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SOLUTION OF RATIONAL DIFFERENCE EQUATION

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Abstract. In this work we investigated the solution for the following difference equation

$$x_{n+1} = \frac{x_{n-17}}{1 + \prod_{t=0}^{4} x_{n-3t-2}}$$

where $x_{-17}, x_{-16}, ..., x_{-1}, x_0 \in (0, \infty)$. Moreover, we gave a numerical example of to the solution the related difference equation.

1. INTRODUCTION

In the recent times, nonlinear difference equations have a critical role in the fields of pyhsics, economy, ecology and computational science engineering, etc. Many researchers have investigated the behavior of the solution of nonlinear difference equations. So, recently there has been an increasing interest in the study of qualitative analysis of rational difference equations. Although difference equations are very simple in form, it is extremely difficult to understand thoroughly the behaviors of their solutions. (see [1]-[44]).

Cinar, studied the following problem with positive initial values

$$x_{n+1} = \frac{x_{n-1}}{-1 + a x_n . x_{n-1}}$$

for $n = 0, 1, 2, ...$ in [3] respectively.

Simsek et. al., studied the following problems with positive initial values

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}$$

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}}$$

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-1} x_{n-3}}$$

$$x_{n+1} = \frac{x_{n-11}}{1 + x_{n-1} x_{n-3} x_{n-5} x_{n-7} x_{n-9}}$$

for $n = 0, 1, 2, ...$ in [7, 8, 9, 10] respectively.

Elsayed et. al [22], studied the periodic nature and the form of the solutions of nonlinear difference equations systems of order three

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$$x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n\left(\pm 1 \pm x_{n-2}y_{n-1}\right)}$$

with initial conditions are nonzero real numbers.

Papaschinopoulos et. al [30], studied the asymptotic behavior of the positive solutions of the systems of the two difference equations

$$x_{n+1} = a + by_{n-1}e^{-x_n}, \quad y_{n+1} = c + dx_{n-1}e^{-y_n},$$

$$x_{n+1} = a + by_{n-1}e^{-y_n}, \quad y_{n+1} = c + dx_{n-1}e^{-x_n},$$

where the constants a, b, c, d are positive real numbers, and the initial values x_0, x_{-1}, y_0, y_{-1} are also positive real numbers.

Matsunaga et. al [34], deals with a system of rational difference equations

$$x_{n+1} = \frac{ay_n + b}{cy_n + d}, \quad y_{n+1} = \frac{ax_n + b}{cx_n + d}, \quad n = 0, 1, 2, ...$$

where a, b, c, d are real numbers with $c \neq 0$ and $ad - bc \neq 0$. Them establish a representation formula of solutions of the system and classify global behavior of solutions when no initial values belong to the forbidden set of the system.

Almatrafi et. al [36], demonstrates the existence of boundedness, asymptotic behavior and the periodicity of the following quadratic second-order rational difference equation:

$$x_{n+1} = ax_n + \frac{bx_n^2 + c_n x_{n-1} + dx_{n-1}^2}{\alpha x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}^2}$$

where the constants $a, b, c, d, \alpha, \beta, \gamma$ are positive real numbers, and the initial values x_0 and x_{-1} are also positive real numbers.

ElDessoky et. al [39], studied the global stability character and the periodic nature of the solutions of the difference equation

$$x_{n+1} = ax_{n-l} + \frac{b + c_{n-k}}{dx_{n-s} + ex_{n-t}}, n = 0, 1, ...,$$

where the initial conditions $x_{-r}, x_{-r+1}, x_{-r+2}, ..., x_0$ are arbitrary positive real numbers, $r = max\left\{l, k, s, t\right\}$ is nonnegative integer and a, b, c, d, e are positive constants.

In this work, the following non-linear difference equation was studied

$$x_{n+1} = \frac{x_{n-17}}{1 + \prod_{t=0}^{4} x_{n-3t-2}}, \quad n = 0, 1, 2, ... \tag{1.1}$$

where $x_{-17}, x_{-16}, ..., x_{-1}, x_0 \in (0, \infty)$.

2. MAIN RESULTS

Theorem 2.1. Consider the difference equation 1.1. Then the following statements are true.

a) The sequences $(x_{18n-17}, x_{18n-16}, x_{18n-15}, ..., x_{18n-1}, x_{18n})$ are decreased and $a_1, a_2 ..., a_{17}, a_{18} \geq 0$ is existed in such that:

$$\lim_{n\to\infty} x_{18n-17+t} = a_{1+t}, \quad t = 0, 1, ..., 17.$$

b) $(a_1, a_2, ..., a_{17}, a_{18}, ...)$ is a solution of 1.1 having period 18.

c)

$$\prod_{t=0}^{4} \lim_{n\to\infty} x_{18n-3t-2} = 0, \prod_{t=0}^{4} \lim_{n\to\infty} x_{18n-3t-1} = 0, \prod_{t=0}^{4} \lim_{n\to\infty} x_{18n-3t} = 0,$$

or

$$\prod_{t=0}^{4} a_{3t+2} = 0, \prod_{t=0}^{4} a_{3t+1} = 0, \prod_{t=0}^{4} a_{3t} = 0.$$

d) If there exist $n_0 \in \mathbb{N}$ such that $\prod_{t=0}^{4} x_{n-3t-2} \geq x_{n+1} \prod_{t=0}^{4} x_{n-3t-2}$, for all $n \geq n_0$, then

$$\lim_{n\to\infty} x_n = 0.$$

e) The following formulas can be generated:

$$x_{18n+k+1} = x_{-17+k}\left(1 - \frac{x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}}{1 + x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}} \times \sum_{j=0}^{n} \prod_{i=1}^{6j} \frac{1}{1 + x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right), \quad k = \overline{0,2},$$

$$x_{18n+k+4} = x_{-14+k}\left(1 - \frac{x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-17+k}}{1 + x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}} \times \sum_{j=0}^{n} \prod_{i=1}^{6j+1} \frac{1}{1 + x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right), \quad k = \overline{0,2},$$

$$x_{18n+k+7} = x_{-11+k}\left(1 - \frac{x_{-2+k}x_{-5+k}x_{-8+k}x_{-14+k}x_{-17+k}}{1 + x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}} \times \sum_{j=0}^{n} \prod_{i=1}^{6j+2} \frac{1}{1 + x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right), \quad k = \overline{0,2},$$

$$x_{18n+k+10} = x_{-8+k}\left(1 - \frac{x_{-2+k}x_{-5+k}x_{-11+k}x_{-14+k}x_{-17+k}}{1 + x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}} \times \sum_{j=0}^{n} \prod_{i=1}^{6j+3} \frac{1}{1 + x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right), \quad k = \overline{0,2},$$

$$x_{18n+k+13} = x_{-5+k}\left(1 - \frac{x_{-2+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}}{1 + x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}} \times \sum_{j=0}^{n} \prod_{i=1}^{6j+4} \frac{1}{1 + x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right), \quad k = \overline{0,2},$$

$$x_{18n+k+16} = x_{-2+k}\Bigg(1 - \frac{x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}}{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}} \times \sum_{j=0}^{n}\prod_{i=1}^{6j+5}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\Bigg), \quad k=\overline{0,2}.$$

f) If $x_{18n+k} \to a_k \neq 0$, $x_{18n+k+3} \to a_{k+3} \neq 0$, $x_{18n+k+6} \to a_{k+6} \neq 0$, $x_{18n+k+9} \to a_{k+9} \neq 0$, $x_{18n+k+12} \to a_{k+12} \neq 0$ then $x_{18n+k+15} \to 0$ as $n \to \infty$, $k = \overline{1,3}$.

Proof. a) Firstly, from 1.1, we get

$$x_{n+1}\left(1 + x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}\right) = x_{n-17}.$$

If $x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14} \in (0,+\infty)$, then

$$\left(1 + x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}\right) \in (1,+\infty).$$

Since $x_{n+1} < x_{n-17}$, $n \in \mathbb{N}$, we obtain that there exist

$$\lim_{n\to\infty} x_{18n-17+k} = a_{1+k}, \quad k = \overline{0,17}.$$

b) $(a_1, a_2, ..., a_{18}, a_1, a_2, ..., a_{18}, ...)$ is a solution of 1.1 having period 18.

c) In view of the 1.1,

$$x_{18n+1} = \frac{x_{18n-17}}{1+\prod_{t=0}^{4} x_{18n-3t-2}}.$$

Passing to the limit as $n \to \infty$ on both sides of the above equality, we get

$$\lim_{n\to\infty} x_{18n+1} = \lim_{n\to\infty}\frac{x_{18n-17}}{1+\prod_{t=0}^{4} x_{18n-3t-2}}.$$

Then

$$\prod_{t=0}^{5}\lim_{n\to\infty} x_{18n-3t-2} = 0, \prod_{t=1}^{6} a_{3t-2} = 0.$$

Similarly way we can obtain,

$$\prod_{t=0}^{5}\lim_{n\to\infty} x_{18n-3t-1} = 0, \prod_{t=1}^{6} a_{3t-1} = 0, \quad \prod_{t=0}^{5}\lim_{n\to\infty} x_{18n-3t} = 0, \prod_{t=1}^{6} a_{3t} = 0.$$

d) If there exist $n_0 \in \mathbb{N}$ such that $x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14} \geq x_{n+1}x_{n-2}x_{n-5}x_{n-8}x_{n-11}$ for all $n \geq n_0$, then $a_1 \leq a_4 \leq a_7 \leq a_{10} \leq a_{13} \leq a_{16} \leq a_1$, $a_2 \leq a_5 \leq a_8 \leq a_{11} \leq a_{14} \leq a_{17} \leq a_2$, $a_3 \leq a_6 \leq a_9 \leq a_{12} \leq a_{15} \leq a_{18} \leq a_3$. Since

$$\prod_{t=0}^{5} a_{3t+2} = 0, \prod_{t=0}^{5} a_{3t+1} = 0, \prod_{t=1}^{6} a_{3t} = 0,$$

we obtain the required result.

e) Subtracting x_{n-17} from the left and right-hand sides of equation 1.1, we obtain:

$$x_{n+1} - x_{n-17} = \frac{1}{1+x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}}\left(x_{n-2} - x_{n-20}\right). \tag{2.1}$$

From 2.1, for $n \geq 3$ following formula is produced.

$$x_{6n-17} - x_{6n-35} = (x_1 - x_{-17}) \prod_{i=1}^{n-6} \frac{1}{1+x_{3i-2}x_{3i-5}x_{3i-8}x_{3i-11}x_{3i-14}},$$
$$x_{6n-16} - x_{6n-34} = (x_2 - x_{-16}) \prod_{i=1}^{n-6} \frac{1}{1+x_{3i-1}x_{3i-4}x_{3i-7}x_{3i-10}x_{3i-13}}, \quad (2.2)$$
$$x_{6n-15} - x_{6n-33} = (x_3 - x_{-15}) \prod_{i=1}^{n-6} \frac{1}{1+x_{3i}x_{3i-3}x_{3i-6}x_{3i-9}x_{3i-12}},$$

Replacing n by $6j$ in 2.2 and summing from $j = 0$ to $j = n$, we obtain:

$$x_{18n+1+k} - x_{-17+k} = \left(x_{1+k} - x_{-17+k}\right) \times \sum_{j=0}^{n} \prod_{i=1}^{6j} \frac{1}{1 + x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}, \quad k = \overline{0,2},$$

Also, replacing n by $6j+1$ in 2.2 and summing from $j = 0$ to $j = n$, we obtain:

$$x_{18n+4+k} - x_{-14+k} = \left(x_{4+k} - x_{-14+k}\right) \times \sum_{j=0}^{n} \prod_{i=1}^{6j+1} \frac{1}{1 + x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}, \quad k = \overline{0,2},$$

Also, replacing n by $6j+2$ in 2.2 and summing from $j = 0$ to $j = n$, we obtain:

$$x_{18n+7+k} - x_{-11+k} = \left(x_{7+k} - x_{-11+k}\right) \times \sum_{j=0}^{n} \prod_{i=1}^{6j+2} \frac{1}{1 + x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}, \quad k = \overline{0,2},$$

Also, replacing n by $6j+3$ in 2.2 and summing from $j = 0$ to $j = n$, we obtain:

$$x_{18n+10+k} - x_{-8+k} = \left(x_{10+k} - x_{-8+k}\right) \times \sum_{j=0}^{n} \prod_{i=1}^{6j+3} \frac{1}{1 + x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}, \quad k = \overline{0,2},$$

Also, replacing n by $6j+4$ in 2.2 and summing from $j = 0$ to $j = n$, we obtain:

$$x_{18n+13+k} - x_{-5+k} = \left(x_{10+k} - x_{-8+k}\right) \times \sum_{j=0}^{n} \prod_{i=1}^{6j+4} \frac{1}{1 + x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}, \quad k = \overline{0,2},$$

Also, replacing n by $6j+5$ in 2.2 and summing from $j=0$ to $j=n$, we obtain:

$$x_{18n+16+k}-x_{-2+k}=\left(x_{16+k}-x_{-2+k}\right)\times\sum_{j=0}^{n}\prod_{i=1}^{6j+5}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}},\quad k=\overline{0,2}$$

Now, we obtained of the above formulas:

$$x_{18n+k+1}=x_{-17+k}\left(1-\frac{x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}}{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}}\times\sum_{j=0}^{n}\prod_{i=1}^{6j}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right),\quad k=\overline{0,2},$$

$$x_{18n+k+4}=x_{-14+k}\left(1-\frac{x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-17+k}}{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}}\times\sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right),\quad k=\overline{0,2},$$

$$x_{18n+k+7}=x_{-11+k}\left(1-\frac{x_{-2+k}x_{-5+k}x_{-8+k}x_{-14+k}x_{-17+k}}{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}}\times\sum_{j=0}^{n}\prod_{i=1}^{6j+2}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right),\quad k=\overline{0,2},$$

$$x_{18n+k+10}=x_{-8+k}\left(1-\frac{x_{-2+k}x_{-5+k}x_{-11+k}x_{-14+k}x_{-17+k}}{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}}\times\sum_{j=0}^{n}\prod_{i=1}^{6j+3}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right),\quad k=\overline{0,2},$$

$$x_{18n+k+13}=x_{-5+k}\left(1-\frac{x_{-2+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}}{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}}\times\sum_{j=0}^{n}\prod_{i=1}^{6j+4}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right),\quad k=\overline{0,2},$$

$$x_{18n+k+16}=x_{-2+k}\left(1-\frac{x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}}{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}}\times\sum_{j=0}^{n}\prod_{i=1}^{6j+5}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right),\quad k=\overline{0,2}.$$

f) Suppose that $a_1 = a_4 = a_7 = a_{10} = a_{13} = a_{16} = 0$. By e, we have

$$\lim_{n\to\infty} x_{18n+k+1} = \lim_{n\to\infty} x_{-17+k}\left(1 - \frac{x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}}{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}} \times \sum_{j=0}^{n}\prod_{i=1}^{6j}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right),$$

$$a_{1+k} = x_{-17+k}\left(1 - \frac{x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}}{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}} \times \sum_{j=0}^{\infty}\prod_{i=1}^{6j}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right),$$

$$a_{1+k} = 0 \Rightarrow \left(\frac{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}}{x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right). \tag{2.3}$$

Similarly,

$$\lim_{n\to\infty} x_{18n+4+k} = \lim_{n\to\infty} x_{-14+k}\left(1 - \frac{x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-17+k}}{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}} \times \sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right),$$

$$a_{4+k} = 0 \Rightarrow \left(\frac{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}}{x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-17+k}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+1}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right). \tag{2.4}$$

Similarly,

$$\lim_{n\to\infty} x_{18n+k+7} = \lim_{n\to\infty} x_{-11+k}\left(1 - \frac{x_{-2+k}x_{-5+k}x_{-8+k}x_{-14+k}x_{-17+k}}{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}} \times \sum_{j=0}^{n}\prod_{i=1}^{6j+2}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right),$$

$$a_{7+k} = 0 \Rightarrow \left(\frac{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}}{x_{-2+k}x_{-5+k}x_{-8+k}x_{-14+k}x_{-17+k}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+2}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right). \tag{2.5}$$

Similarly,

$$lim_{n\to\infty}x_{18n+k+10} = lim_{n\to\infty}x_{-8+k}\Bigg(1 - \frac{x_{-2+k}x_{-5+k}x_{-11+k}x_{-14+k}x_{-17+k}}{1 + x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}} \times \sum_{j=0}^{n}\prod_{i=1}^{6j+3}\frac{1}{1 + x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\Bigg), \quad k = \overline{0,2},$$

$$a_{10+k} = 0 \Rightarrow \Bigg(\frac{1 + x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}}{x_{-2+k}x_{-5+k}x_{-11+k}x_{-14+k}x_{-17+k}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+3}\frac{1}{1 + x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\Bigg). \quad (2.6)$$

Similarly,

$$\lim_{n\to\infty} x_{18n+k+13} = \lim_{n\to\infty} x_{-5+k}\Bigg(1 - \frac{x_{-2+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}}{1 + x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}} \times \sum_{j=0}^{n}\prod_{i=1}^{6j+4}\frac{1}{1 + x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\Bigg), \quad k = \overline{0,2},$$

$$a_{13+k} = 0 \Rightarrow \Bigg(\frac{1 + x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}}{x_{-2+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+4}\frac{1}{1 + x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\Bigg). \quad (2.7)$$

Similarly,

$$\lim_{n\to\infty} x_{18n+k+16} = \lim_{n\to\infty} x_{-2+k}\Bigg(1 - \frac{x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}}{1 + x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}} \times \sum_{j=0}^{n}\prod_{i=1}^{6j+5}\frac{1}{1 + x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\Bigg), \quad k = \overline{0,2}.$$

$$a_{16+k} = 0 \Rightarrow \Bigg(\frac{1 + x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}}{x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+5}\frac{1}{1 + x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\Bigg). \quad (2.8)$$

From 2.3 and 2.4

$$\left(\frac{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}}{x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right) >$$
$$\left(\frac{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}}{x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-17+k}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+1}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right).$$

Thus, $x_{-17+k} > x_{-14+k}$. From 2.4 and 2.5

$$\left(\frac{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}}{x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-17+k}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+1}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right) >$$
$$\left(\frac{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}}{x_{-2+k}x_{-5+k}x_{-8+k}x_{-14+k}x_{-17+k}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+2}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right).$$

Thus, $x_{-14+k} > x_{-11+k}$. From 2.5 and 2.6

$$\left(\frac{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}}{x_{-2+k}x_{-5+k}x_{-8+k}x_{-14+k}x_{-17+k}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+2}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right) >$$
$$\left(\frac{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}}{x_{-2+k}x_{-5+k}x_{-11+k}x_{-14+k}x_{-17+k}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+3}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right).$$

Thus, $x_{-11+k} > x_{-8+k}$. From 2.6 and 2.7

$$\left(\frac{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}}{x_{-2+k}x_{-5+k}x_{-11+k}x_{-14+k}x_{-17+k}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+3}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right) >$$

$$\left(\frac{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}}{x_{-2+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+4}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right).$$

Thus, $x_{-8+k} > x_{-5+k}$. From 2.7 and 2.8

$$\left(\frac{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}}{x_{-2+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+4}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right) >$$

$$\left(\frac{1+x_{-2+k}x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}}{x_{-5+k}x_{-8+k}x_{-11+k}x_{-14+k}x_{-17+k}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+5}\frac{1}{1+x_{3i-2+k}x_{3i-5+k}x_{3i-8+k}x_{3i-11+k}x_{3i-14+k}}\right).$$

Thus, $x_{-5+k} > x_{-2+k}$. We obtain $x_{-17+k} > x_{-14+k} > x_{-11+k} > x_{-8+k} > x_{-5+k} > x_{-2+k}$. Similarly we can obtain $x_{-16+k} > x_{-13+k} > x_{-10+k} > x_{-7+k} > x_{-4+k} > x_{-1+k}$ and $x_{-15+k} > x_{-12+k} > x_{-9+k} > x_{-6+k} > x_{-3+k} > x_{+k}$. We arrive at a contradiction, which completes the proof of the theorem.

□

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