach to illustrate the usage of presented method. All computations were carried out by using Wolfram Mathematica 11 software.

Example 1. Let's consider the nonlinear Volterra integral equation (NVIE) given in [27]

$$u(x) = \exp(x) - \frac{1}{3}\exp(3x) + \frac{1}{3} + \int_0^x \big(u(t)\big)^3\, dt. \tag{5.1}$$

which has the exact solution $u(x) = \exp(x)$. The airfoil polynomials approach was applied to solve Eq.5.1. Table 1 presents computed values of $u(x)$ in several grid points using present method and Legendre wavelets numerical approach [20] together with the exact values.

Table 1. *Approximate solution vs. Legendre wavelets method* [20] *and exact solution*

x_i	Present method	Legendre wavelets [20]	Exact solution
0.0	1	1	1
0.2	1.2214027538	1.2214027582	1.2214027582
0.4	1.4918246782	1.4918246976	1.4918246976
0.6	1.8221188001	1.8221188004	1.8221188004
0.8	2.2255409481	2.2255409485	2.2255409485
1.0	2.7182818269	2.7182818258	2.7182818258

Example 2. Consider the following NVIE given in [28]

$$u(x) = -x^2 - \frac{x}{3}\big(2\sqrt{2} - 1\big) + 2 + \int_0^1 xt\sqrt{u(t)}\, dt.$$

The exact solution of this problem is $u(x) = 2 - x^2$. The comparison of the approximate solution with the exact solution is shown in Table 2.

Table 2. *Comparison of exact and approximate solutions for Example 2.*

x_i	Present method	Exact solution	Absolute error $\lvert x - x_N\rvert$
0.0	2	2	0.00000000
0.1	1.99942354	1.99932275	0.00010079
0.2	1.96723443	1.96783109	0.00005666
0.3	1.91017823	1.91012378	0.00005445
0.4	1.84175342	1.84171840	0.00003502
0.5	1.75354327	1.75378438	0.00024111
0.6	1.64789512	1.64792181	0.00002669
0.7	1.51218943	1.51278348	0.00059405
0.8	1.36085916	1.36074881	0.00011035
0.9	1.19048534	1.19023484	0.00025050
1.0	1	1	0.00000000

Example 3. Consider the NFVIE given in [27] by

$$\begin{aligned} u(x) \quad &= \quad -\frac{1}{30}x^6 + \frac{1}{3}x^4 - x^2 + \frac{5}{3}x - \frac{5}{4}\int_0^x (x-t)\big(u(t)\big)^2\,dt + \\ &\int_0^1 (x+t)u(t)\,dt, \quad 0 \leqslant x,t \leqslant 1. \end{aligned} \tag{5.2}$$

with the exact solution $u(x) = x^2 - 2$. The comparison among the airfoil polynomials solutions beside the exact solutions are shown in Table 3.

Table 3. *Approximate and exact solutions for Example 3.*

x_i	Solution	Method in [29]	Method in [30]	exact solution	Absolute error $\lvert x - x_N\rvert$
0.0	-1.99021	-2.00034	-1.99983	-2.00	0.00979
0.1	-1.99048	-1.99042	-1.99019	-1.98	0.01048
0.2	-1.96034	-1.96051	-1.96020	-1.96	0.00034
0.3	-1.91067	-1.91059	-1.91023	-1.91	0.00067
0.4	-1.84036	-1.84068	-1.84025	-1.84	0.00036
0.5	-1.75021	-1.75066	-1.75027	-1.75	0.00021
0.6	-1.64032	-1.64084	-1.64029	-1.64	0.00032
0.7	-1.49087	-1.51083	-1.51031	-1.51	0.01913
0.8	-1.35054	-1.36074	-1.36034	-1.36	0.00946
0.9	-1.18032	-1.19057	-1.19036	-1.19	0.00968
1.0	-1.00032	-1.00033	-1.00034	-1.00	0.00032

6. Conclusion

In this paper, an efficient and accurate method for solving NFVIE is designed. The properties of the airfoil polynomials, Clenshaw–Curtis quadrature and uniformly distributed collocation nodes help to convert the NFVIE to a nonlinear system of alegbraic equations. The numerical examples and compared results support our claim in the paper. This numerical technique can readily be extended to multi–dimensional dimensional problems with some modifications. Three illustrative examples demonstrate the capabilities of the proposed numerical technique. An issue of complexity of the proposed approach was discussed as well.

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